

MTH101 Practice Questions/ Solutions Lecture No. 23 to 45

Lecture No. 23: Maximum and Minimum Values of Functions

Q 1: Find the minimum value of the function $f(x) = x^3 - 27x + 4$ attains in the interval $[-4, 4]$.

Solution:

First of all we have to find critical points by putting $f'(x) = 0$

$$f(x) = x^3 - 27x + 4$$

$$f'(x) = 3x^2 - 27$$

$$\text{put } f'(x) = 0$$

$$\Rightarrow 3x^2 - 27 = 0$$

$$\Rightarrow 3(x^2 - 9) = 0$$

$$\Rightarrow x^2 - 9 = 0 \Rightarrow x = -3, 3$$

now we have points $-4, -3$ and 3 we will check on all these points

$$f(4) = (4)^3 - 27(4) + 4 = -40$$

$$f(-2) = (-4)^3 - 27(-4) + 4 = 48$$

$$f(3) = (3)^3 - 27(3) + 4 = -50$$

$$f(-3) = (-3)^3 - 27(-3) + 4 = 58$$

So minimum value = -50

Q 2: Find the maximum value of the function $f(x) = x^3 + 3x^2 - 9x$ attains in the interval $[-4, 3]$.

Solution:

First of all we have to find critical points by putting $f'(x) = 0$

$$f(x) = x^3 + 3x^2 - 9x$$

$$f'(x) = 3x^2 + 6x - 9$$

$$\text{put } f'(x) = 0$$

$$\Rightarrow 3x^2 + 6x - 9 = 0$$

$$\Rightarrow x^2 + 2x - 3 = 0$$

$$\Rightarrow x^2 + 3x - x - 3 = 0$$

$$\Rightarrow x(x+3) - (x-3) = 0$$

$$\Rightarrow (x-1)(x+3) = 0 \Rightarrow x = 1, x = -3$$

now we have points 1,-3, 3 and -4 we will check on all these points

$$f(1) = (1)^3 + 3(1)^2 - 9(1) = -5$$

$$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) = 27$$

$$f(3) = (3)^3 + 3(3)^2 - 9(3) = 27$$

$$f(-4) = (-4)^3 + 3(-4)^2 - 9(-4) = 20$$

$$\text{Hence maximum value} = f(-3) = f(3) = 27$$

Q 3: Find the absolute maximum and absolute minimum values of the function

$$f(x) = 4 - x^2 \text{ on interval } -3 \leq x \leq 1.$$

Solution:

First we will find critical points:

$$\text{put } f'(x) = 0$$

$$\Rightarrow -2x = 0$$

$$\Rightarrow x = 0$$

Now we find value of $f(x)$ at critical point and at the end points of interval :

$$f(0) = 4 - (0)^2 = 4$$

$$f(1) = 4 - (1)^2 = 3$$

$$f(-3) = 4 - (-3)^2 = -5$$

$$\text{Hence absolute maximum} = 4 \text{ and absolute minimum} = -5$$

Q 4: Find the absolute maximum and absolute minimum values of the function

$$f(x) = 2 + x \text{ on interval } -2 \leq x \leq 2.$$

Solution:

First we will find critical points:

Since $f'(x) = 1$, so there is no critical points.

Now we find value of $f(x)$ at the end points of interval :

$$f(-2) = 2 - 2 = 0$$

$$f(2) = 2 + 2 = 4$$

$$\text{Hence absolute maximum} = 4 \text{ and absolute minimum} = 0$$

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Q 5: Find the maximum and minimum value of the function $f(x) = 3x^4 - 24x^2 + 1$ on the interval $(-\infty, +\infty)$.

Solution:

This is a continuous function on the given interval and

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} (3x^4 - 24x^2 + 1) = +\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} (3x^4 - 24x^2 + 1) = +\infty$$

So f has a minimum but no maximum value in the interval $(-\infty, +\infty)$. To find the minimum value put $f'(x) = 0$ i.e

$$f'(x) = 12x^3 - 48x = 0 \Rightarrow 12x(x^2 - 4) = 0 \Rightarrow x = 0 \text{ and } x = \pm 2 \text{ are the critical points.}$$

$$\text{At } x = 0, f(0) = 1,$$

$$\text{at } x = 2, f(2) = 3(2)^4 - 24(2)^2 + 1 = -47 \text{ and}$$

$$\text{at } x = -2, f(-2) = 3(-2)^4 - 24(-2)^2 + 1 = -47$$

so minimum value occurs at $x = \pm 2$ and it is equal to ' $f(x) = -47$ '

Lecture No. 24: Newton's Method, Rolle's Theorem and Mean Value Theorem

Lecture No. 25: Integrations

Q 1: Check the validity of Mean Value Theorem for $f(x) = 3x - x^3$ on the interval $[0, 2]$. Also find 'c' if possible

Solution: Because f is a polynomial so continuous and differentiable everywhere hence on $[0, 2]$. As hypothesis of the mean value theorem is satisfied so we can

Find a 'c' such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{where } f(a) = f(0) = 0 \quad \text{and} \quad f(b) = f(2) = -2$$

$$f'(x) = 3 - 3x^2$$

$$f'(c) = 3 - 3c^2$$

$$\frac{-2 - 0}{2 - 0} = 3 - 3c^2$$

$$3 - 3c^2 = -1$$

$$\Rightarrow 3(1 - c^2) = -1$$

$$\Rightarrow 1 - c^2 = -\frac{1}{3}$$

$$\Rightarrow c^2 = \frac{4}{3}$$

$$\Rightarrow c = \pm \frac{2}{\sqrt{3}}$$

$$\text{As } c = \frac{2}{\sqrt{3}} \in (0, 2) \text{ and } c = -\frac{2}{\sqrt{3}} \notin (0, 2) \text{ so value of } c = \frac{2}{\sqrt{3}}.$$

Q 2: To estimate the solution of the equation $x^3 - 7x - 1 = 0$ by using the newton's method if we start with $x_1 = 2$ then find x_2 .

Solution:

$$f(x) = x^3 - 7x - 1,$$

$$f'(x) = 3x^2 - 7,$$

$$x_1 = 2 \Rightarrow f(2) = (2)^3 - 7(2) - 1 = -7,$$

$$f'(2) = 3(2)^2 - 7 = 5.$$

$$\text{By Newton's Method } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{where } n = 1, 2, 3, \dots,$$

$$\text{So, for } n = 1, \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{(-7)}{5} = \frac{17}{5}.$$

Q 3: Evaluate the following integrals.

I. $\int \sec x(\sec x + \cos x)dx$

II. $\int \frac{5\sin x - 3\cos^2 x}{\cos^2 x} dx$

Solution:

$$\begin{aligned}\text{(I)} \quad & \int \sec x(\sec x + \cos x)dx \\ &= \int (\sec^2 x + \sec x \cos x)dx \\ &= \int \sec^2 x dx + \int \sec x \cos x dx \\ &= \tan x + \int \frac{\cos x}{\cos x} dx \\ &= \tan x + \int dx \\ &= \tan x + x + c\end{aligned}$$

$$\begin{aligned}\text{(II)} \quad & \int \frac{5\sin x - 3\cos^2 x}{\cos^2 x} dx \\ &= \int \left[5 \frac{\sin x}{\cos x \cos x} - 3 \right] dx \\ &= \int 5 \frac{\sin x}{\cos x} \frac{1}{\cos x} dx - 3 \int dx \\ &= \int 5 \tan x \sec x dx - 3 \int dx \\ &= 5 \sec x - 3x + c\end{aligned}$$

Q 4: Given that $f'(x) = 16x^3 - 4x$, find $f(3)$.

Solution:

First of all, we need to find the function $f(x)$ for which the derivative is given.

For that we will integral the derivative function which gives

$$\begin{aligned}f(x) &= \int f'(x) dx \\ &= \int (16x^3 - 4x) dx \\ &= \frac{16x^4}{4} - \frac{4x^2}{2} + c \\ f(x) &= 4x^4 - 2x^2 + c \\ f(3) &= 4(3)^4 - 2(3)^2 + c \\ f(3) &= 306 + c\end{aligned}$$

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Lecture No. 26: Integration by Substitution**Lecture No. 27: Sigma Notation****Lecture No. 28: Area as Limit**

Q 1: Evaluate the integral by using substitution method: $\int \frac{1}{t^2} \cos\left(\frac{1}{t}-1\right) dt$.

Solution:

$$\text{Let } u = \frac{1}{t} - 1,$$

$$\Rightarrow du = -\frac{1}{t^2} dt \Rightarrow \frac{1}{t^2} dt = -du,$$

$$\begin{aligned} \therefore \int \frac{1}{t^2} \cos\left(\frac{1}{t}-1\right) dt &= \int \cos u (-du), \\ &= -\sin u + C, \\ &= -\sin\left(\frac{1}{t}-1\right) + C. \quad (\because \text{by replacing with the original value}) \end{aligned}$$

Q 2: Evaluate the indefinite integral by substitution method: $\int \frac{(1+\ln x)^3}{x} dx$.

Solution:

$$\text{Let } t = 1 + \ln x,$$

$$\Rightarrow dt = \frac{1}{x} dx,$$

$$\text{Hence, } \int \frac{(1+\ln x)^3}{x} dx = \int t^3 dt = \frac{t^4}{4} + C = \frac{(1+\ln x)^4}{4} + C.$$

Q 3: Evaluate the sum: $\sum_{k=1}^7 (k^2 - 6)$.

Solution:

$$\begin{aligned} \sum_{k=1}^7 (k^2 - 6) &= \sum_{k=1}^7 k^2 - \sum_{k=1}^7 6, \\ &= \frac{7(7+1)(2(7)+1)}{6} - 6(7), \\ &= \frac{840}{6} - 42, \\ &= 98. \end{aligned}$$

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Q 4: Express $\sum_{k=2}^5 3^{k-2}$ in sigma notation so that the lower limit is '0' rather than '2'.

Solution:

We will define a new summation index 'j' by the relation

$$j = k - 2 \text{ or } j + 2 = k,$$

Now when

$$k = 2, j = 2 - 2 = 0,$$

When

$$k = 5, j = 5 - 2 = 3,$$

So the new summation will become $\sum_{j=0}^3 3^{j+2-2} = \sum_{j=0}^3 3^j$.

Q 5: Find the area of the k^{th} rectangle below the curve $y = x^2$ on the interval $[0, 2]$ by taking x_k^* as right end point and left end point.

Solution:

In order to find the area of k^{th} rectangle, first of all we will find the width or base of the rectangle that is Δx .

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

Right end point

$$x_k^* = a + k\Delta x,$$

$$= 0 + k \cdot \frac{2}{n} = \frac{2k}{n}.$$

The height of the rectangle is

$$f(x_k^*) = \left(\frac{2k}{n}\right)^2 = \frac{4k^2}{n^2}.$$

Thus, the area of the k^{th} rectangle will be

Area = height $\times$ base

$$= f(x_k^*) \cdot \Delta x,$$

$$= \frac{4k^2}{n^2} \cdot \frac{2}{n} = \frac{8k^2}{n^3}.$$

Left end point

$$x_k^* = a + (k-1)\Delta x,$$

$$= 0 + (k-1) \cdot \frac{2}{n} = \frac{2(k-1)}{n}.$$

The height of the rectangle is

$$f(x_k^*) = \left(\frac{2(k-1)}{n}\right)^2 = \frac{4(k-1)^2}{n^2}.$$

Thus, the area of the k^{th} rectangle will be

Area = height $\times$ base

$$= f(x_k^*) \cdot \Delta x,$$

$$= \frac{4(k-1)^2}{n^2} \cdot \frac{2}{n} = \frac{8(k-1)^2}{n^3}.$$

Q 6: Find the approximate area under the graph of function $y = x$ over the interval $[0, 2]$

by taking $\Delta x = \frac{2}{n}$ and $x_k^* = \frac{2k}{n}$.

Solution:

Given that $\Delta x = \frac{2}{n}$ and $x_k^* = \frac{2k}{n}$,

$$f(x_k^*) = \frac{2k}{n},$$

$$f(x_k^*) \cdot \Delta x = \frac{2k}{n} \cdot \frac{2}{n} = \frac{4k}{n^2},$$

$$\sum_{k=1}^n f(x_k^*) \cdot \Delta x = \sum_{k=1}^n \frac{4k}{n^2},$$

$$= \frac{4}{n^2} \sum_{k=1}^n k,$$

$$= \frac{4}{n^2} \frac{n(n+1)}{2} = \frac{2(n+1)}{n} = 2 \left(1 + \frac{1}{n} \right).$$

$$\text{Area} = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n f(x_k^*) \cdot \Delta x \right) = \lim_{n \rightarrow \infty} 2 \left(1 + \frac{1}{n} \right) = 2.$$

Lecture No. 29: Definite Integral**Lecture No. 30: First Fundamental Theorem of Calculus****Lecture No. 31: Evaluating Definite Integral by Substitution**

Q 1: Find $\int_2^5 f(x)dx$ if $\int_3^2 f(x)dx = -5$, $\int_3^4 f(x)dx = 3$, $\int_5^4 f(x)dx = 4$

Solution:

$$\begin{aligned}\int_5^4 f(x)dx = 4 &\Rightarrow \int_4^5 f(x)dx = -4 \text{ and } \int_3^2 f(x)dx = -5 \Rightarrow \int_2^3 f(x)dx = 5 \\ \therefore \int_2^5 f(x)dx &= \int_2^3 f(x)dx + \int_3^4 f(x)dx + \int_4^5 f(x)dx \\ &= 5 + 3 - 4 = 4\end{aligned}$$

Q 2: Express the area of the region below the line $7x + 2y = 25$, above x-axis and between the lines $x = 0$, $x = 4$ as a definite integral. Also express this integral as a limit of the Riemann Sum.

Solution:

$$7x + 2y = 25$$

$$2y = 25 - 7x \Rightarrow y = \frac{25 - 7x}{2}$$

$$\int_0^4 \frac{25 - 7x}{2} dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n \left(\frac{25 - 7x_k^*}{2} \right) \Delta x_k$$

Q 3: Evaluate the integral $\int_1^3 \frac{x^3 - 1}{x - 1} dx$.

Solution:

$$\begin{aligned}\int_1^3 \frac{x^3 - 1}{x - 1} dx &= \int_1^3 \frac{(x - 1)(x^2 + x + 1)}{x - 1} dx \\ &= \int_1^3 (x^2 + x + 1) dx \\ &= \left. \frac{x^3}{3} + \frac{x^2}{2} + x \right|_1^3 \\ &= \left(\frac{27}{3} + \frac{9}{2} + 3 \right) - \left(\frac{1}{3} + \frac{1}{2} + 1 \right) = \left(12 + \frac{9}{2} \right) - \left(\frac{2 + 3 + 6}{6} \right) \\ &= \left(\frac{24 + 9}{2} \right) - \frac{11}{6} = \frac{33}{2} - \frac{11}{6} = \frac{99 - 11}{6} = \frac{88}{6} = \frac{44}{3}\end{aligned}$$

Q 4: Evaluate the definite integral $\int_1^4 (3 + 3\sqrt{x}) dx$.

Solution:

$$\begin{aligned}\int_1^4 (3 + 3\sqrt{x}) dx &= 3x + 3 \frac{x^{3/2}}{3/2} \bigg|_1^4 \\&= 3x + 2x^{3/2} \bigg|_1^4 \\&= \left(3(4) + 2(4)^{3/2} \right) - \left(3(1) + 2(1)^{3/2} \right) = (12 + 2(8)) - (3 + 2) = 28 - 5 = 23\end{aligned}$$

Q 5: Evaluate the integral $\int \frac{1}{\sqrt{x} \cdot (2 + \sqrt{x})^3} dx$ by using proper substitution.

Solution:

$$\text{Let } u = 2 + \sqrt{x}$$

$$\Rightarrow du = \frac{1}{2\sqrt{x}} dx$$

$$\Rightarrow 2du = \frac{1}{\sqrt{x}} dx$$

Putting these values in the given integral, it becomes

$$\begin{aligned}\int \frac{1}{\sqrt{x} \cdot (2 + \sqrt{x})^3} dx &= 2 \int \frac{1}{u^3} du = \\&= 2 \int u^{-3} du \\&= 2 \left| \frac{u^{-3+1}}{-3+1} \right| = \frac{2}{-2u^2} \\&= \frac{-1}{(2 + \sqrt{x})^2} + c\end{aligned}$$

Q 6: Use the Substitution method to express the following definite integrals in terms of the variable 'u' but do not evaluate the integrals.

i. $\int_0^{\frac{\pi}{2}} e^{\sin t} \cos t \, dt$

ii. $\int_0^1 4t \sqrt{1-t^2} \, dt$

Solution:

i.

Let $u = \sin t$

$\Rightarrow du = \cos t \, dt$

$t = 0 \Rightarrow u = \sin 0 = 0$

and $t = \frac{\pi}{2} \Rightarrow u = \sin \frac{\pi}{2} = 1$

as t goes from 0 to $\frac{\pi}{2}$

so u goes from 0 to 1

$$\int_0^{\frac{\pi}{2}} e^{\sin t} \cos t \, dt = \int_0^1 e^u (du) = \int_0^1 e^u \, du$$

ii.

Let

$u = 1 - t^2$

$\Rightarrow du = -2t \, dt$

$-2du = 4t \, dt$

$t = 1 \Rightarrow u = 1 - (1)^2 = 0$

and $t = 0 \Rightarrow u = 1 - (0)^2 = 1$

as t goes from 0 to 1

so u goes from 1 to 0

$$\int_0^1 4t \sqrt{1-t^2} \, dt = -2 \int_1^0 \sqrt{u} \, du = 2 \int_0^1 \sqrt{u} \, du \quad \left(\begin{array}{l} \because -2t \, dt = du \\ 4t \, dt = -2 \, du \end{array} \right)$$

Lecture No. 32: Second Fundamental Theorem of Calculus**Lecture No. 33: Application of Definite Integral****Lecture No. 34: Volume by slicing; Disks and Washers**

Q 1: Find a definite integral indicating the area enclosed by the curves $y = x^2 - 6x - 7$ and $y = x + 1$. But do not evaluate it further.

Solution:

Here $y = x^2 - 6x - 7$ _____ (1),

and $y = x + 1$ _____ (2),

equating (1) and (2)

$$\Rightarrow x^2 - 6x - 7 = x + 1,$$

$$x^2 - 6x - 7 - x - 1 = 0,$$

$$x^2 - 7x - 8 = 0,$$

$$x^2 - 8x + x - 8 = 0,$$

$$x(x - 8) + 1(x - 8) = 0,$$

$$\therefore (x - 8)(x + 1) = 0,$$

$$\Rightarrow x = -1 \text{ and } x = 8,$$

as $x + 1 \geq x^2 - 6x - 7$ for $-1 \leq x \leq 8$,

$$\therefore \int_{-1}^8 ((x + 1) - (x^2 - 6x - 7)) dx.$$

Q 2: Find the area of the region enclosed by the curves $y = x^3$ and $y = x$ between $x = 0$ and $x = \frac{1}{3}$.

Solution:

Since we have $y = x^3$ and $y = x$

$$\Rightarrow x \geq x^3 \text{ in the interval } \left[0, \frac{1}{3}\right].$$

$$\begin{aligned} \text{Thus required area is } A &= \int_0^{\frac{1}{3}} (x - x^3) dx = \left. \frac{x^2}{2} - \frac{x^4}{4} \right|_0^{\frac{1}{3}} \\ &= \frac{1}{2} \left(\frac{1}{3} \right)^2 - \frac{1}{4} \left(\frac{1}{3} \right)^4 = \frac{1}{18} - \frac{1}{324} \\ &= \frac{17}{324} \end{aligned}$$

Q 3: Evaluate $\int_{\pi}^0 \frac{1 + \sin 2t}{2} dt$.

Solution:

$$\begin{aligned}\int_{\pi}^0 \frac{1 + \sin 2t}{2} dt &= \int_{\pi}^0 \left(\frac{1}{2} + \frac{\sin 2t}{2} \right) dt \\&= \int_{\pi}^0 \frac{1}{2} dt + \int_{\pi}^0 \frac{\sin 2t}{2} dt \\&= \frac{1}{2} \left| t \right|_{\pi}^0 - \frac{1}{4} \left| \cos 2t \right|_{\pi}^0 \\&= \frac{1}{2} (0 - \pi) - \frac{1}{4} (\cos 0 - \cos 2\pi) \\&= -\frac{\pi}{2} - \frac{1}{4} (1 - 1) = -\frac{\pi}{2}\end{aligned}$$

Q 4: Evaluate $\frac{d}{dx} \left[\int_1^x \cos t dt \right]$ by using second fundamental theorem of calculus. Also

Check the validity of this theorem.

Solution:

Since $f(x) = \cos x$ is continuous on $(1, x)$, so by second fundamental theorem of calculus

$$\frac{d}{dx} \left[\int_1^x \cos t dt \right] = \cos x$$

Now we will verify this result by evaluating this integral.

$$\int_1^x \cos t dt = \left| \sin t \right|_1^x = \sin x - \sin 1$$

$$\Rightarrow \frac{d}{dx} \int_1^x \cos t dt = \frac{d}{dx} (\sin x - \sin 1) = \cos x = f(x)$$

Q 5: Find the volume of a solid bounded by two parallel planes perpendicular to the x-axis at $x = 0$ and $x = 3$, where the cross section perpendicular to the x-axis is a rectangle having dimensions 'a' and 'b'

Solution:

Here dimensions are a & $b \Rightarrow A = ab$

$$V = \int_0^3 ab \, dx \quad \left(\because V = \int_c^d A(x) \, dx \right)$$

$$= ab \int_0^3 dx \Rightarrow ab \cdot x \Big|_0^3 \Rightarrow 3ab$$

Q 6: Find the volume of the solid generated when the region between the graphs of

$f(x) = \frac{1}{2} + x$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

Solution:

Since the volume enclosed between two curves $y = f(x)$ and $y = g(x)$ revolve about the x-axis so

$$V = \int_a^b \pi (g^2(x) - f^2(x)) \, dx$$

Here $f(x) = \frac{1}{2} + x$ and $g(x) = x$ as $f(x) \geq g(x)$ for $0 \leq x \leq 2$

$a = 0$, $b = 2$, so

$$V = \int_0^2 \pi \left(\left(\frac{1}{2} + x \right)^2 - x^2 \right) dx$$

$$V = \int_0^2 \pi \left(\frac{1}{4} + x^2 + x - x^2 \right) dx$$

$$= \int_0^2 \pi \left(\frac{1}{4} + x \right) dx \Rightarrow \pi \left[\frac{x}{4} + \frac{x^2}{2} \right]_0^2 \Rightarrow \frac{5\pi}{2}$$

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Lecture No. 35: Volume by Cylindrical Shells**Lecture No. 36: Length of Plane Curves****Lecture No. 37: Area of Surface of Revolution**

Q 1: Use cylindrical shells to find the volume of the solid generated when the region 'R' in the first quadrant enclosed between $y = 4x$ and $y = x^3$ is revolved about the y -axis.

Solution:

To find the limit of integration, we will find the point of intersection between two curves.

Equating $y = 4x$ and $y = x^3$, we get

$$\begin{aligned}x^3 &= 4x, \\ \Rightarrow x^3 - 4x &= 0, \\ \Rightarrow x(x^2 - 4) &= 0, \\ \Rightarrow x &= 0, x = \pm 2.\end{aligned}$$

Since our region R is in first quadrant so we ignore $x = -2$. Hence limit of integration is $x = 0$ to $x = 2$.

$$\begin{aligned}\text{So } V &= \int_0^2 2\pi x(4x - x^3) dx, \\ &= 2\pi \int_0^2 (4x^2 - x^4) dx, \\ &= 2\pi \left[4\frac{x^3}{3} - \frac{x^5}{5} \right]_0^2.\end{aligned}$$

After simplification, we get

$$V = \frac{128}{15} \pi.$$

Q 2: Use cylindrical shell method to find the volume of the solid generated when the region enclosed between $y = x^3$ and the x -axis in the interval $[0, 3]$ is revolved about the y -axis.

Solution:

$$\begin{aligned}V &= \int_0^3 2\pi x(x^3) dx = 2\pi \int_0^3 x^4 dx, & \left(\begin{array}{l} \because \text{Here } f(x) = x^3, \quad a = 0; \quad b = 3. \\ \text{Cylindrical shells revolved about the } y\text{-axis: } V = \int_a^b 2\pi x f(x) dx \end{array} \right) \\ &= 2\pi \cdot \frac{1}{5} \left[x^5 \right]_0^3.\end{aligned}$$

After simplification, we get $V = \frac{486}{5} \pi.$

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Q 3: Find the arc length of the curve $y = \frac{2}{3}(x-1)^{\frac{3}{2}}$ from $x = 0$ to $x = \frac{1}{2}$.

Solution:

$$\begin{aligned}\therefore y &= \frac{2}{3}(x-1)^{\frac{3}{2}}, \\ \Rightarrow \frac{dy}{dx} &= \frac{2}{3} \left(\frac{3}{2} \right) (x-1)^{\frac{1}{2}} = (x-1)^{\frac{1}{2}}. \\ \therefore L &= \int_a^b \sqrt{1 + [f'(x)]^2} dx, \\ \Rightarrow L &= \int_0^{\frac{1}{2}} \sqrt{1 + \left[(x-1)^{\frac{1}{2}} \right]^2} dx = \int_0^{\frac{1}{2}} \sqrt{(1+x-1)} dx = \int_0^{\frac{1}{2}} \sqrt{x} dx, \\ &= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^{\frac{1}{2}}, \\ &= \frac{2}{3} \left(\frac{1}{2} \right)^{\frac{3}{2}}, \\ &= \frac{1}{3\sqrt{2}}.\end{aligned}$$

Q 4: If f is a smooth function on $[1, 4]$, then find a definite integral indicating the arc length of the curve $x = y^{\frac{2}{3}}$ from $y = 1$ to $y = 4$. **Note:** Do not evaluate it further.

Solution:

$$\begin{aligned}\text{Here } x &= g(y) = y^{\frac{2}{3}}, \\ \Rightarrow g'(y) &= \frac{2}{3} y^{-\frac{1}{3}}. \\ \therefore \text{Arc length } L &= \int_a^b \sqrt{1 + [g'(y)]^2} dy, \\ \therefore L &= \int_1^4 \sqrt{1 + \frac{4}{9} y^{-\frac{2}{3}}} dy.\end{aligned}$$

Q 5: Find the area of the surface generated by revolving the curve $y = \sqrt{1-x^2}$; $0 \leq x \leq 1$ about the x – axis.

Solution:

$$\because f(x) = y = \sqrt{1-x^2},$$

$$\Rightarrow f'(x) = -\frac{x}{\sqrt{1-x^2}}.$$

$$\because S = \int_a^b 2\pi f(x) \sqrt{1+[f'(x)]^2} dx,$$

$$\Rightarrow S = \int_0^1 2\pi \sqrt{1-x^2} \sqrt{1+\left[-\frac{x}{\sqrt{1-x^2}}\right]^2} dx,$$

$$= \int_0^1 2\pi \sqrt{1-x^2} \sqrt{1+\frac{x^2}{1-x^2}} dx,$$

$$= \int_0^1 2\pi \sqrt{1-x^2} \sqrt{\frac{1-x^2+x^2}{1-x^2}} dx,$$

$$= \int_0^1 2\pi dx,$$

$$= 2\pi x \Big|_0^1,$$

$$= 2\pi.$$

Q 6: Write a definite integral indicating the area of the surface generated by revolving the curve $y = x^2$; $0 \leq x \leq 2$ about the x – axis. **Note:** Do not evaluate it further.

Solution:

$$\because y = f(x) = x^2 ; 0 \leq x \leq 2,$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2x.$$

$$\therefore S = \int_0^2 2\pi (x^2) \sqrt{1+(2x)^2} dx, \quad \left(\because S = \int_c^d 2\pi f(x) \sqrt{1+[f'(x)]^2} dx \right)$$

$$= 2\pi \int_0^2 x^2 \sqrt{1+4x^2} dx.$$

Lecture No. 38: Work and Definite Integral**Lecture No. 39: Improper Integral****Lecture No. 40: L'Hopital's Rule**

Q 1: Find the work done by the force $500x$ if an object moves in the positive direction over the interval $[0.16, 0.19]$?

Solution:

Here

$$F(x) = 500x; \quad [0.16, 0.19]$$

$$W = \int_{0.16}^{0.19} 500x \, dx \quad \text{since } W = \int_a^b F(x) \, dx$$

$$= \left| \frac{500x^2}{2} \right|_{0.16}^{0.19}$$

$$= \frac{500}{2} |x^2|_{0.16}^{0.19}$$

$$= 250[(0.19)^2 - (0.16)^2]$$

$$= 2.625 \text{ J}$$

Q 2: Find the spring constant if a force took 1800 J of work to stretch a spring from its natural length of 5m to a length of 8m ?

Solution:

The work done by F is $W = \int_0^3 F(x) \, dx$

$$1800 = \int_0^3 Kx \, dx$$

$$= k \int_0^3 x \, dx$$

$$= k \left| \frac{x^2}{2} \right|_0^3$$

$$= \frac{9k}{2}$$

$$3600 = 9k$$

or

$$k = 400$$

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Q 3: Evaluate the improper integral:

$$\int_{-\infty}^0 \frac{1}{(2x+1)^3} dx$$

Solution:

$$\begin{aligned} & \int_{-\infty}^0 \frac{1}{(2x+1)^3} dx \\ &= \lim_{t \rightarrow \infty} \int_t^0 \frac{1}{(2x+1)^3} dx \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_t^0 \frac{1}{(2x+1)^3} (2dx) \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_t^0 (2x+1)^{-3} (2dx) \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \left| \frac{(2x+1)^{-2}}{-2} \right|_t^0 \\ &= \lim_{t \rightarrow \infty} \frac{-1}{4} |(2x+1)^{-2}|_t^0 \\ &= \frac{-1}{4} \lim_{t \rightarrow \infty} [1 - (2t+1)^{-2}] \\ &= \frac{-1}{4} \lim_{t \rightarrow \infty} \left[1 - \frac{1}{(2t+1)^2} \right] \\ &= \frac{-1}{4} [1 - 0] \\ &= \frac{-1}{4} \end{aligned}$$

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Q 4: Solve the improper integral: $\int_1^5 \frac{1}{(x-2)^{\frac{2}{3}}} dx$.

Solution:

$$\begin{aligned}
 & \int_1^5 \frac{1}{(x-2)^{\frac{2}{3}}} dx \\
 &= \int_1^2 \frac{dx}{(x-2)^{\frac{2}{3}}} + \int_2^5 \frac{dx}{(x-2)^{\frac{2}{3}}} \\
 &= \int_1^2 (x-2)^{-\frac{2}{3}} dx + \int_2^5 (x-2)^{-\frac{2}{3}} dx \\
 &= 3 \left| (x-2)^{\frac{1}{3}} \right|_1^2 + 3 \left| (x-2)^{\frac{1}{3}} \right|_2^5 \\
 &= 3[0 - (-1)^{1/3}] + 3[(3)^{1/3} - 0] \\
 &= 3[(3)^{1/3} - (-1)^{1/3}] \\
 &= 3[\sqrt[1/3]{3} - \sqrt[1/3]{-1}]
 \end{aligned}$$

Q 5: Use L'Hopital's rule to evaluate the limit:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\cos 2x + 1}$$

Solution:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\cos 2x + 1} \quad \frac{0}{0} \text{ form}$$

By L'Hopital's rule

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{-2 \sin 2x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{-2 \sin 2x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{-4 \sin x \cos x} \quad \text{since } \sin 2x = 2 \sin x \cos x$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{-4 \sin x}$$

$$= \frac{-1}{4}$$

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Q 6: Evaluate the limit: $\lim_{x \rightarrow \infty} \sqrt{x^2 - 5x} - x$.

Solution:

$$\begin{aligned} & \lim_{x \rightarrow \infty} \sqrt{x^2 - 5x} - x \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 5x} - x}{\sqrt{x^2 - 5x} + x} \times \sqrt{x^2 - 5x} + x \\ &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 - 5x})^2 - x^2}{\sqrt{x^2 - 5x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - 5x - x^2}{\sqrt{x^2 - 5x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{-5x}{\sqrt{x^2 - 5x} + x} \\ &= \lim_{x \rightarrow \infty} \frac{-5x}{\sqrt{x^2(1 - \frac{5}{x})} + x} \\ &= \lim_{x \rightarrow \infty} \frac{-5x}{x\sqrt{(1 - \frac{5}{x})} + x} \\ &= \lim_{x \rightarrow \infty} \frac{-5}{x(\sqrt{(1 - \frac{5}{x})} + 1)} \\ &= \lim_{x \rightarrow \infty} \frac{-5}{\sqrt{(1 - \frac{5}{x})} + 1} \\ &= \frac{-5}{2} \end{aligned}$$

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Lecture No. 41: Sequence**Lecture No. 42: Infinite Series****Lecture No. 43: Additional Convergence tests**

Q 1: Determine whether the sequence $\{a_n\}$ converges or diverges? Where $a_n = \frac{n+1}{n^2+3}$.

Solution:

$$\text{Here } a_n = \frac{n+1}{n^2+3}$$

$$\text{We will calculate the limit } \lim_{n \rightarrow \infty} \frac{n+1}{n^2+3}.$$

Dividing by 'n²' in denominator and numerator

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{\frac{n}{n^2} + \frac{1}{n^2}}{\frac{n^2}{n^2} + \frac{3}{n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{3}{n^2}} = \frac{\frac{1}{\infty} + \frac{1}{\infty}}{1 + \frac{3}{\infty}} = \frac{0}{1} = 0 \end{aligned}$$

The sequence is convergent and its limit is '0'.

Q 2: Determine whether the following sequence is strictly monotone or not. Justify your answer.

$$a_n = \left\{ \frac{2}{n^2} \right\}$$

Solution:

Here

$$a_n = \frac{2}{n^2} \quad \text{and} \quad a_{n+1} = \frac{2}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{2}{(n+1)^2} \times \frac{n^2}{2} = \frac{n^2}{(n+1)^2} < 1$$

$$\frac{a_{n+1}}{a_n} < 1 \Rightarrow a_{n+1} < a_n$$

So this is a decreasing sequence and hence strictly monotonic.

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Q 3: Determine whether the sequence $\{a_n\}$ converges or diverges; if it converges then find its limit; where

$$a_n = [\ln 2n - \ln(n+2)]$$

Solution:

$$\lim_{n \rightarrow \infty} [\ln 2n - \ln(n+2)] = \lim_{n \rightarrow \infty} \left[\ln \frac{2n}{n+2} \right]$$

$$= \ln \lim_{n \rightarrow \infty} \left[\frac{2n}{n+2} \right]$$

divide numerator and denominator by n ,

$$= \ln \lim_{n \rightarrow \infty} \left[\frac{2}{1 + \frac{2}{n}} \right]$$

$$= \ln(2)$$

$$\left(\therefore \lim_{n \rightarrow \infty} \left[\frac{2}{1 + \frac{2}{n}} \right] = 2 \right)$$

The sequence is convergent and its limit is $\ln 2$.

Q 4: Determine whether the following series converges; if so then find the sum:

$$5 + \frac{5}{2} + \frac{5}{2^2} + \dots + \frac{5}{2^{k-1}} + \dots$$

Solution:

Since $5 + \frac{5}{2} + \frac{5}{2^2} + \dots + \frac{5}{2^{k-1}} + \dots$ is a geometric series with $a=5$ and $r = \frac{1}{2}$.

And $|r| = \frac{1}{2} < 1$, therefore the series converges and the sum is

$$\begin{aligned} \frac{a}{1-r} &= \frac{5}{1-\frac{1}{2}} \left(\because \text{if } |r| < 1, \text{ the series converges then the sum is } \right. \\ &\quad \left. \frac{a}{1-r} = a + ar + ar^2 + \dots + ar^{k-1} + \dots \right) \\ &= 5 \left(\frac{2}{1} \right) \\ &= 10 \end{aligned}$$

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Q 5: Use the divergence test to show that the given series diverges.

$$\sum_{k=1}^{\infty} \frac{k^2 + k + 2}{3k^2 + 1}$$

Solution:

To show that the given series diverges by using divergence test we have to show that

$$\lim_{k \rightarrow \infty} \frac{k^2 + k + 2}{3k^2 + 1} \neq 0$$

Dividing in numerator and denominator by k^2

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{(k^2 + k + 2) / k^2}{(3k^2 + 1) / k^2} &= \lim_{k \rightarrow \infty} \frac{1 + \frac{1}{k} + \frac{2}{k^2}}{3 + \frac{1}{k^2}} \\ &= \frac{1}{3} \neq 0 \end{aligned}$$

So according to divergence test the series diverges.

$$\sum_{k=1}^{\infty} \frac{k^2 + k + 2}{3k^2 + 1}$$

Q 6: Check the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)!}$

by using the Ratio Test.

Solution:

$\therefore$ The Ratio Test: $\rho = \lim_{k \rightarrow +\infty} \frac{u_{k+1}}{u_k}$ for the series $\sum u_k$

$$\therefore \sum_{k=2}^{\infty} \frac{1}{(k+1)!} \Rightarrow u_{k+1} = \frac{1}{(k+2)!}; \quad u_k = \frac{1}{(k+1)!}$$

$$\rho = \lim_{k \rightarrow +\infty} \left(\frac{\frac{1}{(k+2)!}}{\frac{1}{(k+1)!}} \right) = \lim_{k \rightarrow +\infty} \left(\frac{(k+1)!}{(k+2)!} \right)$$

$$= \lim_{k \rightarrow +\infty} \left(\frac{1}{k+2} \right)$$

$$= 0 < 1 \Rightarrow \text{the series converges} \quad (\because \rho < 1 \Rightarrow \text{the series converges})$$

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Q 7: Find ρ by using the Limit Comparison Test where $\sum a_n = \frac{1}{2n-1}$ and $\sum b_n = \frac{1}{2n}$.

Solution:

$$\sum a_n = \frac{1}{2n-1} \quad \text{and} \quad \sum b_n = \frac{1}{2n}$$

$$\rho = \lim_{n \rightarrow +\infty} \frac{\frac{1}{2n-1}}{\frac{1}{2n}} = \lim_{n \rightarrow +\infty} \frac{2n}{2n-1} \quad \left(\because \text{The Limit Comparison test: } \rho = \lim_{n \rightarrow +\infty} \frac{a_n}{b_n} \right)$$

$$= \lim_{n \rightarrow +\infty} \frac{2}{2 - \frac{1}{n}}$$

$$= \frac{2}{2 - \frac{1}{\infty}} = \frac{2}{2} \quad \left(\because \frac{a}{\infty} = 0 \right)$$

$$= 1 \quad (\because \rho \text{ is finite and } \rho > 0)$$

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Lecture No. 44: Alternating Series; Conditional Convergence
Lecture No. 45: : Taylor and Maclaurin Series

Q 1: Find the radius of convergence for the following power series: $\sum_1^{\infty} \frac{n! \cdot x^n}{(3n)!}$.

Solution:

Here $a_n = \frac{n! \cdot x^n}{(3n)!}$,

so $a_{n+1} = \frac{(n+1)! \cdot x^{n+1}}{(3n+3)!}$.

Now,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! \cdot x^{n+1}}{(3n+3)!} \cdot \frac{(3n)!}{n! \cdot x^n} \right|, \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)n!}{(3n+3)(3n+2)(3n+1)(3n)!} \cdot \frac{(3n)!}{n!} |x|, \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{(3n+3)(3n+2)(3n+1)} |x|, \\ &= 0. \end{aligned}$$

Thus the series converges absolutely $\forall x$ and radius of convergence $= \infty$.

Q 2: Show that $\sum_1^{\infty} |a_n|$ is divergent for the following alternating series: $\sum_1^{\infty} \frac{(-1)^n \cdot n^n}{2n!}$.

Solution:

Here $a_n = \frac{(-1)^n \cdot n^n}{2n!}$,

so $|a_n| = \frac{n^n}{2n!}$, and $|a_{n+1}| = \frac{(n+1)^{n+1}}{2(n+1)!}$.

$$\begin{aligned} \text{Now } \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{2(n+1)!} \cdot \frac{2n!}{n^n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n, \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n, \\ &= e > 1. \end{aligned}$$

Thus the given series is diverges.

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Q 3: Find the first two terms of Taylor series for $f(x) = \ln x$ at $x = 2$.

Solution:

$$\because f(x) = \ln x, \quad f(2) = \ln 2,$$

$$\Rightarrow f'(x) = \frac{1}{x}, \quad f'(2) = \frac{1}{2}.$$

$\therefore$ Taylor polynomial for f about $x = a$:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

$$\Rightarrow p_0(x) = f(2) = \ln 2,$$

$$\Rightarrow p_1(x) = f(2) + f'(2)(x-2) = \ln 2 + \frac{1}{2}(x-2).$$

Q 4: Find the first four terms of the Taylor series generated by f at $x = 2$ where

$$f(x) = \frac{1}{x+1}.$$

Solution:

$$\because f(x) = \frac{1}{x+1} = (x+1)^{-1} \quad \text{and} \quad f(2) = \frac{1}{2+1} = \frac{1}{3},$$

$$\Rightarrow f'(x) = -(x+1)^{-2} = -\frac{1}{(x+1)^2} \quad \text{and} \quad f'(2) = -\frac{1}{(2+1)^2} = -\frac{1}{3^2},$$

$$\Rightarrow f''(x) = 2(x+1)^{-3} = \frac{2}{(x+1)^3} \quad \text{and} \quad f''(2) = \frac{2}{(2+1)^3} = \frac{2}{3^3},$$

$$\Rightarrow f'''(x) = -6(x+1)^{-4} = -\frac{6}{(x+1)^4} \quad \text{and} \quad f'''(2) = -\frac{6}{(2+1)^4} = -\frac{6}{3^4}.$$

$\therefore$ Taylor Series

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$= \frac{1}{3} + \frac{-\frac{1}{3^2}}{1!}(x-2) + \frac{\frac{2}{3^3}}{2!}(x-2)^2 + \frac{-\frac{6}{3^4}}{3!}(x-2)^3 + \dots$$

$$\Rightarrow \frac{1}{x+1} = \frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3 + \dots$$

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