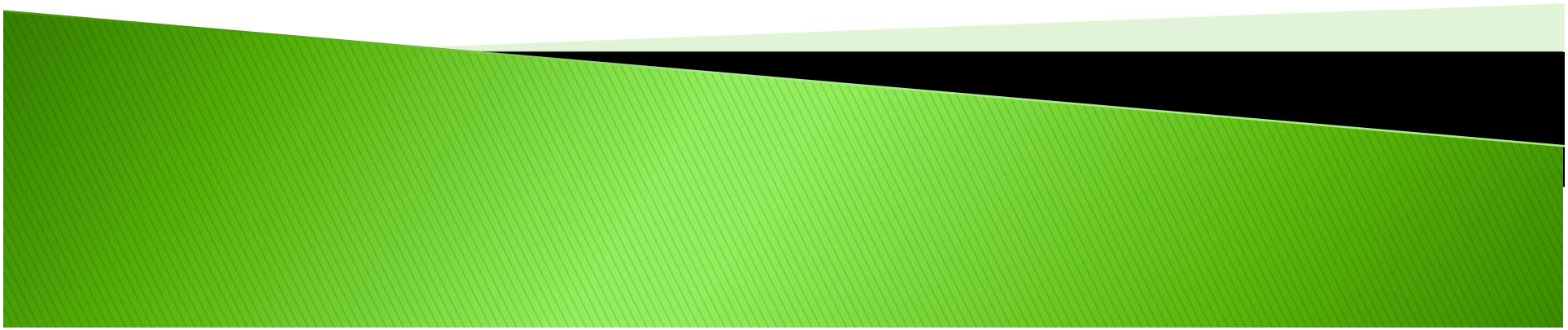
Real Numbers and the Power of Continuum
Part 1



Theorem 1

The unit interval $I = [0, 1]$ is nondenumerable.

$0 = 0.000\ldots$
$0.1 = 0.0999\ldots$
$0.5 = 0.4999\ldots$
$1 = 0.999\ldots$

Proof. Let I is denumerable. Then

$$I = \{x_1, x_2, x_3, \ldots\}$$

that is, the elements of I can be written in a sequence. So

$$x_1 = 0.a_{11}a_{12}a_{13}\ldots a_{1n}\ldots$$
$$x_2 = 0.a_{21}a_{22}a_{23}\ldots a_{2n}\ldots$$
$$x_3 = 0.a_{31}a_{32}a_{33}\ldots a_{3n}\ldots$$
$$\vdots$$
$$x_n = 0.a_{n1}a_{n2}a_{n3}\ldots a_{nm}\ldots$$

Here $a_{ij} \in \{0, 1, 2, \ldots 9\}$

Theorem 1

Now construct the real number

$$y = 0.b_1 b_2 b_3 \ldots b_n \ldots$$

which will belong to I in the following way:

choose b_1 so $b_1 \neq a_{11}$ and $b_1 \neq 0$

choose b_2 so $b_2 \neq a_{22}$ and $b_2 \neq 0$

choose b_3 so $b_3 \neq a_{33}$ and $b_3 \neq 0$ and so on.

Note that $y \neq x_1$ since $b_1 \neq a_{11}$.

and $y \neq x_2$ since $b_2 \neq a_{22}$ and so on.

That is $y \neq x_n$ for all $n \in P$.

Thus $y \notin I$, which contradicts the fact that $y \in I$. Thus the assumption that I is denumerable is wrong. Hence I is nondenumerable.

Power of the Continuum

A set A is said to have the power of the continuum if A has the same cardinality as the unit interval $I = [0, 1]$.

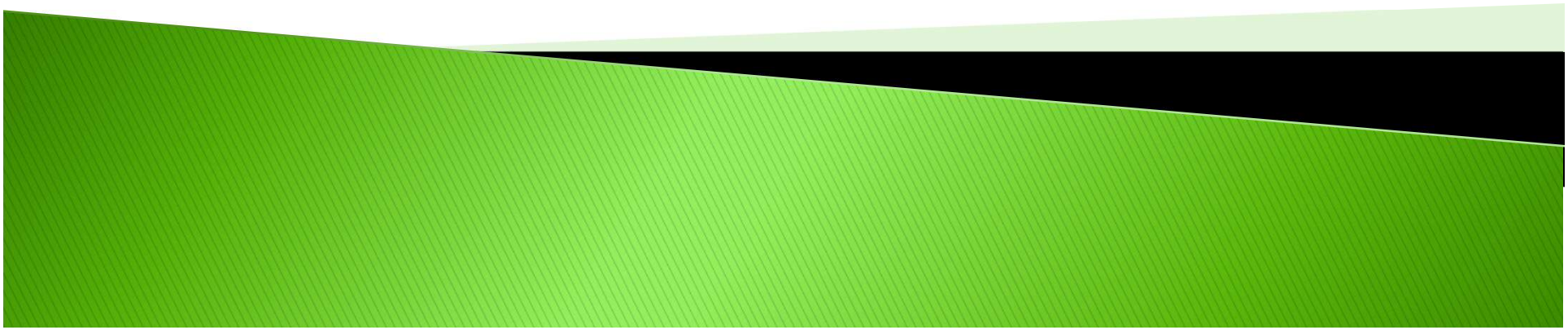
For example, $[a, b] \approx [0, 1]$.
Hence closed interval $[a, b]$ has the power of continuum. Similarly, $[a, b)$, $(a, b]$ and (a, b) also have the power of continuum.



Power of the Continuum



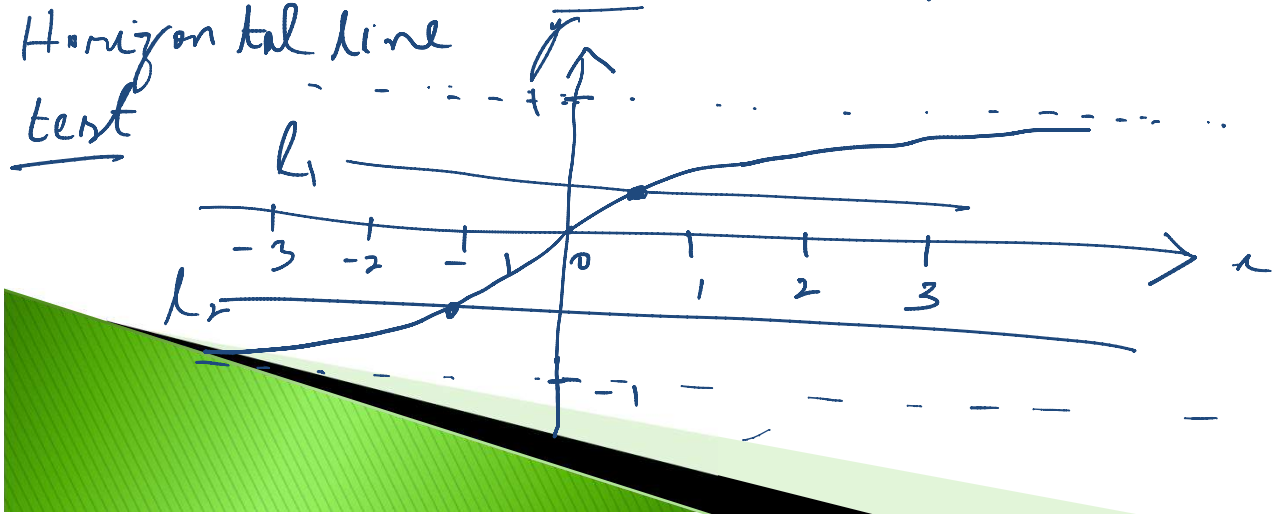
Real Numbers and the Power of Continuum Part 2



Theorem 1

Prove that the set R of real numbers has the power of continuum.

Proof. Consider the function $f : R \to (-1,1)$ defined by $f(x) = \frac{x}{1+|x|}$. Clearly values of f belong to $(-1,1)$ since $|x| < 1 + |x|$.



① $R \simeq (-1,1)$ ✓

② $(-1,1) \simeq [0,1]$ ✓

Then $R \simeq [0,1]$.

$f(1) = \frac{1}{1+1} = \frac{1}{2} \in (-1,1)$

$f(-2) = \frac{-2}{1+2} = \frac{-2}{3} \in (-1,1)$

By horizontal line test, $f(x)$ is bijective. Hence, R has the power of continuum.

Theorem 1



Cardinal Numbers



Definition

Each set A is assigned a symbol in such a way that two sets A and B are assigned the same symbol if and only if they are equipotent. This symbol is called the cardinality or cardinal number of A, and it is denoted by

$$|A|,\ n(A) \text{ or } \operatorname{card}(A)$$

Thus, $|A| = |B|$ iff $A \approx B$.



Finite Cardinal Numbers

0 is assigned to the empty set ø, and *n* is assigned to the set {1, 2, … *n*}. Thus,

$|A| = n \quad A \approx \{1, 2, \ldots, n\}$

$|A| = 3 \quad A \approx \{1, 2, 3\}$, e.g. $A = \{2, 3, 4\}$

Alternatively, the symbols 0, 1, 2, 3,… are assigned, respectively, to the sets

$\emptyset, \{\emptyset\}, [\emptyset, \{\emptyset\}], [\emptyset, \{\emptyset\}], \{\emptyset, \{\emptyset\}\}, \ldots$

Transfinite Cardinal Numbers

Cardinal numbers of infinite sets are called infinite or transfinite cardinal numbers.

The cardinal number of the infinite set P of positive integers is $\aleph_0$ (aleph-nought)

Thus,

$$|A| = \aleph_0 \quad \text{iff} \quad A \approx P.$$

In particular,

$$|Z| = \aleph_0 \text{ since } Z \approx P.$$

and $|Q| = \aleph_0$ since $Q \approx P$



Transfinite Cardinal Numbers

The cardinal number of the unit interval $I = [0, 1]$ is denoted by **c** and it is called the power of continuum. Thus: $|A| = c$ iff $A \approx I$.

In particular, we have $|R| = \mathbf{c}$, and the cardinal number of any interval is **c**.

For example, $|(2,3)| = c$ since $(2,3) \approx [0,1]$

$|[-1,5]| = c$ // $[-1,5] \approx [0,5]$

Some Important Statements

(a) A is denumerable or countably infinite means $|A| = \aleph_0$

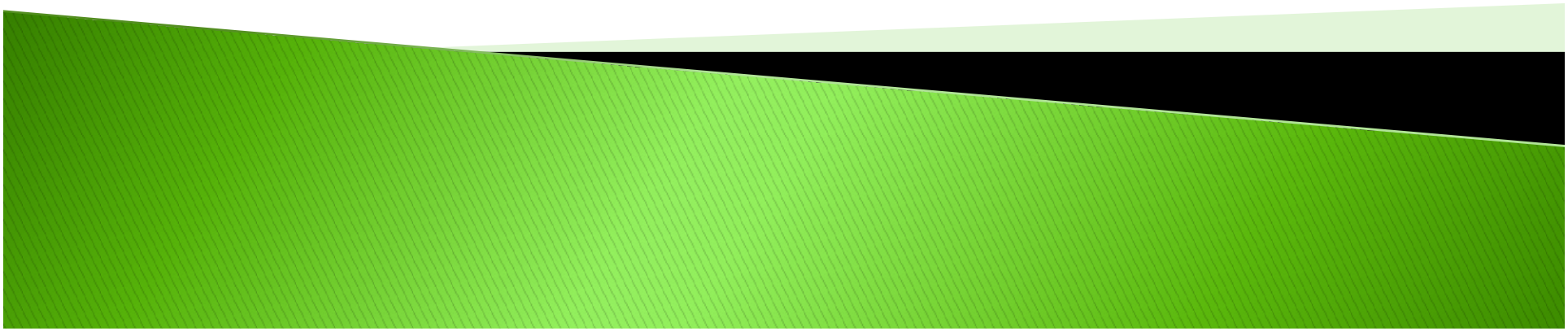
(b) A is countable means $|A|$ is finite or $|A| = \aleph_0$

(c) A has the power of continuum if $|A| = \mathbf{c}$.

$|R| = c$



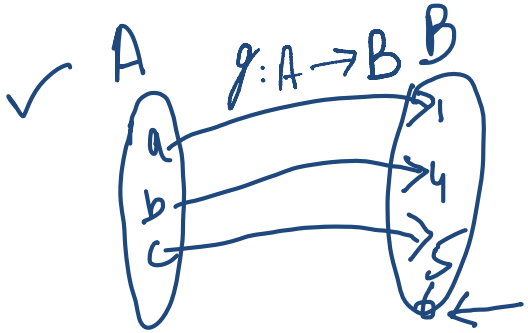
Ordering of Cardinal Numbers



Definition

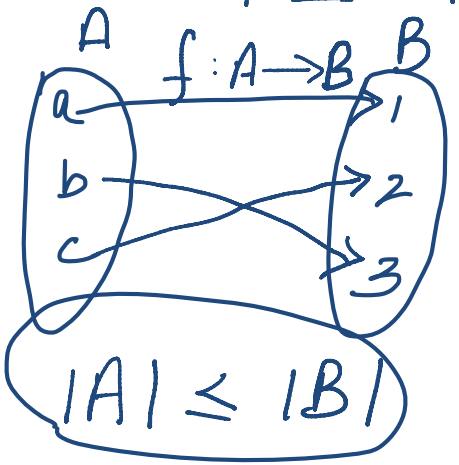
Let A and B be two sets. We say that $|A| \leq |B|$ if there exists a one-to-one(injective) function $f : A \to B$.

The above relation between cardinal numbers are well defined, that is, the relations are independent of the particular sets involved. Namely, if $A \approx A'$ and $B \approx B'$, $|A| \leq |B|$ iff $|A'| \leq |B'|$ $|A| = |B|$



$|A| < |B|$

Here g is one-to-one.
but g is not onto.



Example 1

Let ~~A~~ B be a proper subset of a finite set ~~B~~ A.

For example, $A = \{1,2,3,4\}$, then $B = \{1,2\}$

Here $|B| < |A|$.

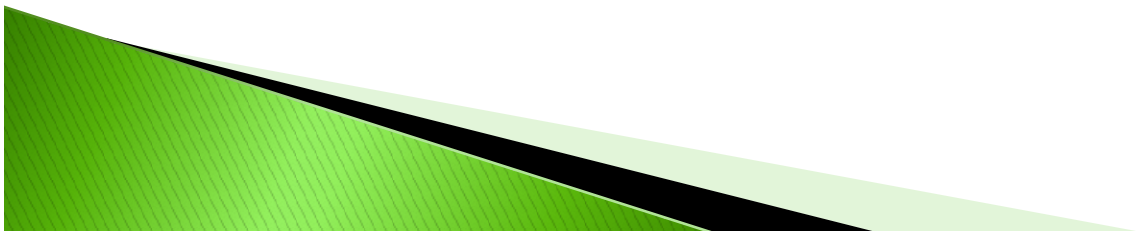
Example 2

Let n be a finite cardinal. Let A be a finite subset of $P = \{1,2,3,\cdots\}$. Then $|A| \neq |P|$. Thus $|A| < |P|$.

$n < \aleph_0$

We may write

$$0 < 1 < 2 < \cdots < \aleph_0$$



Example 3

Consider the set P of positive integers and the unit interval I, that is, consider the sets

$P = \{1, 2, 3, \ldots\}$ and $I = [0, 1] = \{x \in R : 0 < x \leq 1\}$

The function $f : P \rightarrow I$ defined by $f(n) = 1/n$ is one-to-one.(prove it.)

Let $f(n_1) = f(n_2)$

$\Rightarrow 1/n_1 = 1/n_2$

$\Rightarrow n_1 = n_2$ Proved

$f(1) = 1/1 = 1 \in I$

$f(2) = 1/2 = 0.5 \in I$

$f(3) = 1/3 = 0.3.. \in I$

$f(4) =$ different

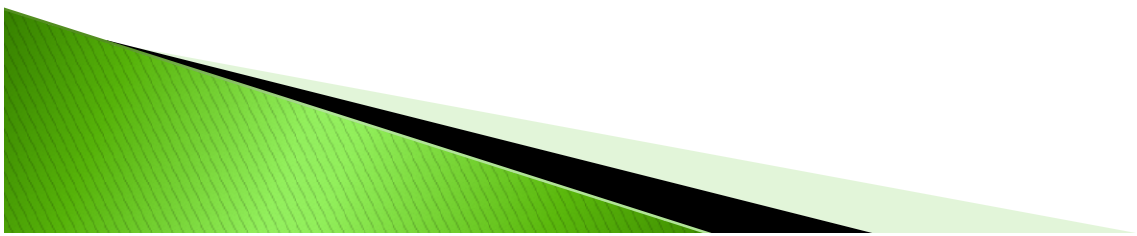
$|P| = \aleph_0$, $|I| = c$

Therefore, $|P| \leq |I|$. ✓

On the other hand $|P| \neq |I|$. Therefore $|P| < |I|$.

Accordingly we may write $\Rightarrow \aleph_0 < c$.

$0 < 1 < 2 < \cdots < \aleph_0 < c$.



Example 4

Let A be any infinite set. So A contains a subset which is denumerable. Accordingly, for any infinite set A, we always have

$\aleph_0 \leq |A|$.

$P = \{1, 2, 3, \ldots\}$

$E \subseteq P$.

$|P| = |E|$

$\aleph_0 = \aleph_0$

$\mathbb{R}$ = infinite set,

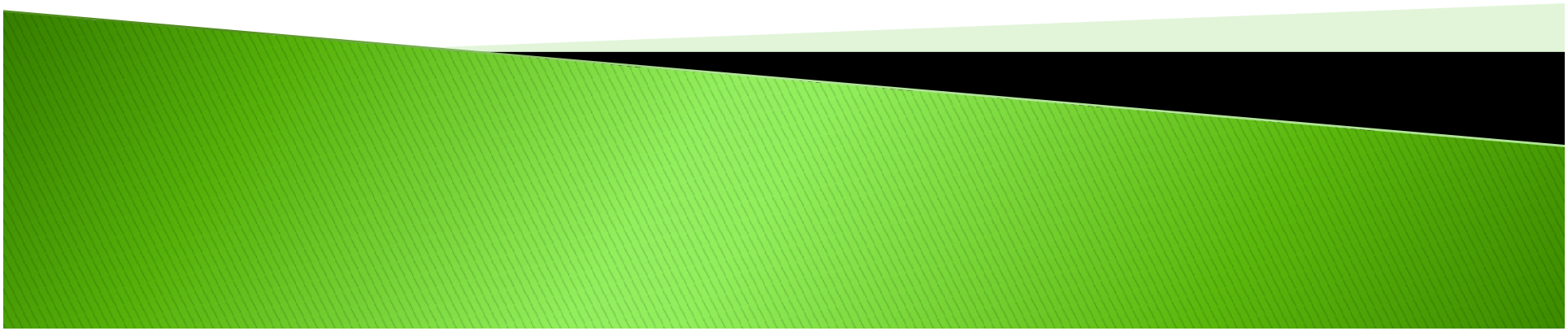
$P \subseteq \mathbb{R}$.

Here P is denumerable.

$|P| \leq |\mathbb{R}|$

$\aleph_0 < c$.

Cantor's Theorem



Theorem 1

If A has 3 elements, then $P(A)$ has 8 elements. $2^3 = 8$

For any set A, we have $|A| < |P(A)|$.

Proof. Let $f : A \to P(A)$ defined by $f(x) = \{x\}$, $\forall x \in A$.

We need to prove that f is one-to-one but not onto.

Let $f(x_1) = f(x_2)$

$\Rightarrow \{x_1\} = \{x_2\}$

$\Rightarrow x_1 = x_2$

Hence f is one-to-one.

$|P| < |P(P)|$

$\aleph_0 < 2^{\aleph_0}$

$|\mathbb{R}| < |P(\mathbb{R})|$

$c < 2^c$

Theorem 1

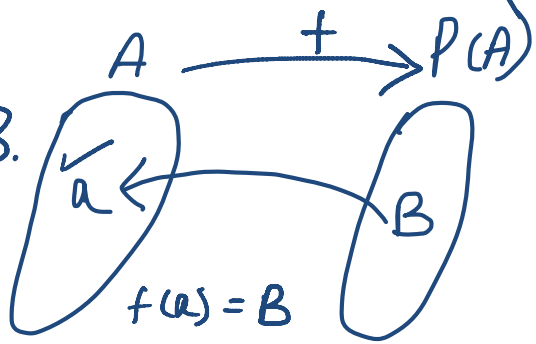
Suppose $f : A \to P(A)$ is onto. Now consider the set

$$B = \{x \in A : x \notin f(x)\}$$

For example, if $A = \{1, 2, 3\}$, then

$$1 \to \phi \,, \quad 2 \to \{1, 3\}, \quad 3 \to \{2\}$$

clearly B is a subset of A, so $B \in P(A)$. Also since f is onto, so there exists $a \in A$ such that $f(a) = B$. Now we have two possibilities, either $a \in B$ or $a \notin B$.



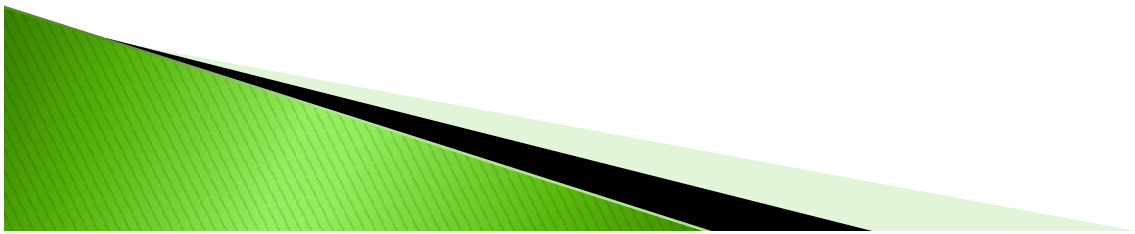
Theorem 1

1) If $a \in B$, then $a \in f(a)$. Then, by the definition of set B, we have $a \notin B$, which is a contradiction.

2) If $a \notin B$, then $a \notin f(a)$. Then $a \in B$, which is a contradiction.

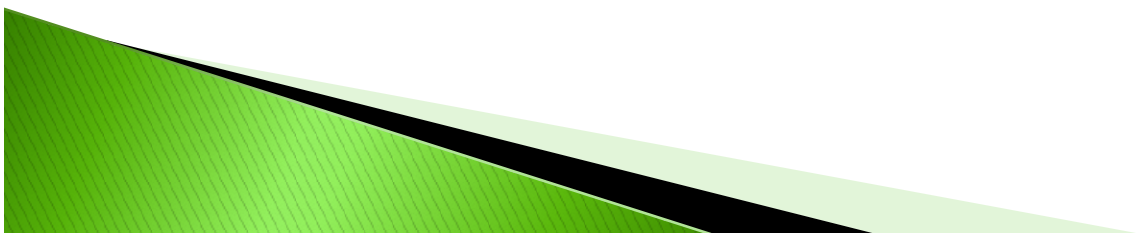
Hence, our assumption that f is onto is wrong. Thus f is not onto. Therefore,

$$|A| < |P(A)|.$$



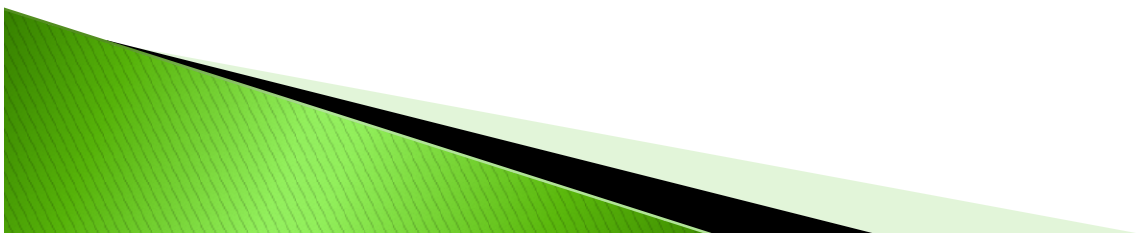
Theorem 2

Prove that $2^{\aleph_0} = c$.



Theorem 2

$$2^{\aleph_0} = c$$



Continuum Hypothesis

There exists no cardinal number β such that

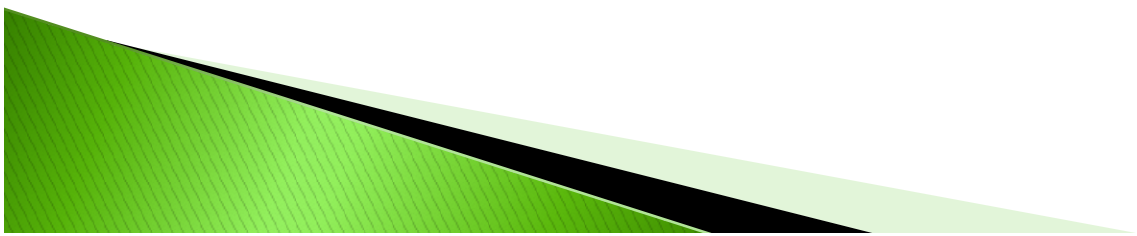
$$\aleph_0 < \beta < c.$$

$\aleph_0, c$

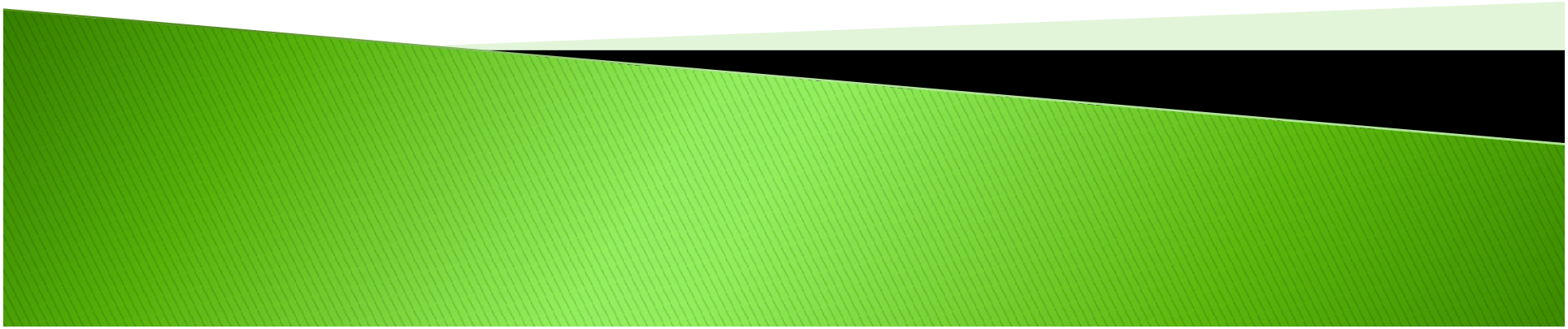
$\aleph_0 < c.$

$\aleph_0 < 2^{\aleph_0} = c.$

same

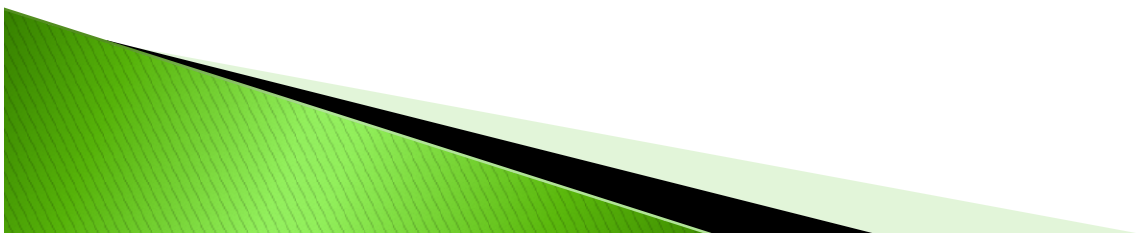


Cardinal Arithmetic Part 1



Cardinal Addition

Addition and multiplication of the counting numbers N are sometimes treated from the point of view of set theory. The interpretation of $2+3 = 5$, for example, is the union of two disjoint sets, one having two elements and the other having three elements, is a set with five elements. This idea leads to a completely general definition of addition of cardinal numbers.



Cardinal Addition

Definition: Let α and β be cardinal numbers and let A and B be disjoint sets with $\alpha = |A|$ and $\beta = |B|$. Then the sum of α and β is denoted and defined by

$$\alpha + \beta = |(A \cup B)|$$

For example, Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$.
Here $|A| = 3$, $|B| = 3$. Now, $A \cup B = \{1, 2, 3, a, b, c\}$, $|A \cup B| = 6$
$= |A| + |B|$

Note:(1) This definition is well-defined.

Let A' and B' with same cardinal numbers.
means $|A'| = \alpha$, $|B'| = \beta$. Then $|A' \cup B'| = |A \cup B|$

(2) If A and B are any two sets, then

$A \times \{1\}$ and $B \times \{2\}$ are disjoint sets.

Examples

(a) Let m and n be finite cardinal numbers. Then $m + n$ corresponds to the usual addition in N.

(b) Let n be a finite cardinal number.

Let $A = \{1, 2, 3, \ldots, n\}$. Then $|A| = n$

$B = \{n+1, n+2, n+3, \ldots\}$. Then $|B| = \aleph_0$

Here $B \approx P$

Now, $A \cup B = \{1, 2, 3, \ldots, n, n+1, n+2, \ldots\}$. Then $|A \cup B| = \aleph_0$

But $|A \cup B| = n + \aleph_0$

$\Rightarrow \boxed{\aleph_0 = n + \aleph_0}$

Examples

c) Let $C = [0, \frac{1}{2}]$, $D = (\frac{1}{2}, 1]$

Here $C \approx I$ and $D \approx I = [0,1]$.

So, $|C| = c$, $|D| = c$.

Now, $C \cup D = [0, \frac{1}{2}] \cup (\frac{1}{2}, 1] = [0,1]$

$\Rightarrow |C \cup D| = c$.

Thus, $\boxed{|C + C = C}$

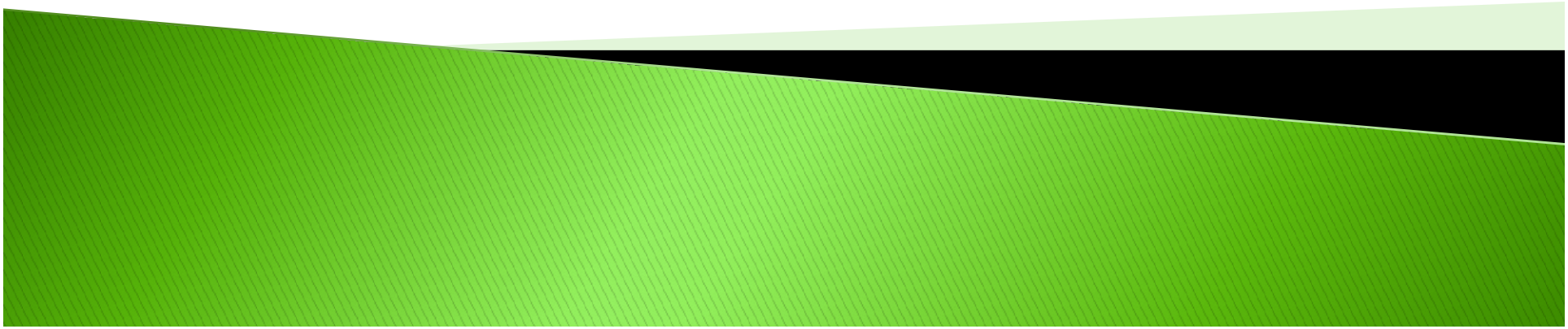
d) Let $E = \{2, 4, 6, \ldots\}$ and $F = \{1, 3, 5, \ldots\}$

$\Rightarrow |E| = \aleph_0$ and $|F| = \aleph_0$

Now, $E \cup F = \{1, 2, 3, 4, 5, 6, \ldots\} = P$

$|E \cup F| = \underline{\aleph_0}$ $\boxed{\aleph_0 + \aleph_0 = \aleph_0}$

Cardinal Arithmetic Part 2



Cardinal Multiplication

Let α and β be cardinal numbers and let A and B be sets with $\alpha = |A|$ and $\beta = |B|$. *Then the product* α *and* β is denoted and defined by

$$\alpha\beta = |A \times B|$$

For example, let $A = \{2, 3, 4\}$, $B = \{4, 5\}$.

Here $|A| = 3$ and $|B| = 2$.

Now, $A \times B = \{(2,4), (2,5), (3,4), (3,5), (4,4), (4,5)\}$

$\therefore |A \times B| = 6 = 2 \times 3 = |A| \times |B| = \alpha\beta$

Examples

(a) Let m and n be finite cardinal numbers. Them mn corresponds to the usual multiplication in N.

(b) Since $N \times N$ is denumerable, so $|N \times N| = \aleph_0$

$N \times N \approx P$ or $|N \times N| = |P| \Rightarrow \aleph_0 \times \aleph_0 = \aleph_0$ ✓

(c) The Cartesian plane $R^2 = R \times R$ has the same cardinality as R. That is, **cc = c.**

$|R \times R| = R$

$cc = c$

Theorem 1

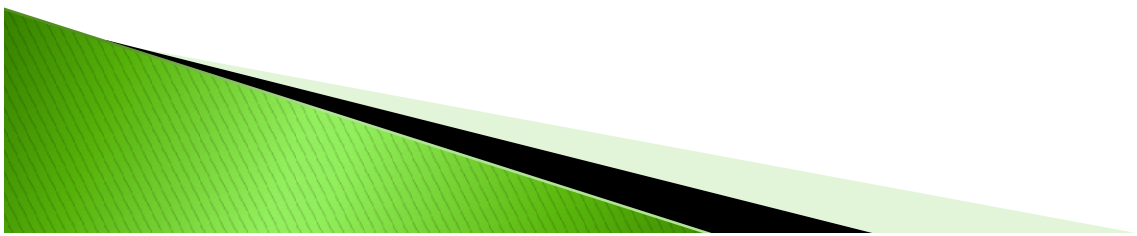
Let α and β be nonzero cardinal numbers such that β is infinite and $\alpha \leq \beta$. Then

$$\alpha + \beta = \alpha\,\beta = \beta$$

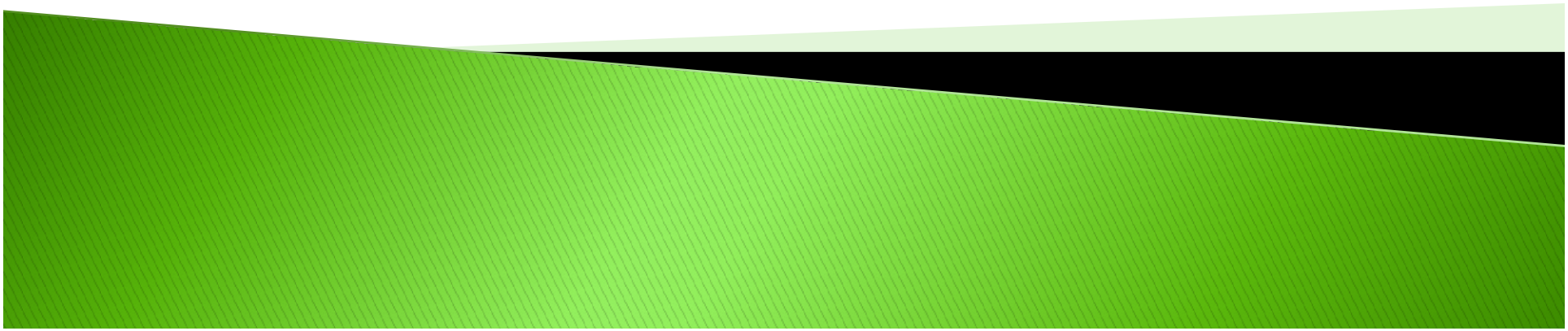
We know that $\aleph_0 < c$.

It means, $\aleph_0 + c = c$.

and $\aleph_0 c = c$



Ordered Sets

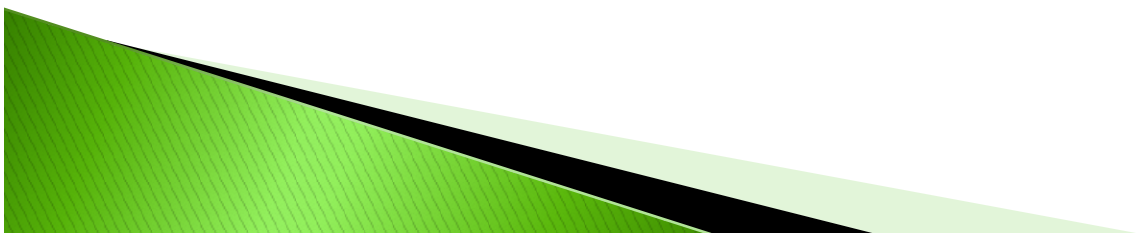


Definition

Suppose R is a relation on a set S satisfying the following three properties:

1) (Reflexive) For any $a \in S$, we have $a \; R \; a$.
2) (Antisymmetric) If $a \; R \; b$ and $b \; R \; a$, then $a = b$.
3) (Transitive) If $a \; R \; b$ and $b \; R \; c$, then $a \; R \; c$.

Then R is called a partial order or, simply an order relation, and R is said to define a partial ordering of S. The set S with partial ordering R is called a partially ordered set or, simply an ordered set. Poset



Notation

1) If $a \in P$, then $a \leq a$.
2) If $a \leq b$ and $b \leq a$, then $a = b$.
3) If $a \leq b$ and $b \leq c$, then $a \leq c$.

The most familiar order relation, called the usual order, is the relation $\leq$ on the positive integers P or, more generally on any subset of the real numbers R. For this reason, a partial ordering relation is frequently denoted by $\leq = R$

With this notation, the above three properties of a partial order appear in the following usual form:

1) (Reflexive) For any $a \in S$, we have $a \leq a$. or aRa.
2) (Antisymmetric) If $a \leq b$ and $b \leq a$, then $a = b$.
3) (Transitive) If $a \leq b$ and $b \leq c$, then $a \leq c$.

Notation

Although an ordered set consists of two things, a set S and the partial ordering R, one usually simply write S to denote the ordered sets as long as the partial ordering is fixed in the context of the discussion; otherwise the ordered set is denoted by the pair $(S, \leq)$

Suppose S is an ordered set. Then the statement

$a \leq b$ is read "a precedes b".

In this context we also write:

$a < b$ means $a \leq b$ and $a \neq b$. (a strictly precedes b)

$b \geq a$ read "b succeeds a"

$b > a$ means $b < a$ and $a \neq b$. (b strictly succeed a")

3) If $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$.

Example 1

1) Let C be any collection of sets. The relation of set inclusion is a partial ordering of C. Specifically,

   1) For any $A \in C$, $A \subseteq A$ or $A R A$.
   2) For any $A, B \in C$, we have $A \subseteq B$ and $B \subseteq A$, then $A = B$.

2) Consider the set P of positive integers. We say "a *divides* b", written $a|b$, if there exists an integer c such that $ac = b$. For example, 2|4, 3|12, 7|21, and so on. This relation of divisibility is a partial ordering of P.

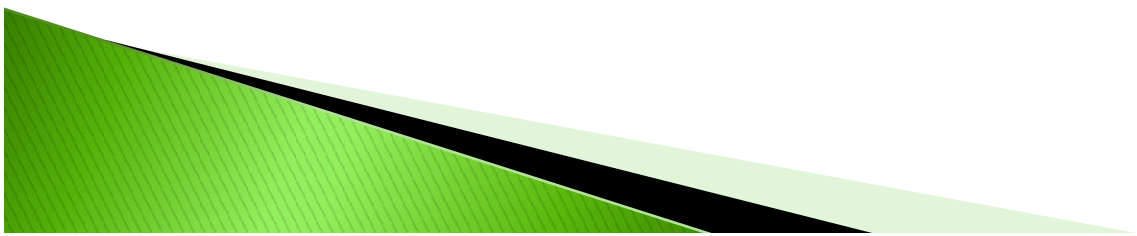
   1) If $a \in P$, $a|a$ or $a R a$.
   2) If $a|b$ and $b|a$, then $a = b$.
   3) If $a|b$, $b|c$, then $a|c$. $\left| \begin{array}{l} 3|12, 12|24 \\ \Rightarrow 3|24. \end{array} \right.$

Example 1

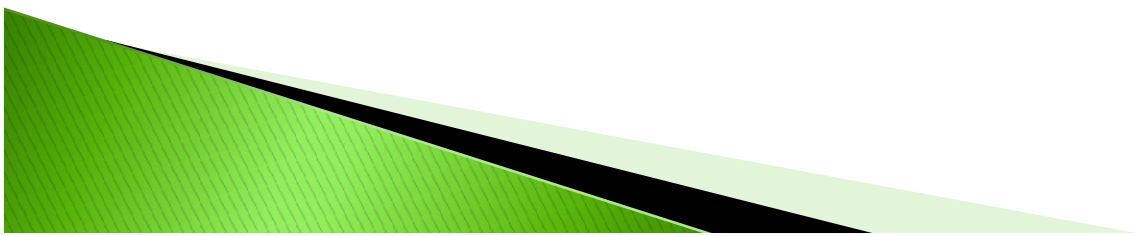
3) The relation "|" of divisibility is not an ordering of the set Z of integers. Specifically, the relation is not antisymmetric. For instance,

For example, $a = 2 \mid -2 = b$ and $-2 \mid 2$

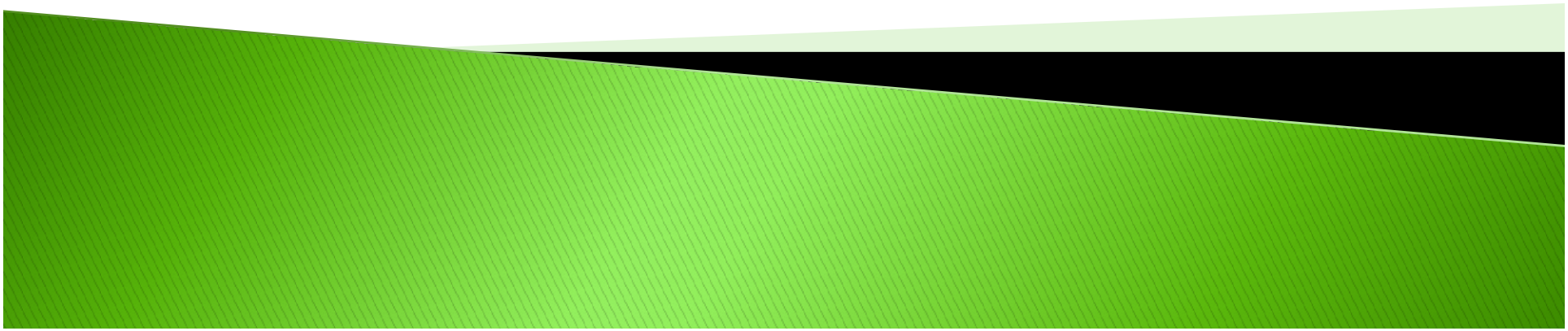
but $2 \neq -2$.



Example 1



Comparability and Linearly Ordered Sets



Comparability

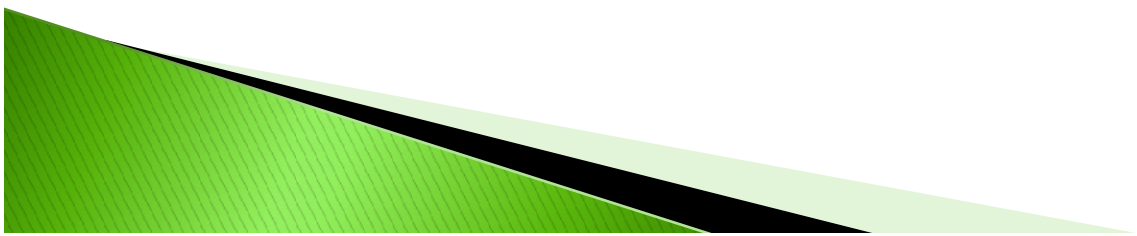
Suppose a and b are distinct elements in a partially ordered set S. We say a and b are comparable if

$a \preceq b$ or $b \preceq a$.

Thus a and b are noncomparable written

$a \parallel b$

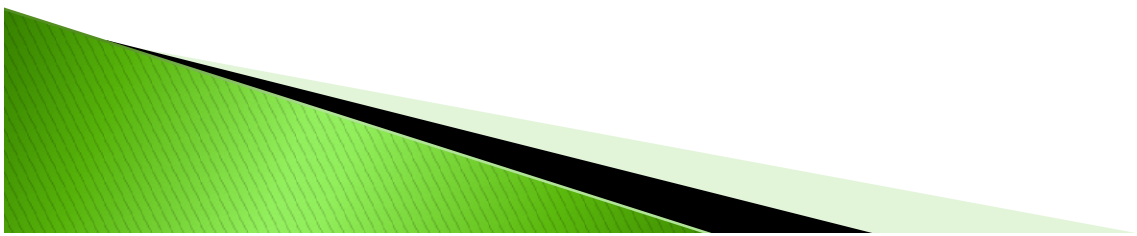
if $a \not\preceq b$ or $b \not\preceq a$.



Linearly Ordered Sets and Chain

Suppose that every pair of elements of S are comparable. Then S is said to be linearly or totally ordered.

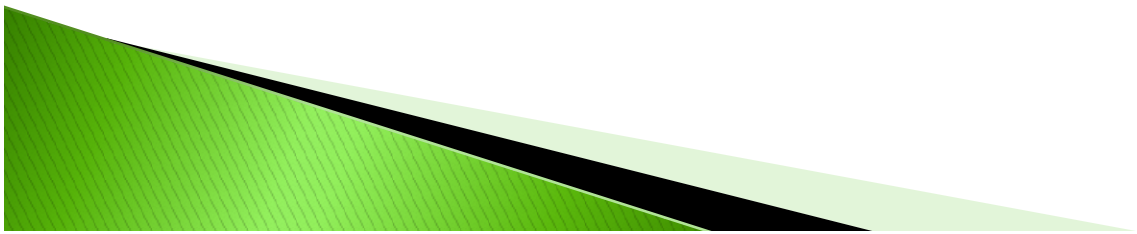
Although an ordered set S may not be linearly ordered, it is still possible for a subset A of S to be linearly ordered. Such a linearly ordered subset A of an ordered set S is called a chain in S.



Example 1

a) Consider the set P of positive integers ordered by divisibility. Then 21 and 7 are comparable since $7 \mid 21$. Here 3 and 5 are non comparable because neither $3 \mid 5$ nor $5 \mid 3$. Thus P is not linearly ordered by divisibility. Observe that $A = \{2, 6, 12, 36\}$ is a chain.

b) The set P of positive integers with the usual order $\leq$ is linearly ordered and hence every ordered subset of P is also linearly ordered. $P = \{1, 2, 3, \ldots\}$, now $B = \{2, 5, 7\}$



Example 1

c) The power set $P(A)$ of a set A with 2 or more elements is not linearly ordered by set inclusion. For example, $a, b \in A$, then $\{a\}$ and $\{b\}$ are non comparable. Let $B \subseteq P(A)$ such that $B = \{\phi, \{a\}, A\}$. Here $\phi < \{a\} < A$ or $\phi \subseteq \{a\} \subseteq A$ so set B is a chain.

d) Consider the set $A = \{1, 2, 3, 4, 6, 12, 24\}$ ordered by divisibility. Then clearly A is not linearly ordered as for instance, 3 and 4 are not comparable. Observe that the sets $\{1, 2, 4, 12, 24\}$, $\{1, 2, 6, 12, 24\}$ and $\{1, 3, 6, 12, 24\}$ are all chains.

Example 2

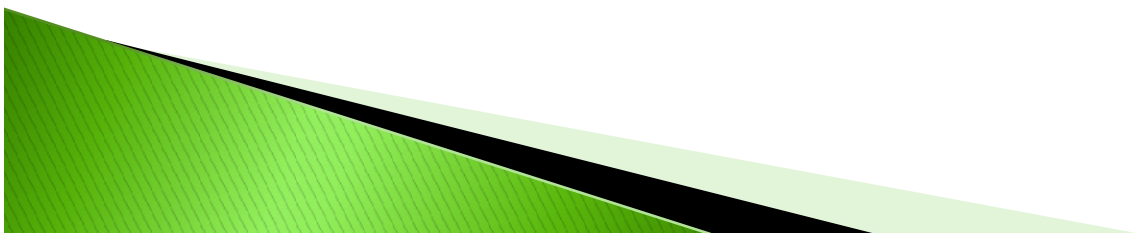
Let $P = \{1, 2, 3, \ldots\}$ be ordered by divisibility. State whether each of the following is a chain in P.

a) $A = \{24, 2, 6\}$, it is a chain.
Here $2 < 6 < 24$ or $2 | 6 | 24$

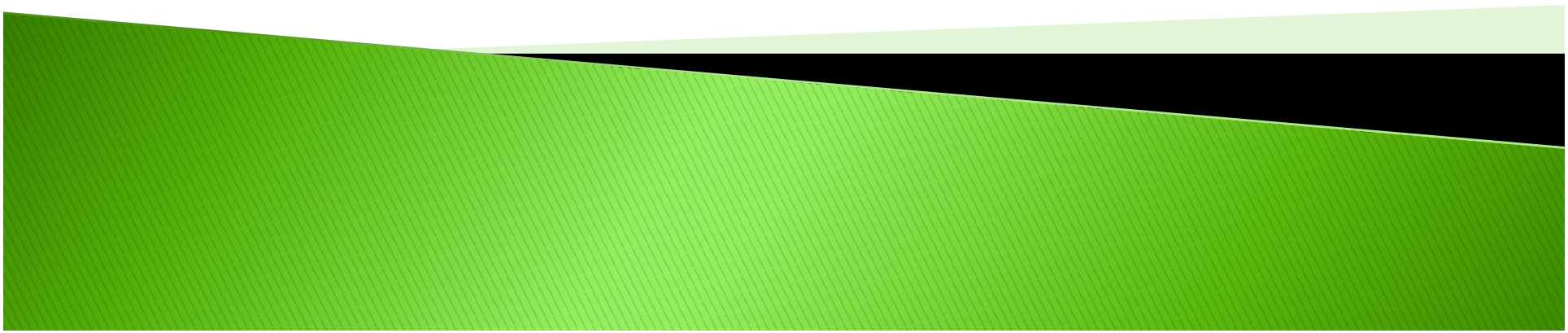
b) $B = \{3, 15, 5\}$, so it is not a chain.
Here $3 \nmid 5$

c) $C = \{2, 8, 32, 4\}$, it is a chain.
$2 < 4 < 8 < 32$

d) $D = \{7\}$, it is a chain.



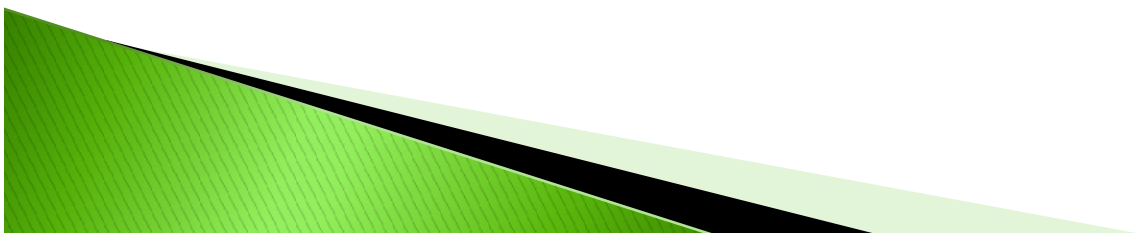
Partially Ordered Sets and Hasse Diagrams



Immediate Predecessor and Immediate Successor

Let S be a partially ordered set, and suppose $a, b \in S$. We say that a is an immediate predecessor of b, or that b is an immediate successor of a, or that b is a cover of a, written $a << b$, if $a < b$ but no element in S lies between a and b, that is, there exists no element c in S such that $a < c < b$.

For example, consider the set $P = \{1, 2, 3, 4, \ldots\}$ with the order $<$. Then we can see that $1 << 2$, $2 << 3$, and so on.



Immediate Predecessor and Immediate Successor

Suppose S is a finite partially ordered set. Then the order on S is completely known once we know all pairs a, b in S such that $a << b$. This follows from the fact that $x < y$ if and only if $x << y$ or there exists elements

$$x << a_1 << a_2 << \cdots << a_m << y$$

For example, consider the set $A = \{1, 2, 3, 4, 6, 12, 24\}$ ordered by the relation "x *divides* y". Then we can write

$2 << 4$, $2 << 6$, $3 << 6$, $4 << 12$, $6 << 12$, $12 << 24$.

Also $1 << 2 << 4 << 12 << 24$ or $1 << 2 << 6 << 12 << 24$ or $1 << 3 << 6 << 12 << 24$.

Hasse Diagrams



The Hasse diagram of a finite partially ordered set S is a graphical representation of S as follows. The elements of S are represented by points in the plane and there is a directed line segment(arrow) drawn from a to b whenever $a << b$ in S. Instead of drawing an arrow from a to b, we sometimes place b higher than a and draw a line between them.

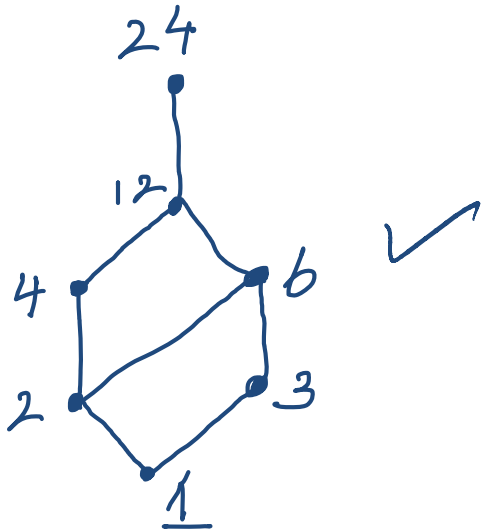
The Hasse diagram of an ordered set S is a picture of S. Sometimes we define a partially ordered set by simply presenting its Hasse diagram.

Example 1

Draw the Hasse diagram representing the divisibility relation on set $A = \{1, 2, 3, 4, 6, 12, 24\}$.

Solution. As we saw $1 << 2 << 4 << 12 << 24$ or $1 << 2 << 6 << 12 << 24$ or $1 << 3 << 6 << 12 << 24$. We can draw its Hasse diagram as follows:

Here $1 < 24$



Example 2

Let $A=\{a,b,c\}$. Draw the Hasse diagram representing the subset relation $\subseteq$ on the power set $P(A)$.

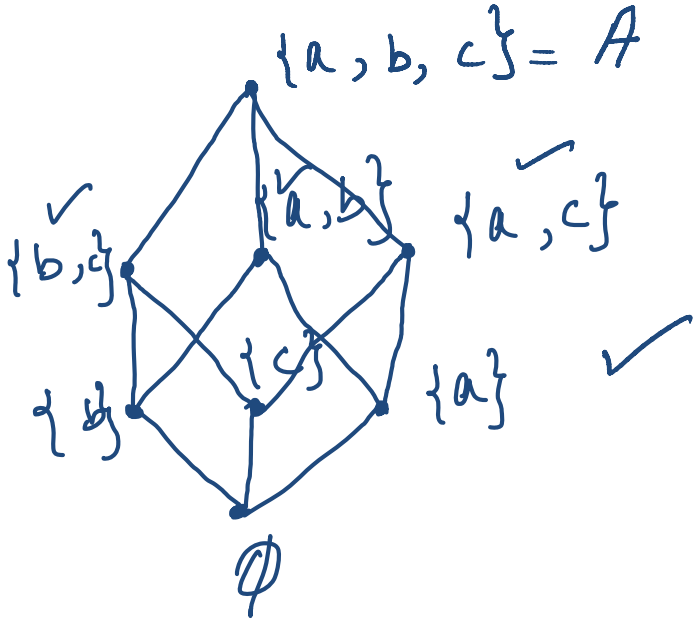
- Solution. The power set $P(A)$ contains all subsets of A:
- $P(A)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$.
- We place the empty set $\emptyset$ at the bottom of the diagram. The subsets with one element $\{a\},\{b\},\{c\}$ are placed on the first level, and the subsets with two elements $\{a,b\},\{b,c\},\{a,c\}$ are placed on the next level. The element $\{a,b,c\}$ occupies the top of the diagram. Finally we connect the subsets with their immediate successor with respect to the inclusion relation. The resulting diagram is shown in the next slide.

Example 2

Here $\{b\} \lessdot\!\!\cdot \{b,c\}$ because $\{b\} \subseteq \{b,c\}$

$\{b\} << \{a,b\}$, $\{c\} << \{b,c\}$, $\{c\} << \{a,c\}$

$\{a\} << \{a,b\}$, $\{a\} << \{a,c\}$



Example 3

Let $A=\{1,2,3,4,5\}$ and $R=\{(1,1),(1,3),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),(4,4),(5,5)\}$. Show that the relation R is a partial order and draw its Hasse diagram.

Solution.

- The relation R is reflexive since it contains all reflexive pairs: $(1,1),(2,2),(3,3),(4,4),(5,5)$.
- R is antisymmetric since all non-reflexive elements do not have the corresponding inverse pairs: $(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5)$.

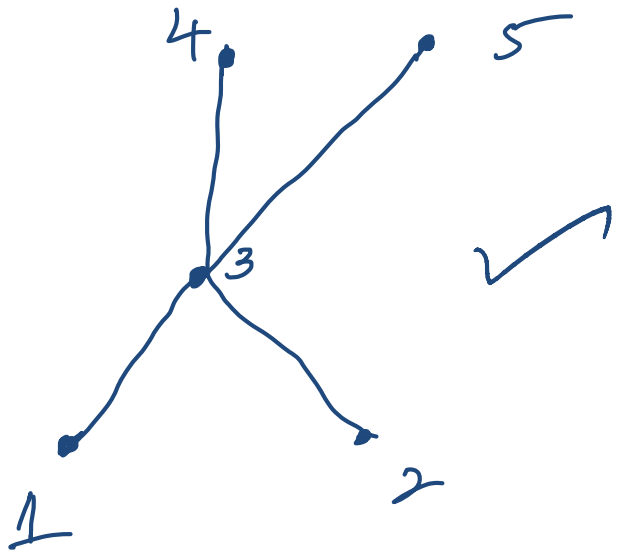
Example 3

- R is transitive:
- $(1,3) \in R, (3,4) \in R \rightarrow (1,4) \in R;$
- $(1,3) \in R, (3,5) \in R \rightarrow (1,5) \in R;$
- $(2,3) \in R, (3,4) \in R \rightarrow (2,4) \in R;$
- $(2,3) \in R, (3,5) \in R \rightarrow (2,5) \in R.$

Hence, the relation R is a partial order and we can draw its Hasse diagram, which is represented below.

Example 3

$R = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5)\}$



Minimal and Maximal Elements

Minimal Element

Let S be a partially ordered set. An element $a \in S$ is called a minimal element of S if no element of S strictly precedes(is less than) a; that is, if

$$x \leq a \text{ implies } x = a$$

For example, let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by divisibility. Then we can see that A has only one minimal element, 1.

Here $1 \leq 2$, $1 < 3$, $1 < 4$ and so on.

Maximal Element

Let S be a partially ordered set. An element $b \in S$ is called a maximal element of S if no element of S strictly succeeds(is greater than) b; that is, if

$$x \geq b \text{ implies } x = b$$

For example, let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be a set ordered by divisibility. Then we can see that A has two maximal elements, 18 and 24.

Here no $x \in A$ exists such that

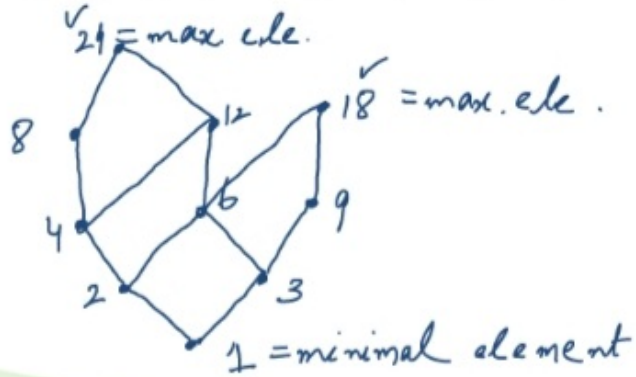
$x > 18$ or $x > 24$

or $18 < x$ or $24 < x$

Geometrical Meaning

Geometrically speaking, a is a minimal element of S if no edge enters a(from below), and b is a maximal element of S if no edge leaves b (in an upward direction).

Note: S can have more than one minimal and more than one maximal element. Hasse diagram of divisibility relation on set $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ is given below:



Minimal and Maximal Element of an Infinite and Finite Set

If S is infinite, then S may have no minimal and maximal element. For instance, the set Z of integers with the usual order $\leq$ has no minimal and maximal element. On the other hand, if S is finite, then S has atleast one minimal and one maximal element.

$$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

First and Last Element

An element $a \in S$ is called a first element of S if $a \leq x$ for every $x \in S$, that is, if a precedes every other element in S.

Similarly, an element $b \in S$ is called a last element of S if $y \leq b$ for every $y \in S$, that is, b succeeds every other element in S.

For example, let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be a set ordered by divisibility. 1 is minimal element of A which is also the first element of A. Set A has two maximal element and neither is a last element.

3:03

First and Last Element

Note: We note that S can have at most one first element which must be a minimal element of S, and S can have at most one last element which must be a maximal element of S. Generally speaking, S may have neither a first nor a last element, even when S is finite.

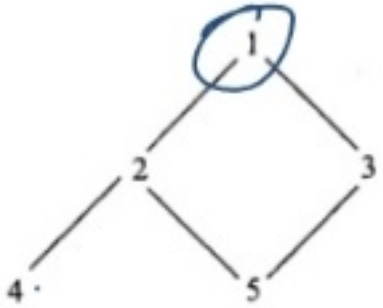
Examples

a) Consider the set $P = \{1, 2, 3, \ldots\}$ with the usual order $\leq$. Here 1 is a first element and only minimal element. P has no last and no maximal element.

b) Let A be any nonempty set and let $P(A)$ be the power set of A ordered by set inclusion. ($\subseteq$) Then the empty set $\emptyset$ is a first element of $P(A)$. Moreover, A is a last element of $P(A)$, since every set Y in $P(A)$ is a subset of A, that is, $Y \subseteq A$.

Examples

c) Consider the ordered set A in the figure below:



1) Find all minimal and maximal elements of A.
2) Does A have a first element or a last element?

So.: 1) 4 and 5 are minimal elements.
1 is a maximal element.
2) No first element of A.
1 is a last element.

Supremum and Infimum

Upper Bound and Supremum

$A \subseteq S$

Let S be a partially ordered set, and let A be a subset of S. An element m in S is called an upper bound of A if m succeeds every element of A, that is, for every $x \in A$, we have

$$x \leq m$$

If an upper bound of A precedes every other upper bound of A, then it is called the supremum of A and it is denoted by $sup(A)$.

Lower Bound and Infimum

Let S be a partially ordered set, and let A be a subset of S. An element n in S is called a lower bound of A if n precedes every element of A, that is, for every $y \in A$, we have

$$y \geq n$$

If a lower bound of A succeeds every other lower bound of A, then it is called the infimum of A and it is denoted by $inf(A)$.

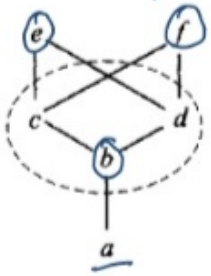
Bounded Above and Bounded Below

If A has an upper bound we say A is bounded above, and if A has a lower bound we say A is bounded below. In particular, A is bounded if A has an upper and lower bound.

Example 1

Let $S = \{a, b, c, d, e, f\}$ be ordered as in the Hasse diagram below and let $A = \{b, c, d\}$.

The upper bounds of A are e and f.

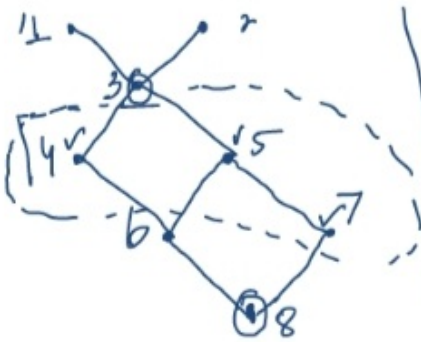


Hence
inf(A) = b

The lower bounds of are a and b.
Here e and f are non comparable, hence sup (A) does not exist.

Example 2

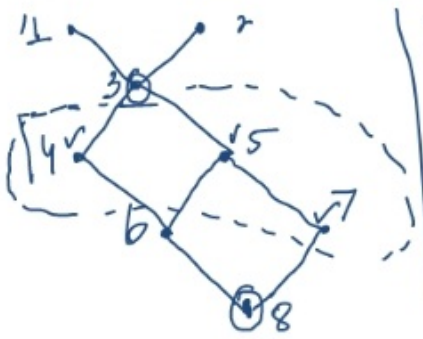
Let $S = \{1, 2, 3, \ldots, 8\}$ be ordered as in the Hasse diagram below and let $A = \{4, 5, 7\}$.



The upper bounds of A are 1, 2, 3.
Here sup (A) = 3.
Now, the lower bounds of A ~~is~~ 8. Hence inf(A) = 8.

Example 2

Let $S = \{1, 2, 3, \ldots, 8\}$ be ordered as in the Hasse diagram below and let $A = \{4, 5, 7\}$.



The upper bounds of A are 1, 2, 3.
Here sup (A) = 3.
Now, the lower bounds of A are 8. Hence inf(A) = 8.

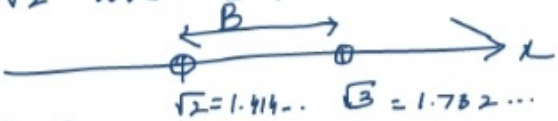
Example 3

$2 < x^2 < 3$
$\Rightarrow \sqrt{2} < x < \sqrt{3}$

Consider the set Q of rational numbers, and its subset

$$B = \{x \in Q : x > 0 \text{ and } 2 < x^2 < 3\}$$

B consists of those rational numbers which lie between $\sqrt{2}$ and $\sqrt{3}$.



{1.2, 1.1.1, 1.10, 1.79}

upper bounds of B = {1.8, 1.81, 1.82, ...} are infinite.
Lower bounds of B are also infinite.
Hence inf(B) and sup(B) do not exists.

Similar Ordered Sets

Similar Mapping

Suppose X and Y are partially ordered sets. A one-to-one(injective) function $f : X \rightarrow Y$ is called a similar mapping from X into Y if f preserves the order relation, that is, if the following conditions holds for any pair a, $b \in X$:

$a \leq b$ in X if and only if $f(a) \leq f(b)$ in Y

The above condition is equivalent to the following two conditions:

1) If $a \leq b$ then $f(a) \leq f(b)$.

2) If $a \| b$ (noncomparable), then $f(a) \| f(b)$.

Similar Ordered Sets

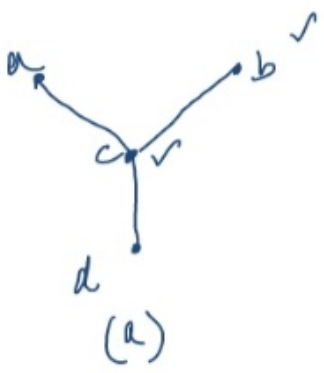
Two ordered sets X and Y are said to be order-isomorphic or isomorphic or similar, written

$$X \simeq Y$$

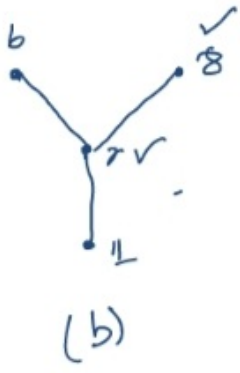
if there exists a one-to-one correspondence $f : X \to Y$ which preserves the order relations. Such a function f is then called an order-isomorphism or isomorphism from X onto Y.

Example 1

Suppose $S = \{a, b, c, d\}$ is ordered by the diagram below and suppose $T = \{1, 2, 6, 8\}$ is ordered by divisibility. Determine whether $S \simeq T$?



(a)



(b)

Define a function $f : S \to T$ such that

$f(a) = b$, $f(b) = 8$,

$f(c) = 2$, $f(d) = 1$

Here $d \le c \Rightarrow f(d) \le f(c)$

$\Rightarrow 1 \le 2$

Also, $c \le b \Rightarrow 2 \le 8$

Hence $S \simeq T$.

Example 2

The set of positive integers $P = \{1, 2, 3, \ldots\}$ is order-isomorphic to the set of even positive integers $E = \{2, 4, 6, \ldots\}$ since the function $f: P \to E$ defined by $f(x) = 2x$ is an isomorphism between P and E.

Example 3

Consider the usual ordering $\leq$ of the positive integers $P = \{1, 2, 3, \ldots\}$ and the negative integers $A = \{-1, -2, -3, \ldots\}$. Then P is not order-isomorphic to A. ✓

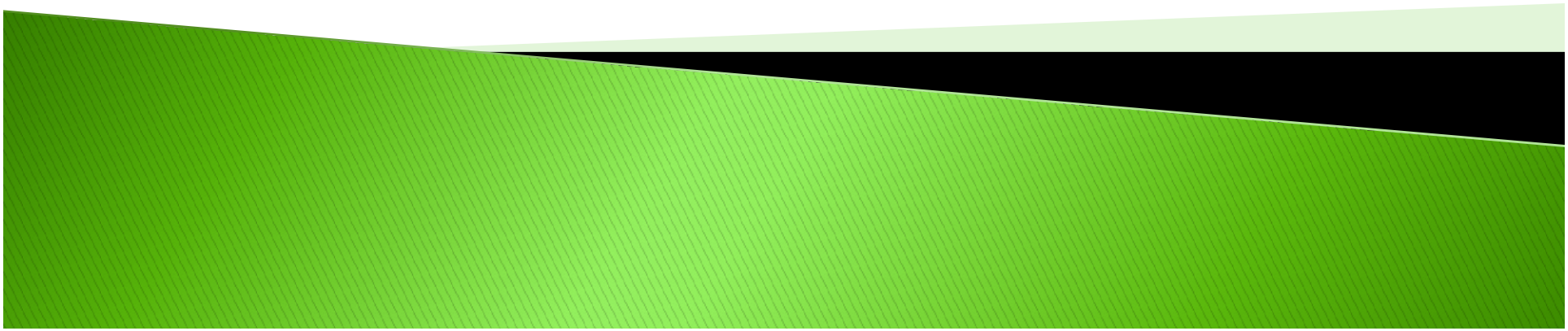
Proof. Let $f: P \to A$ is an isomorphism, for every $n \in P$, we have

$1 \leq n$ should imply

$f(1) \leq f(n)$ for every $f(n) \in A$

Here $f(1) \in A$, since A has no first element, so f cannot exist.

Well Ordered Sets and Initial Segment

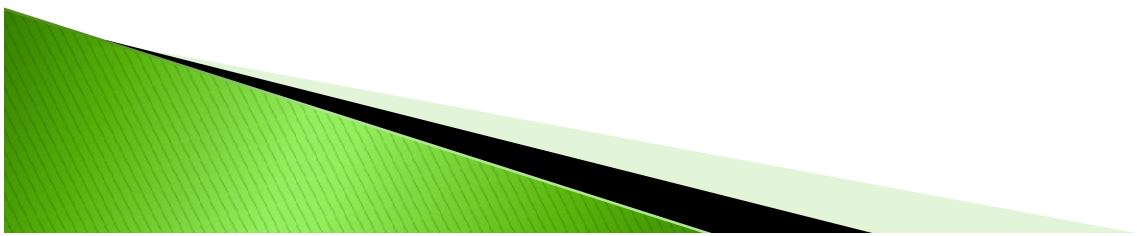


Definition

Let A be an ordered set. Then A is said to be well-ordered if every subset of A contains a first element.

Note: Any well-ordered set A is linearly ordered.

Proof. Let A be a well ordered set. Also, Let $a, b \in A$, then the subset $\{a, b\}$ of A has a first, which, therefore, must precede the other. Hence any two elements of A are comparable.

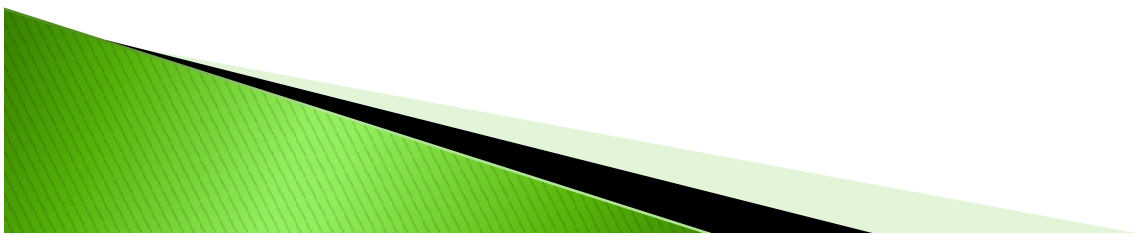


Theorem 1

Let A be a well-ordered set. Then:

(i) Every subset of A is well-ordered.

(ii) If B is similar to A, then B is well-ordered.



Example 1

Consider the following two subsets of well-ordered set P:

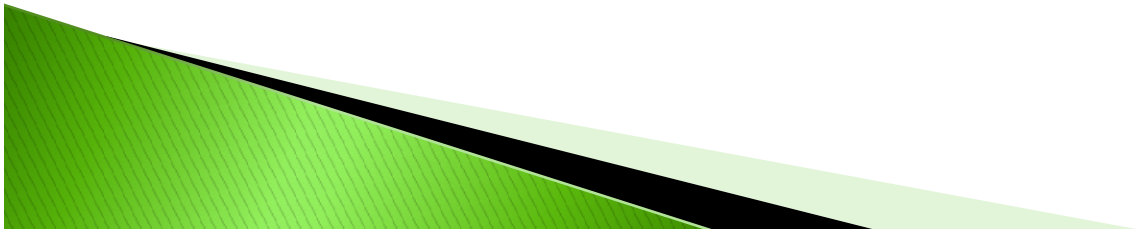
$A_1 = \{1, 3, 5, \ldots\}$ and $A_2 = \{2, 4, 6, \ldots\}$

Then A_1 and A_2 are also well-ordered.

Suppose the union

$S = A_1 \cup A_2 = \{1, 3, 5, \ldots; 2, 4, 6, \ldots\}$

is ordered from left to right. Then S is also well-ordered.



Initial Segments

Let A be a well-ordered set. The initial segment $s(a)$ of an element $a \in A$ consists of all elements in A which strictly precedes a. In other words,

$$s(a) = \{x : x \in A, x < a\}$$

Consider the well-ordered set S as in previous example, that is, $S = \{1, 3, 5, \cdots; 2, 4, 6, \cdots\}$ ✓

Then $s(1) = \emptyset$, $s(5) = \{1, 3\}$,

$s(2) = \{1, 3, 5, \cdots\}$

$s(8) = \{1, 3, 5, \cdots, 2, 4, 6\}$

Theorem 2

Let $S(A)$ denote the collection of all initial segments of elements in a well-ordered set A, and let $S(A)$ be ordered by set inclusion. Then A is similar to $S(A)$ and, in particular, the function $f : A \to S(A)$ defined by $f(x) = s(x)$ is a similarity mapping between A and $S(A)$.

Consider the set $A = \{1, 2, 3\}$ with usual ordering.

Here $s(1) = \phi$, $s(2) = \{1\}$, $s(3) = \{1, 2\}$

Now $S(A) = \{\phi, \{1\}, \{1, 2\}\}$.

Let $f : A \to S(A)$ defined by $f(x) = s(x)$.

Means $f(1) = s(1) = \phi$, $f(2) = s(2) = \{1\}$,
$f(3) = s(3) = \{1, 2\}$.

Theorem 2

$A = \{1,2,3\}$, $S(A) = \{\phi, \{1\}, \{1,2\}\}$

Here $1 < 2 \Rightarrow f(1) < f(2)$

$\phi \subset \{1\}$

$2 < 3 \Rightarrow f(2) < f(3)$

$\Rightarrow \{1\} \subseteq \{1,3\}$

$1 < 3 \Rightarrow f(1) < f(3)$

$\Rightarrow \phi \subseteq \{1,3\}$