

mtch: 201/301

Imp. Questions of Chain Rule. Date: 1/20

①
Chain Rule :-

I,

Let $Z = f(x, y)$

$$x = x(t)$$

$$y = y(t)$$

Then,

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}$$

II, Let,

$$Z = f(x, y) \text{ where}$$

$$x = x(u, v)$$

$$y = y(u, v)$$

Then,

$$\frac{\partial Z}{\partial u} = \frac{\partial Z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial Z}{\partial y} \frac{\partial y}{\partial u}$$

and

$$\frac{\partial Z}{\partial v} = \frac{\partial Z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial Z}{\partial y} \frac{\partial y}{\partial v}$$

Q,

If $Z = x^2 + xy e^x$

$$x = t \sin t, y = t \cos t$$

Find $\frac{dZ}{dt}$.

Muhammad Kholil (B.Sc. dual math, Phy) & (bs-math)

BZU

V.U

Mon [] Tue [] Wed [] Thu [] Fri [] Sat []

Sol: Here $Z = f(x, y)$
 $x = x(t), y = y(t)$

$$\therefore \frac{dZ}{dt} = \frac{\partial Z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial Z}{\partial y} \cdot \frac{dy}{dt} \quad \text{--- (A)}$$

$$\frac{\partial Z}{\partial x} = ?$$

$$Z = x^2 + 2y e^x$$

$$\frac{dx}{dt} = ?$$

$$\frac{\partial Z}{\partial x} = 2x + 2y e^x$$

$$\frac{d}{dt}(t \sin t) = t \cos t + \sin t$$

$$\frac{\partial Z}{\partial y} = ?$$

$$\frac{dx}{dt} = t \cos t + \sin t$$

$$\frac{\partial Z}{\partial y} = 0 + 2e^x (1)$$

$$\frac{dy}{dt} = ?$$

$$\frac{\partial Z}{\partial y} = 2e^x$$

$$\frac{d}{dt}(t \cos t) = -t \sin t + \cos t$$

$$\frac{dy}{dt} = \cos t - t \sin t$$

$$\therefore \begin{matrix} x = t \sin t, \\ y = t \cos t \end{matrix}$$

Putting all these values in eq. (A)

$$\frac{dZ}{dt} = (2x + 2y e^x)(t \cos t + \sin t) + (2e^x)(-t \sin t + \cos t)$$

$$\frac{dZ}{dt} = (2t \sin t + 2t \cos t e^{t \sin t})(t \cos t + \sin t) + (2e^{t \sin t})(\cos t - t \sin t)$$

$$\text{Qf } Z = \ln(u^2 + v) \quad u = e^{x+y^2}$$

$$v = x^2 + y$$

$$\text{Find } \frac{\partial Z}{\partial x} \text{ and } \frac{\partial Z}{\partial y}$$

Sol. Here $Z = f(u, v)$, $u = u(x, y)$
 $v = v(x, y)$

Then,

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial Z}{\partial v} \cdot \frac{\partial v}{\partial x} \quad \text{--- (I)}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial Z}{\partial v} \cdot \frac{\partial v}{\partial y} \quad \text{--- (II)}$$

$$\because u = e^{x+y^2}, \quad v = x^2 + y$$

$$\because Z = \ln(u^2 + v) \Rightarrow \frac{\partial Z}{\partial u} = \frac{1}{u^2 + v} (2u)$$

$$\frac{\partial Z}{\partial u} = \frac{2u}{u^2 + v} \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial x} = e^{x+y^2} (1) = e^{x+y^2} \quad \text{--- (ii)}$$

$$\frac{\partial Z}{\partial v} = \frac{1}{u^2 + v} (1) = \frac{1}{u^2 + v} \quad \text{--- (iii)}$$

$$\frac{\partial v}{\partial x} = 2x + 0 = 2x$$

$$\frac{\partial v}{\partial x} = 2x \quad \text{--- (iv)}$$

$$\frac{\partial u}{\partial y} = e^{x+y^2} (2y) = 2y e^{x+y^2} \quad \text{--- (v)}$$

$$\frac{\partial v}{\partial y} = 0 + 1 = \frac{\partial v}{\partial y} \quad \text{--- (vi)}$$

"For any information about mth 2nd/3rd and mth 8th" my whatsapp number
 0302-4865858

Putting these values (i) — (vi) in
(I) & (II)

$$\frac{\partial Z}{\partial x} = \frac{\partial u}{u^2+v} (e^{x+y^2}) + \frac{1}{u^2+v} (2x)$$

$$\frac{\partial Z}{\partial x} = \frac{2}{u^2+v} (ue^{x+y^2} + x)$$

&

$$\frac{\partial Z}{\partial y} = \left(\frac{\partial u}{u^2+v} \right) (2ye^{x+y^2}) + \frac{1}{(u^2+v)} (1)$$

$$\frac{\partial Z}{\partial y} = \frac{1}{u^2+v} (4uy e^{x+y^2} + 1)$$

(3) NO :-

Using Chain Rule to derive Formula
for diff 'u^v' where u and v
are differentiating-ble of 'x'
Hence differentiable at x^x
and x^{x^x}

Sol.

$$\text{Let } Z = u^v \text{ — (I)}$$

$$\text{Here } Z = Z(u, v), u = u(x)$$

$$v = v(x)$$

$$\Rightarrow \frac{dZ}{dx} = \frac{\partial Z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial Z}{\partial v} \cdot \frac{dv}{dx} \text{ — (II)}$$

By (I), Taking $(\ln)$ both sides.

$$\ln Z = \ln u^v$$

$$= v \ln u$$

$$\frac{\partial}{\partial u} (\ln Z) = \frac{\partial}{\partial u} (v \ln u) \quad \& \quad \frac{\partial}{\partial v} (\ln Z) =$$

$$\frac{1}{Z} \cdot \frac{\partial Z}{\partial u} = \frac{v}{u} \qquad \frac{\partial}{\partial v} (v \ln u)$$

$$\frac{1}{Z} \frac{\partial Z}{\partial v} = \ln u$$

$$\Rightarrow \frac{\partial Z}{\partial u} = Z \cdot \frac{v}{u}$$

$$\Rightarrow \frac{\partial Z}{\partial v} = Z \cdot \ln u$$

$$\frac{\partial Z}{\partial u} = u^v \cdot \frac{v}{u} \quad \text{--- (i)} \quad \Rightarrow \quad \frac{\partial Z}{\partial v} = u^v \ln u \quad \text{--- (ii)}$$

Using (i) & (ii) (II) become's

$$\frac{dZ}{dx} = u^v \cdot \frac{u}{v} \frac{dv}{dx} + u^v \ln u \frac{du}{dx}$$

$$\frac{dZ}{dx} = u^v \left(\frac{v}{u} \frac{du}{dx} + \ln u \frac{du}{dx} \right)$$

that Required formula is

$$\frac{d(u^v)}{dx} = u^v \left(\frac{v}{u} \cdot \frac{du}{dx} + \ln u \frac{du}{dx} \right) \quad \text{(A)}$$

(ii) Now we find $\frac{d(x^x)}{dx}$ by formula

$$\therefore u = x, \quad v = x$$

$$\therefore \frac{d}{dx}(x^x) = x^x \left(\frac{x}{x} \frac{dx}{dx} + \ln x \frac{d}{dx}(x) \right)$$

$$\frac{d}{dx}(x^x) = x^x (1 + \ln x) \quad \text{--- (1)}$$

Now we find $\frac{d}{dx}(x^{x^x})$

Put, $u = x$, $v = x^x$ in (A)

therefore,

$$\frac{d}{dx}(x^{x^x}) = (x^x)^{x^x} \left(\frac{x^x}{x^x} \frac{d}{dx}(x^x) + \ln x^x \frac{d}{dx}(x^x) \right)$$

By (1)

$$= (x^x)^{x^x} \left(\frac{x^x}{x^x} + \ln x^x \{ x^x (1 + \ln x) \} \right)$$

$$= (x^x)^{x^x} \left(\frac{x^x}{x^x} + x^x \ln x + x^x (\ln x)^2 \right)$$

$$= (x^x)^{x^x} \cdot x^x \left(\frac{1}{x^x} + \ln x + (\ln x)^2 \right)$$

$$\cancel{= (x^x)^{x^x} \cdot x^x \cdot \frac{1}{x^x} (1 + x \ln x + x (\ln x)^2)}$$

$$\frac{d}{dx}(x^{x^x}) = x^x \cdot \frac{1}{x^x} (1 + x \ln x + x (\ln x)^2)$$

$$\frac{d}{dx}(x^{x^x}) = x^{x-1} (1 + x \ln x (1 + \ln x))$$

(4) NO

Let $u = f(x, y) = xy - \cos y$

$$x(t) = e^t, \quad y(t) = \sqrt{t^2 + 1}$$

Use chain rule to find $\frac{du}{dt}$ and show that it is a function of 't'

Sol.

Using chain Rule:-

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \quad \text{--- (1)}$$

$$\therefore u = xy - \cos y, \quad x = e^t, \quad y = \sqrt{t^2 + 1}$$

$$\frac{\partial u}{\partial x} = y(1) - 0 \Rightarrow \frac{\partial u}{\partial x} = y \quad \text{(i)}$$

$$\frac{\partial u}{\partial y} = ?$$

$$\frac{\partial u}{\partial y} = x(1) - (\sin y)(1)$$

$$\frac{dx}{dt} = ? \quad , \quad x = e^t$$

$$\frac{\partial u}{\partial y} = x + \sin y \quad \text{--- (iii)}$$

$$\frac{d(e^t)}{dt} = e^t \quad \text{--- (ii)}$$

$$\frac{dy}{dt} = \frac{d}{dt} (\sqrt{t^2 + 1}) = \frac{1}{2} (t^2 + 1)^{-\frac{1}{2}} \cdot \frac{d}{dt} (t^2 + 1)$$

$$\frac{dy}{dt} = \frac{1}{2} (t^2 + 1)^{-\frac{1}{2}} \cdot 2t + 0$$

$$= \frac{2t}{2\sqrt{t^2 + 1}} = \frac{t}{\sqrt{t^2 + 1}}$$

$$\frac{dy}{dt} = \frac{t}{\sqrt{t^2 + 1}} \quad \text{--- (iv)}$$

Now Putting The values of eq (i) - (v) in equation (1)

$$\frac{du}{dt} = \frac{ye^x + t(x + \sin y)}{\sqrt{t^2 + 1}} \quad \text{--- (II)}$$

Now Putting $x = e^t$, $y = \sqrt{t^2 + 1}$ in equation (II)

$$\frac{du}{dt} = \frac{\sqrt{t^2 + 1} \cdot e^x + t(e^x + \sin \sqrt{t^2 + 1})}{\sqrt{t^2 + 1}}$$

$$\frac{du}{dt} = \frac{(\sqrt{t^2 + 1})(\sqrt{t^2 + 1}) \cdot e^x + t(e^x + \sin \sqrt{t^2 + 1})}{\sqrt{t^2 + 1}}$$

$$\frac{du}{dt} = \frac{e^x(t^2 + 1) + t(e^x + \sin \sqrt{t^2 + 1})}{\sqrt{t^2 + 1}}$$

(5) NO

$$\text{Let } w = g(u, v) = \ln(u + v)$$

$$u(t) = e^{2t}$$

$$v(t) = t^2 + t$$

use Chain Rule to find $\frac{dw}{dt}$ and express it as a function of 't'

Sol.

Using Chain rule:-

$$\frac{dw}{dt} = \frac{\partial w}{\partial u} \cdot \frac{du}{dt} + \frac{\partial w}{\partial v} \cdot \frac{dv}{dt} \quad (1)$$

$$\therefore w = \ln(u+v)$$

$$\frac{\partial w}{\partial u} = ?$$

$$\frac{\partial \ln(u+v)}{\partial u} = \frac{1}{u+v} \cdot 1$$

$$\frac{\partial w}{\partial u} = \frac{1}{u+v} \quad (i)$$

$$\therefore u = e^{2t}, \frac{du}{dt} = ?$$

$$= \frac{d}{dt}(e^{2t}) = e^{2t} \cdot 2$$

$$\frac{du}{dt} = 2e^{2t} \quad (ii)$$

$$\frac{\partial w}{\partial v} = \frac{1}{u+v}$$

$$\frac{\partial w}{\partial v} = \frac{1}{u+v} \cdot 1$$

$$\frac{\partial w}{\partial v} = \frac{1}{u+v} \quad (iii)$$

$$\therefore v = t^2 + t$$

$$\frac{dv}{dt} = \frac{d}{dt}(t^2 + t)$$

$$\frac{dv}{dt} = 2t + 1 \quad (iv)$$

Putting The values of eq (i) — (iv) in eq (1)

$$\frac{dw}{dt} = \frac{1}{u+v} \cdot 2e^{2t} + \frac{1}{u+v} (2t+1)$$

$$\frac{dw}{dt} = \frac{2e^{2t}}{u+v} + \frac{2t+1}{u+v} = \frac{2e^{2t} + 2t+1}{u+v} \quad (2)$$

Now Put $u = e^{2t}$, $v = t^2 + t$ in eq (2)

$$\frac{dw}{dt} = \frac{2e^{2t} + 2t+1}{e^{2t} + t^2 + t}$$

Q NO :-

Let $Z = f(x, y) = e^{xy}$, $x(s, t) = 3s \cos t$ and $y(s, t) = 4t^2 s$. Use Chain rule to find $\frac{\partial Z}{\partial t}$ and simplify it as a function of s & t .

Sol: As $Z = e^{xy}$, $x = 3s \cos t$
 $y = 4t^2 s$

Using Chain Rule:-

$$\frac{\partial Z}{\partial t} = ?$$

$$\therefore \frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial Z}{\partial y} \cdot \frac{\partial y}{\partial t} \quad \text{--- (1)}$$

$$Z = e^{xy}$$

$$\frac{\partial Z}{\partial x} = \frac{\partial}{\partial x} (e^{xy}) = e^{xy} \cdot y$$

$$\frac{\partial Z}{\partial x} = y e^{xy} \quad \text{--- (i)}$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t} (3s \cos t)$$

$$\frac{\partial x}{\partial t} = 3s \cdot (-\sin t) \quad \text{--- (ii)}$$

$$\frac{\partial x}{\partial t} = -3s \sin t \quad \text{--- (iii)}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial}{\partial y} (e^{xy}) = e^{xy} \cdot x$$

$$\frac{\partial Z}{\partial y} = x \cdot e^{xy} \quad \text{--- (iii)}$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t} (4t^2 s)$$

$$\frac{\partial y}{\partial t} = 4s \cdot 2t \quad \text{--- (iv)}$$

$$\frac{\partial y}{\partial t} = 8st \quad \text{--- (iv)}$$

Putting the values of eq (i) — (iv),
 in eq (1) we get,

$$\frac{\partial Z}{\partial t} = y e^{xy} \cdot (-3s \sin t) + x \cdot e^{xy} \cdot 8st$$

$$\frac{\partial Z}{\partial t} = -3y e^{xy} s \sin t + 8sx \cdot e^{xy}$$

$$\frac{\partial Z}{\partial t} = 8xs \cdot e^{xy} - 3y e^{xy} s \sin t$$

$$\frac{\partial Z}{\partial t} = s \cdot e^{xy} (8x - 3y \sin t)$$

$$\frac{\partial Z}{\partial t} = e^{(3s \cos t)(4t^2s)} \cdot s (8(3s \cos t) - 3(4t^2s \sin t))$$

$$\frac{\partial Z}{\partial t} = s e^{(3s \cos t)(4t^2s)} (24s \cos t - 12st^2 \sin t)$$

(B) NO, ...

Sol. Using Chain Rule:-

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial Z}{\partial y} \cdot \frac{dy}{dt} \quad \text{--- (1)}$$

$$\frac{\partial Z}{\partial x} = \frac{\partial (e^{xy})}{\partial x} = y e^{xy}$$

$$\frac{\partial Z}{\partial y} = \frac{\partial (e^{xy})}{\partial y} = x e^{xy}$$

$$\frac{\partial Z}{\partial x} = y e^{xy} \quad \text{--- (i)}$$

$$\frac{\partial Z}{\partial y} = x e^{xy} \quad \text{--- (iii)}$$

$$\frac{dx}{dt} = \frac{d}{dt} (3s \cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt} (4t^2s)$$

$$\frac{dx}{dt} = 3s \cdot (-\sin t) \quad \text{--- (i)}$$

$$= -3s \sin t \quad \text{--- (ii)}$$

$$\frac{dy}{dt} = 8ts \quad \text{--- (iv)}$$

Putting the values of (i) — (iv) in eq. (1)

$$\frac{dz}{dt} = -3sy \sin t e^{xy} + 8tx e^{xy}$$

$$\frac{dz}{dt} = (8tx e^{xy} - 3sy \sin t e^{xy})$$

$$\frac{dz}{dt} = s \cdot e^{xy} (8tx - 3y \sin t) \rightarrow (2)$$

Put $x = 3s \cos t$ & $y = 4t^2 s$ in eq. (2)

$$\frac{dz}{dt} = s \cdot e^{(3s \cos t)(4t^2 s)} (8t(3s \cos t) - 3(4t^2 s) \sin t)$$

$$\frac{dz}{dt} = s \cdot e^{12s^2 t \cos t} (24t^2 \cos t - 12s t^2 \sin t)$$

$$\frac{dz}{dt} = 12s^2 t \cos t e^{12s^2 t \cos t} (2 \cos t - t \sin t)$$

(7) NO: if $h = g(x, y, z)$

$$u = u(r, s, t)$$

$$v = v(r, s, t)$$

$w = w(r, s, t)$ then state the chain rule for derivative of h with respect to ' r '

mth 201/301

Assignment #1, Fall 2020 (13)

Date: / / 20

(8) NO:-

Assignment #1 Question:

mth 201/301 Fall 2020

"For Given The function $w = x^2 + yz$
Where, $x = r \cos t$, $y = r \sin t$, $z = r$

By using The Chain rule:-

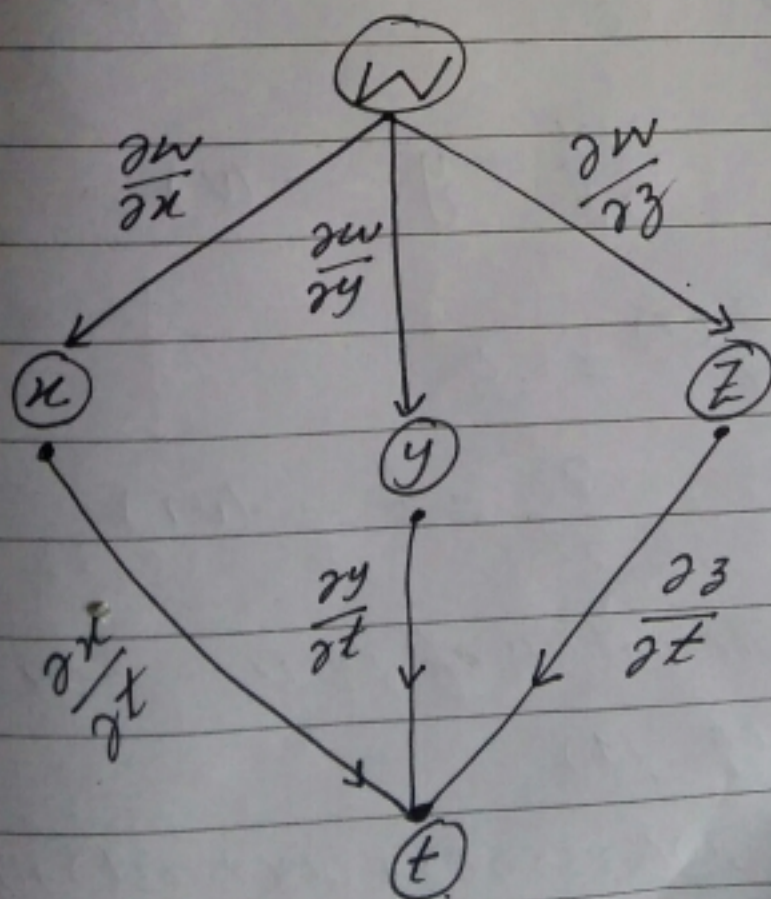
(i) Find The Partial derivative $\frac{\partial w}{\partial t}$

(ii) Find The value of Partial derivative $\frac{\partial w}{\partial t}$ where $r=2$, $s=1$, $t=0$

Solution:-

Using chain Rule:-

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t} \quad (1)$$



M. Khalil (B.Sc & B.S-math)
2302-4865858

"For any information about mth 201/301"

st-I

$$\therefore W = x^2 + yz$$

$$\frac{\partial W}{\partial x} = ?$$

$$\frac{\partial W}{\partial x} = \frac{\partial}{\partial x} (x^2 + yz) = \frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial x} (yz)$$

$\downarrow$
Constant

$$\frac{\partial W}{\partial x} = 2x \cdot \frac{\partial}{\partial x} (x) + 0$$

$$\frac{\partial W}{\partial x} = 2x \cdot (1) \Rightarrow \frac{\partial W}{\partial x} = 2x \quad \text{--- (i)}$$

st-II

$$\frac{\partial x}{\partial t} = ? \quad \therefore x = r \cos t$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t} (r \cos t) \quad \left| \begin{array}{l} r \\ \downarrow \\ \text{Constant} \end{array} \right.$$

$$\frac{\partial x}{\partial t} = r \cdot \frac{\partial}{\partial t} (\cos t)$$

$$\frac{\partial x}{\partial t} = r \cdot (-\sin t) \cdot \frac{\partial}{\partial t} (t)$$

$$\frac{\partial x}{\partial t} = -r \sin t \quad (1)$$

$$\Rightarrow \frac{\partial x}{\partial t} = -r \sin t \quad \text{--- (ii)}$$

st-III $\frac{\partial W}{\partial y} = ?$

$$\frac{\partial W}{\partial y} = \frac{\partial}{\partial y} (x^2 + yz)$$

$$\left| \begin{array}{l} x^2, z \\ \downarrow \\ \text{Constant} \end{array} \right.$$

$$\frac{\partial W}{\partial y} = \frac{\partial}{\partial y} (x^2) + z \cdot \frac{\partial}{\partial y} (y) = 0 + z \cdot (1)$$

$$\frac{\partial W}{\partial y} = z \quad \text{--- (iii)}$$

st-IV $\frac{\partial y}{\partial t} = ?$

$$\therefore y = r \sin t$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t} (r \sin t)$$

$$\left| \begin{array}{l} r \\ \downarrow \\ \text{Constant} \end{array} \right.$$

$$\frac{\partial y}{\partial t} = rs \frac{\partial}{\partial t} (\sin t) = rs \cos t \cdot \frac{\partial t}{\partial t}$$

$$\frac{\partial y}{\partial t} = rs \cos t (1) \Rightarrow \frac{\partial y}{\partial t} = rs \cos t \text{ --- (iv)}$$

st-v $\frac{\partial w}{\partial z} = ? \Rightarrow \frac{\partial (x^2 + yz)}{\partial z}$ $\begin{matrix} x^2, y \\ \downarrow \\ \text{Constant} \end{matrix}$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} (x^2) + y \frac{\partial (z)}{\partial z} = 0 + y (1)$$

$$\frac{\partial w}{\partial z} = y \text{ --- (v)}$$

st-vi $\frac{\partial z}{\partial t} = ? \Rightarrow \frac{\partial z}{\partial t} = \frac{\partial (r)}{\partial t}$ $\begin{matrix} r \\ \downarrow \\ \text{Constant} \end{matrix}$

$$\frac{\partial z}{\partial t} = 0 \text{ --- (vi)}$$

Putting all these values in eq (i) — (vi)

In eq (i)

$$\frac{\partial w}{\partial t} = [(0x)(-rs \sin t) + (r \cos t)(3) + (y)(0)]$$

$$\frac{\partial w}{\partial t} = [-0xrs \sin t + 3r \cos t + 0]$$

$$\frac{\partial w}{\partial t} = [3r \cos t - 0xrs \sin t]$$

$$\frac{\partial w}{\partial t} = rs [3 \cos t - 0x \sin t]$$

$$\frac{\partial w}{\partial t} = rs [3 \cos t - 0(r \cos t)(\sin t)]$$

$$\frac{\partial w}{\partial t} = r \cos t [3 - 0rs \sin t]$$

$$\frac{\partial w}{\partial t} = r \cos t [r - 0rs \sin t] = r^2 \cos t [1 - 0s \sin t]$$

$$\frac{\partial w}{\partial t} = r^2 \cos t [1 - 0s \sin t] \text{ — (II)}$$

Put, $r=2$, $s=1$, $t=0$ in eq (II)

$$\frac{\partial w}{\partial t} = (2)^2 (1) \cos(0) [1 - 0(1) \sin(0)]$$

$$\frac{\partial w}{\partial t} = 4(1) [1 - 0]$$

$$\frac{\partial w}{\partial t} = 4$$