

## Lecture No. 1

### Practice Questions

#### Important Note:

- ✓ Practice these questions on your rough notebook.
- ✓ Type their solutions in Math Type Software in order to increase your typing speed.
- ✓ For more practice questions, follow the recommended book.
- ✓ Make a list of formulae of this lecture, which will help you for a quick revision in the future.

Q1: What is the difference between quadrant and octant?

Q2: Describe two applications of Differential Calculus and two applications of Integral Calculus.

Q3: In two dimensions, in which quadrant, are the following points located?

- a) (2, 5)
- b) (-3, 7)
- c) (9, -2)
- d) (-8, -1)

Q4: In three dimensions, in which octant, are the following points located?

- a) (1, 2, 3)
- b) (-5, 7, 1)
- c) (9, -2, 6)
- d) (-9, -3, 9)
- e) (-4, -2, -6)
- f) (-1, 2, -3)
- g) (2, 2, -2)

Q5: Identify the functions of single variable, two and three variables.

- a)  $y = 2x + 1$
- b)  $z = 5x + 6y + 10$
- c)  $w = 3xyz + 5$
- d)  $y = x^2 + 5x + 6$
- e)  $A = \pi r^2$

f)  $P = 2(x + y)$

Q6. Write down a function of two variables and a function of three variables.

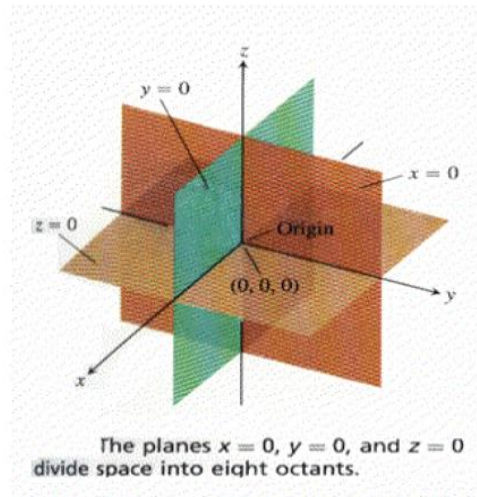
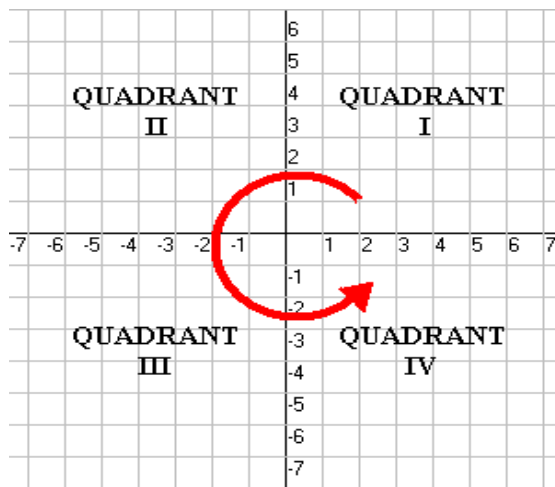
Q7. Identify rational and irrational numbers.

$\pi$ ,  $\sqrt{5}$ , 3.6134613461346134....., 9.185,  $\sqrt{9}$

## Answers

Q1: In two dimension plane, say xy- plane, we divide the plane into four parts. Each part is called quadrant.

In three dimension space, we divide the whole space in eight parts. Each part is called Octant.



Q2: See handouts lecture No. 1. The answer is there.

Q3: (a) First quadrant (b) Second quadrant (c) 4<sup>th</sup> quadrant (d) 3<sup>rd</sup> quadrant

Q4: (a) First octant (b) 2<sup>nd</sup> octant (c) 4<sup>th</sup> octant  
(d) 3<sup>rd</sup> octant (e) 7<sup>th</sup> octant (f) 6<sup>th</sup> octant  
(g) 5<sup>th</sup> octant

Q5: (a) single variable (b) two variables (c) three variables  
(d) single variable (e) single variable (f) two variables

Q6:

$f(x,y) = 3x+4y$  OR  $f(x,y) = x^2+y^2$  OR any function of two variables

$f(x,y,z) = x^2 + y^2 + z^2$  OR any function of three variables

Q7. Identify rational and irrational numbers.

$\pi$  ,  $\sqrt{5}$  irrational numbers

3.6134613461346134..... , 9.185,  $\sqrt{9}$  rational numbers

Lectures # (2 -3)

Practice Questions

Q#1: Let  $f(x, y) = x^2 + 9x^3y - 2y$ . Is the function defined at the point (4, 2)? If yes, what is its value and if not, give the reason.

Sol:

$$\begin{aligned}f(x, y) &= x^2 + 9x^3y - 2y \\f(4, 2) &= (4)^2 + 9(4)^3(2) - 2(2) \\&= 16 + 1152 - 4 = 1164\end{aligned}$$

Q#2: Let  $f(x, y, z) = x^2ye^{xyz}$ . Is the function defined at point (1, 3, 6)? If yes, what is its value and if not, give the reason.

Sol:

$$\begin{aligned}f(x, y, z) &= x^2ye^{xyz} \\f(1, 3, 6) &= (1)^2(3)e^{(1)(3)(6)} = 196979907.4\end{aligned}$$

Q#3: Let  $f(x, y, z)$  such that  $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$ . Is the function defined at origin? If yes, what is its value and if not, give the reason.

Sol:  $f$  is not defined at origin.

Q#4: Find the points where the parabola, represented by the equation  $y = x^2 - 4x$ , meets x-axis.

Sol: See solution of Q#5.

Q#5: Find at least two points that lie on the parabola  $y = x^2 - 6x + 8$ .

Sol: put  $y = 0$ ;  $x^2 - 6x + 8 = 0$  (after factorizing  $x = 2$  and  $x = 4$ )

Q#6: Find the mid point between (6, 2, 4) and (2, 5, 8).

$$\text{Sol: } \left( \frac{6+2}{2}, \frac{2+5}{2}, \frac{4+8}{2} \right) = \left( \frac{8}{2}, \frac{7}{2}, \frac{12}{2} \right) = \left( 4, \frac{7}{2}, 6 \right)$$

Q#7: Find the distance between the points  $A(4, 5, 9)$  and  $B(1, 3, 7)$ .

$$\text{Solution: } \sqrt{(4-1)^2 + (5-3)^2 + (9-7)^2} = \sqrt{17}$$

Q#8: For a line joining the points  $A(3, 2, 6)$  and  $B(0, 3, 4)$ , find

a) Direction ratios of the line AB

$$0-3, 3-2, 4-6 = -3, 1, -2$$

b) Direction cosines of the line AB

$$\frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$$

Q#9:

Find the vertex point of the parabola  $y = 2x^2 + 6x + 3$

Given equation of parabola:

$$y = 2x^2 + 6x + 3$$

$$x - \text{coordinate of vertex} = x_0 = \frac{-b}{2a} = \frac{-6}{2(2)} = \frac{-6}{4} = \frac{-3}{2}$$

$$\begin{aligned} y - \text{coordinate of vertex} = y_0 &= 2\left(\frac{-3}{2}\right)^2 + 6\left(\frac{-3}{2}\right) + 3 \\ &= \frac{18}{4} - \frac{18}{2} + 3 \\ &= \frac{-6}{4} = \frac{-3}{2} \end{aligned}$$

So the vertex point is  $\left(\frac{-3}{2}, \frac{-3}{2}\right)$

Q#10: Find the points where the parabola  $y = x^2 - 5x$  meets  $x$ -axis.

Given equation of the parabola is

$$y = x^2 - 5x \quad \text{---(1)}$$

At  $x$ -axis,  $y = 0$

So eq (1) becomes

$$\begin{aligned} 0 &= x^2 - 5x \\ 0 &= x(x-5) \\ 0 &= x \quad \text{or} \quad x-5 = 0 \\ 0 &= x \quad \text{or} \quad x = 5 \end{aligned}$$

So the parabola meets  $x$ -axis at  $(0, 0)$  and  $(5, 0)$ .

Q#11: Find at least two points that lie on the parabola  $y = x^2 - 7x + 12$ .

put

$$y = 0$$

$$\Rightarrow x^2 - 7x + 12 = 0$$

$$(x - 4)(x - 3) = 0$$

$$x = 4 \text{ or } x = 3$$

so the two points on parabola be (4,0) and (3,0).

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### **Lecture No. 4 (Solutions)**

Q1: a) since  $x = 3, y = 4$ .

Here, we can find the polar coordinates  $(r, \theta)$

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{3}\right).$$

Q2: a) since  $x = -2, y = 2\sqrt{3}$ , and  $z = 4$

Here, we can find the spherical coordinates  $(\rho, \theta, \phi)$

$$\text{Since } \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 12 + 16} = \sqrt{32} = 4\sqrt{2}.$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{4}{4\sqrt{2}}\right) = \frac{\pi}{4}$$

Thus, the point is  $(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4})$  in spherical coordinates.

b) Since  $x = -\sqrt{3}, y = 1$ , and  $z = -2$

Here, we can find the spherical coordinates  $(\rho, \theta, \phi)$

$$\text{Since } \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3 + 1 + 4} = \sqrt{8} = 2\sqrt{2}.$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{-2}{2\sqrt{2}}\right) = \frac{3\pi}{4}$$

Thus, the point is  $(2\sqrt{2}, \frac{5\pi}{6}, \frac{3\pi}{4})$  in spherical coordinates.

Q3: We'll start by acknowledging that  $\theta$  is the same in both coordinate system

For  $\rho$

$$\rho = \sqrt{r^2 + z^2} = \sqrt{6 + 2} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \frac{\pi}{4}$$

Now we find  $\phi$ , for this we can use the conversion for  $z$ .

$$z = \rho \cos \theta \Rightarrow \cos \phi = \frac{z}{\rho} = \frac{\sqrt{2}}{2\sqrt{2}} \Rightarrow \phi = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

So, the spherical coordinate of this point will be  $\left(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3}\right)$

Q4: For Rectangular coordinates

$$x = r \cos \theta = \sqrt{2} \cos \frac{\pi}{4} = \sqrt{2} \frac{1}{\sqrt{2}} = 1$$

$$y = r \sin \theta = \sqrt{2} \sin \frac{\pi}{4} = \sqrt{2} \frac{1}{\sqrt{2}} = 1$$

Thus the rectangular coordinates are (1,1).

Q5:  $(x+y)^2 - z^2 + 4 = 0$

$$x^2 + y^2 + 2xy - z^2 + 4 = 0 \dots\dots\dots (1)$$

For cylindrical coordinates,  $x = r \cos \theta$ ,  $y = r \sin \theta$ , and  $z = z$ , so (1) becomes

$$(r \cos \theta)^2 + (r \sin \theta)^2 + 2(r \cos \theta)(r \sin \theta) - z^2 + 4 = 0$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta + 2r^2 \cos \theta \sin \theta - z^2 + 4 = 0$$

$$r^2(\cos^2 \theta + \sin^2 \theta) + 2r^2 \cos \theta \sin \theta - z^2 + 4 = 0$$

$$r^2(1) + 2r^2 \cos \theta \sin \theta - z^2 + 4 = 0$$

$$r^2(1 + 2 \cos \theta \sin \theta) - z^2 + 4 = 0$$

$$r^2(1 + \sin 2\theta) - z^2 + 4 = 0$$

For spherical coordinates,  $x = p \sin \phi \cos \theta$ ,  $y = p \sin \phi \sin \theta$ , and  $z = p \cos \phi$ , so (1) becomes

$$(p \sin \phi \cos \theta)^2 + (p \sin \phi \sin \theta)^2 + 2(p \sin \phi \cos \theta)(p \sin \phi \sin \theta) - (p \cos \phi)^2 + 4 = 0$$

$$p^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + 2p^2 \sin^2 \phi \cos \theta \sin \theta - p^2 \cos^2 \phi + 4 = 0$$

$$p^2 \sin^2 \phi (1) + 2p^2 \sin^2 \phi \cos \theta \sin \theta - p^2 \cos^2 \phi + 4 = 0$$

$$2p^2 \sin^2 \phi \cos \theta \sin \theta + 4 = p^2 \cos^2 \phi - p^2 \sin^2 \phi$$

$$2p^2 \sin^2 \phi \cos \theta \sin \theta + 4 = p^2 (\cos^2 \phi - \sin^2 \phi)$$

$$p^2 \cos 2\phi - 2p^2 \sin^2 \phi \cos \theta \sin \theta = 4$$

$$p^2 (\cos 2\phi - 2 \sin^2 \phi \cos \theta \sin \theta) = 4$$

$$p^2 (\cos 2\phi - \sin^2 \phi \sin 2\theta) = 4$$

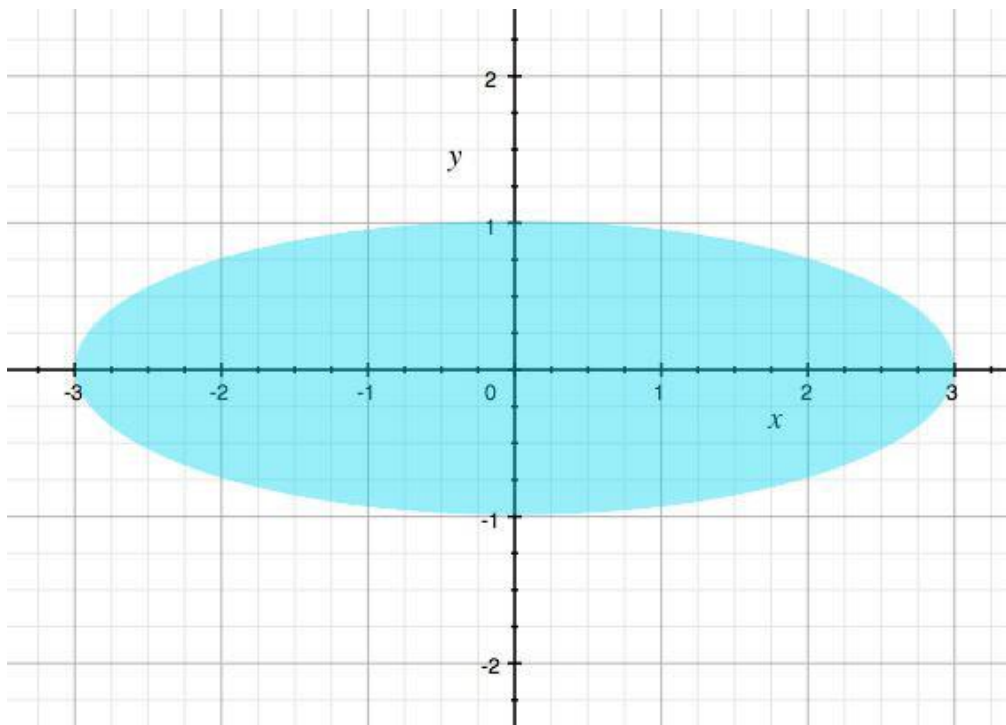
$$2p^2 \sin^2 \phi \cos \theta \sin \theta + 4 = p^2 (\cos^2 \phi - \sin^2 \phi)$$

## **Lecture No. 5 (Solutions)**

Q1: We see that  $9 - x^2 - 9y^2$  must be positive.

$$\text{We have } 9 - (x^2 + 9y^2) > 0 \Rightarrow x^2 + 9y^2 < 9 \Rightarrow \frac{x^2}{9} + y^2 < 1$$

The domain is the set of all points interior to the ellipse centered at the origin with  $a=3$ ,  $b=1$ .



Q2:  $\lim_{(x,y,z) \rightarrow (1,1,1)} \frac{x^2 \ln(xyz)}{e^{xz}}$

By applying limit, we get

$$= \frac{(1)^2 \ln(1)}{e^1} = \frac{1 \times 0}{e} = 0.$$

Q3:  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - y^2)(x^2 + y^2)}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} (x^2 - y^2)$

Along x-axis  $y = 0$ .

$$\lim_{(x,y) \rightarrow (0,0)} x^2 = 0$$

Along y-axis  $x = 0$ .

$$\lim_{(x,y) \rightarrow (0,0)} -y^2 = 0$$

Along the line  $y = mx$ .

$$\lim_{(x,y) \rightarrow (0,0)} x^2 - (mx)^2 = \lim_{(x,y) \rightarrow (0,0)} x^2(1 - m^2) = 0.$$

Along the parabola  $y = x^2$ .

$$\lim_{(x,y) \rightarrow (0,0)} x^2 - (x^2)^2 = \lim_{(x,y) \rightarrow (0,0)} x^2(1 - x^2) = 0$$

Thus the limit exist

Q4: If  $(x, y) \rightarrow (0, 0)$  along the line  $y = x^3$ , then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 x^3}{x^2 + (x^3)^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^6}{x^2 + x^6} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^6}{x^2(1 + x^4)} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^4}{(1 + x^4)} = \frac{0}{1} = 0.$$

## **Lecture No. 6 (Solutions)**

Q1:  $\lim_{(x,y) \rightarrow (1,-1)} e^{-xy} \cos(x+y) = e^{-1(-1)} \cos(1+(-1)) = e^1 \cos(0) = e$

Q2:  $f(x, y)$  is continuous in the entire two dimensional space except at the point

where  $y = \sqrt{x^3}$

Q3: Let  $(x, y) \rightarrow (0, 0)$  along the line  $y = mx$  so that  $y \rightarrow 0$  as  $x \rightarrow 0$ , then

$$f(x, y) = \frac{2x(m^2 x^2)}{x^3 + m^3 x^3}$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{x \rightarrow 0} \frac{2m^2 x^3}{x^3(1 + m^3)}$$

$$= \lim_{x \rightarrow 0} \frac{2m^2}{(1 + m^3)}, \text{ since } x \neq 0$$

$$= \frac{2m^2}{1+m^3}$$

So that limit value will vary as  $m$  changes. Therefore  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  does not exist

so  $f$  is discontinuous at  $(0, 0)$ .

Q4:  $f_x = e^{x-y}, f_y = -e^{x-y}$

Q5:  $f_x = \frac{1}{1+\frac{y^2}{x^2}} \frac{\partial}{\partial x} \left( \frac{y}{x} \right) = \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$

$$f_y = \frac{1}{1+\frac{y^2}{x^2}} \frac{\partial}{\partial y} \left( \frac{y}{x} \right) = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

## Lecture#7

Q1. If the function  $f(x, y) = e^{2x} \cos y$ , then find  $\frac{\partial f}{\partial x}$  at point  $(\ln 3, \pi)$ .

### Solution

$$f(x, y) = e^{2x} \cos y$$

Differentiating w.r.t x,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (e^{2x} \cos y)$$

$$= 2e^{2x} \cos y$$

$$\left. \frac{\partial f}{\partial x} \right|_{(\ln 3, \pi)} = 2e^{2\ln 3} \cos \pi$$

$$= 2e^{2\ln 3} (-1) = -2e^{\ln 3^2}$$

$$= -2(3^2) = -18, \text{ since } e^{\ln x} = x$$

Q2. If  $f(x, y) = \sin(x^4 y^3)$ , then find  $\frac{\partial f}{\partial x}$ .

### Solution

$$\text{As, } f(x, y) = \sin(x^4 y^3)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} [\sin(x^4 y^3)]$$

$$= \cos(x^4 y^3) \frac{\partial}{\partial x} (x^4 y^3)$$

$$= \cos(x^4 y^3) [4x^3 y^3]$$

$$= 4x^3 y^3 \cos(x^4 y^3)$$

Q3. If  $f(x, y) = x \ln(1 + xy^{-1/4})$ , find  $\frac{\partial f}{\partial y}$ .

**Solution**

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \{x \ln(1 + xy^{-1/4})\} \\&= x \frac{\partial}{\partial y} \{\ln(1 + xy^{-1/4})\} \\&= x \frac{1}{(1 + xy^{-1/4})} \frac{\partial}{\partial y} (1 + xy^{-1/4}) \\&= \frac{x}{(1 + xy^{-1/4})} \frac{\partial}{\partial y} (xy^{-1/4}) \\&= \frac{x}{(1 + xy^{-1/4})} \left(-\frac{1}{4} xy^{-5/4}\right) \\&= -\frac{x^2}{4(y^{5/4} + xy)}\end{aligned}$$

Q4. Given that  $h(u, v) = \ln uv^2 + \cos v$ . Find  $\frac{\partial^2 h}{\partial v^2}$ .

**Solution**

$$\begin{aligned}h(u, v) &= \ln uv^2 + \cos v \\ \frac{\partial h}{\partial v} &= \frac{1}{uv^2} 2uv - \sin v = \frac{2}{v} - \sin v \\ \frac{\partial^2 h}{\partial v^2} &= -\frac{2}{v^2} - \cos v\end{aligned}$$

Q5. If  $g(x, y) = (x + 3)\ln(xy)$ , then find both first order partial derivatives.

**Solution**

$$g(x, y) = (x + 3)\ln(xy)$$

$$\begin{aligned}\frac{\partial g}{\partial x} &= (x + 3) \frac{\partial}{\partial x} \ln(xy) + \ln(xy) \frac{\partial}{\partial x} (x + 3) \\ &= (x + 3) \frac{y}{xy} + \ln(xy) = \left(1 + \frac{3}{x}\right) + \ln(xy)\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial y} &= (x + 3) \frac{\partial}{\partial y} \ln(xy) + \ln(xy) \frac{\partial}{\partial y} (x + 3) \\ &= (x + 3) \frac{x}{xy} = \frac{x + 3}{y}\end{aligned}$$

Q6:

Solution:

Euler's theorem is

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Find the partial derivatives and verify it.

## Lecture#8

Q1. Suppose  $f(x, y) = \frac{x}{y}$  where  $x = e^u$  and  $y = \ln u$ . Use chain rule to find  $\frac{df}{du}$  and write the answer in variable  $u$  only.

**Solution**

Since  $f$  is a function of two variables  $x$  and  $y$ , and each  $x$  and  $y$  are function of one variable  $u$ . So by chain rule

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{dx}{du} + \frac{\partial f}{\partial y} \frac{dy}{du} \quad (1)$$

Since  $f(x, y) = \frac{x}{y}$  where  $x = e^u$  and  $y = \ln u$ , so

$$\frac{dx}{du} = e^u, \quad \frac{dy}{du} = \frac{1}{u}$$

$$\frac{\partial f}{\partial x} = \frac{1}{y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}$$

By putting the values of x and y, above equations become

$$\frac{\partial f}{\partial x} = \frac{1}{\ln u} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{e^u}{(\ln u)^2}$$

Put values in equation (1)

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \frac{dx}{du} + \frac{\partial f}{\partial y} \frac{dy}{du} \\ &= \frac{e^u}{\ln u} - \frac{e^u}{u(\ln u)^2} \\ &= \frac{u(\ln u)e^u - e^u}{u(\ln u)^2} \\ &= \frac{[u(\ln u) - 1]e^u}{u(\ln u)^2} \end{aligned}$$

Q2.

If  $h = g(u, v, w)$

$$u = u(t)$$

$$v = v(t)$$

$w = w(t)$ , then state the chain rule for derivative of  $h$  with respect to  $t$ .

**Solution**

$$\frac{dh}{dt} = \frac{\partial h}{\partial u} \frac{du}{dt} + \frac{\partial h}{\partial v} \frac{dv}{dt} + \frac{\partial h}{\partial w} \frac{dw}{dt}$$

Q3. Let  $u = f(x, y) = xy - \cos y$

$$x(t) = e^t$$

$$y(t) = \sqrt{t^2 + 1}$$

Use chain rule to find  $\frac{du}{dt}$  and show it as a function of  $t$ .

**Solution**

Using chain rule

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \quad (1)$$

$$\text{As } \frac{\partial u}{\partial x} = y, \frac{\partial u}{\partial y} = x + \sin y$$

$$\frac{dx}{dt} = e^t \text{ and } \frac{dy}{dt} = \frac{t}{\sqrt{t^2 + 1}}$$

By putting these values in equation (1)

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \\ &= ye^t + \frac{t(x + \sin y)}{\sqrt{t^2 + 1}} \end{aligned}$$

Now put the values of  $x$  and  $y$ ,

$$\begin{aligned} \frac{du}{dt} &= e^t \sqrt{t^2 + 1} + \frac{t(e^t + \sin \sqrt{t^2 + 1})}{\sqrt{t^2 + 1}} \\ &= \frac{e^t(t^2 + 1) + t(e^t + \sin \sqrt{t^2 + 1})}{\sqrt{t^2 + 1}} \end{aligned}$$

Q4. Let  $w = g(u, v) = \ln(u + v)$   
 $u(t) = e^{2t}$   
 $v(t) = t^2 + t$

Use chain rule to find  $\frac{dw}{dt}$  and express it as a function of  $t$ .

**Solution**

Here  $w = g(u, v) = \ln(u + v)$   
 $u(t) = e^{2t}$   
 $v(t) = t^2 + t$

Using chain rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} \quad (1)$$

$$\text{As } \frac{\partial w}{\partial u} = \frac{1}{u+v}, \frac{\partial w}{\partial v} = \frac{1}{u+v}$$

$$\frac{du}{dt} = 2e^{2t} \text{ and } \frac{dv}{dt} = 2t+1$$

By putting these values in equation (1)

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} \\ &= \frac{2e^{2t}}{u+v} + \frac{(2t+1)}{u+v} \\ &= \frac{2e^{2t} + 2t + 1}{u+v} \end{aligned}$$

Now put the values of u and v,

$$\frac{dw}{dt} = \frac{2e^{2t} + 2t + 1}{e^{2t} + t^2 + t}$$

Q5: Find the partial derivatives with respect to x and y at point (1, 2).

$$g(x, y) = x^3 y^2 \ln(xy)$$

Solution:

$$g_x = 3x^2 y^2 \ln(xy) + x^2 y^2$$

$$g_x \text{ at } (1, 2) = 12 \ln 2 + 4$$

$$g_y = 2x^3 y \ln(xy) + x^3 y$$

$$g_y \text{ at } (1, 2) = 4 \ln 2 + 2$$

**Answer**  $g_x(1, 2) = 12 \ln 2 + 4, \quad g_y(1, 2) = 4 \ln 2 + 2$

## Lecture#9

Q1.

Let  $f(x, y) = \frac{\cos x}{y}$ ,  $x = u^2 - v$  and  $y = v + e^u$ . Find  $\frac{\partial f}{\partial u}$  and simplify it as a function of u and v.

**Solution**

Here  $f(x, y) = \frac{\cos x}{y}$ ,  $x = u^2 - v$  and  $y = v + e^u$

$$\begin{aligned}\frac{df}{du} &= \frac{\partial f}{\partial x} \frac{dx}{du} + \frac{\partial f}{\partial y} \frac{dy}{du} \\ &= -\frac{\sin x}{y}(2u) - \frac{\cos x}{y^2} e^u \\ &= -\frac{(2uy \sin x + e^u \cos x)}{y^2}\end{aligned}$$

Putting the values of x and y

$$\begin{aligned}\frac{df}{du} &= -\frac{[2u(v + e^u) \sin(u^2 - v) + e^u \cos(u^2 - v)]}{(v + e^u)^2} \\ &= -\frac{2u \sin(u^2 - v)}{(v + e^u)} - \frac{e^u \cos(u^2 - v)}{(v + e^u)^2}\end{aligned}$$

Q2. Let  $z = f(x, y) = e^{xy}$ ,  $x(s, t) = 3s \cos t$  and  $y(s, t) = 4t^2 s$ . Use chain rule to find  $\frac{\partial z}{\partial t}$  and simplify it as a function of s and t.

### Solution

As,  $z = f(x, y) = e^{xy}$ ,  $x(s, t) = 3s \cos t$  and  $y(s, t) = 4t^2 s$

Using chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad (1)$$

$$\text{As } \frac{\partial z}{\partial x} = ye^{xy}, \quad \frac{\partial z}{\partial y} = xe^{xy}$$

$$\frac{dx}{dt} = -3s \sin t \quad \text{and} \quad \frac{dy}{dt} = 8ts$$

By putting these values in equation (1)

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= -3sy \sin t e^{xy} + 8tsx e^{xy} \\ &= se^{xy} (-3y \sin t + 8tx)\end{aligned}$$

Putting the values of x and y

$$\begin{aligned}\frac{dz}{dt} &= se^{12s^2 t^2 \cos t} (-12st^2 \sin t + 24st \cos t) \\ &= 12s^2 te^{12s^2 t^2 \cos t} (-t \sin t + 2 \cos t)\end{aligned}$$

Q3.

If  $h = g(x, y, z)$

$u = u(r, s, t)$

$v = v(r, s, t)$

$w = w(r, s, t)$ , then state the chain rule for derivative of  $h$  with respect to  $r$ .

**Solution**

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial r}$$

## Practice Questions

### Important Note:

- ✓ Practice these questions on your rough notebook.
- ✓ Type their solutions in Math Type Software in order to increase your typing speed.
- ✓ For more practice questions, follow the recommended book.
- ✓ Make a list of formulae of this lectures, which will help you for a quick revision in the future.

## Lecture#10

**Q#1:** Compute the dot product for each of the following.

1)  $\vec{a} = 4\hat{i} - \hat{j} + 2\hat{k}$  and  $\vec{b} = 6\hat{i} + 2\hat{j} + 5\hat{k}$

**Solution:**

$$\vec{a} = 4\hat{i} - \hat{j} + 2\hat{k} \quad \text{And} \quad \vec{b} = 6\hat{i} + 2\hat{j} + 5\hat{k}$$

$$\vec{a} \cdot \vec{b} = 4(6) + (-1)(2) + (2)(5) = 32$$

2)  $\vec{a} = 0\hat{i} + 3\hat{j} - 7\hat{k}$  and  $\vec{b} = 2\hat{i} + 3\hat{j} - 1\hat{k}$

**Solution:**

$$\vec{a} = 0\hat{i} + 3\hat{j} - 7\hat{k} \quad \text{and} \quad \vec{b} = 2\hat{i} + 3\hat{j} - 1\hat{k}$$

$$\vec{a} \cdot \vec{b} = 0(2) + (3)(3) + (-7)(-1) = 0 + 9 + 7 = 16$$

**Answer:**

1) 32

2) 16

**Q#2:** Find  $\vec{a} \times \vec{b}$

if  $\vec{a} = 2\hat{i} - \hat{j} + 6\hat{k}$  ;  $\vec{b} = -3\hat{i} + 5\hat{j} + \hat{k}$

**Solution:**

$$\vec{a} = 2\hat{i} - \hat{j} + 6\hat{k} ; \vec{b} = -3\hat{i} + 5\hat{j} + \hat{k}$$

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix} = \hat{i}(-1-30) - \hat{j}(2+18) + \hat{k}(10-3) \\ &= -31\hat{i} - 20\hat{j} + 7\hat{k}\end{aligned}$$

**Answer:**  $-31\hat{i} - 20\hat{j} + 7\hat{k}$

**Q#3:** Find area of the parallelogram determined by the vectors  $\vec{a} = 2\hat{i} - 4\hat{k}$  and  $\vec{b} = 3\hat{j} + \hat{k}$

**Solution:**

$$\vec{a} = 2\hat{i} - 4\hat{k} \text{ and } \vec{b} = 3\hat{j} + \hat{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -4 \\ 0 & 3 & 1 \end{vmatrix} = \hat{i}(0+12) - \hat{j}(2+0) + \hat{k}(6-0)$$

$$\vec{a} \times \vec{b} = 12\hat{i} - 2\hat{j} + 6\hat{k}$$

$$\left| \vec{a} \times \vec{b} \right| = \sqrt{12^2 + (-2)^2 + 6^2} = \sqrt{184} \text{ is the area of parallelogram}$$

**Answer:**  $\sqrt{184}$  is the area of parallelogram

**Q#4:** Find the angle between the vectors  $\vec{a} = 3\hat{i} - 5\hat{j} + \hat{k}$  ;  $\vec{b} = -2\hat{i} + 4\hat{j} + 7\hat{k}$

**Solution:**

$$\vec{a} = 3\hat{i} - 5\hat{j} + \hat{k} \text{ and } \vec{b} = -2\hat{i} + 4\hat{j} + 7\hat{k}$$

$$\begin{aligned}
 \vec{a} \cdot \vec{b} &= (\hat{3i} - \hat{5j} + \hat{k})(-\hat{2i} + \hat{4j} + \hat{7k}) \\
 &= 3(-2) + (-5)(4) + (1)(7) \\
 &= -6 - 20 + 7 = -19 \\
 |\vec{a}| &= \sqrt{9 + 25 + 1} = \sqrt{35} \quad \text{and} \quad |\vec{b}| = \sqrt{4 + 16 + 49} = \sqrt{69}
 \end{aligned}$$

$$\theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right) = \cos^{-1} \left( \frac{-19}{\sqrt{35 \times 69}} \right) = 2.2581$$

**Answer: 2.2581**

## Lecture#11

**Q#1:** Given  $\vec{a} \times \vec{b} = 2i + 3j - k$  and  $\vec{c} = 6i - 2k$ . Find scalar triple product of these vectors

**Solution:**

$$\begin{aligned}
 \vec{a} \times \vec{b} &= 2i + 3j - k \quad \text{and} \quad \vec{c} = 6i - 2k \\
 (\vec{a} \times \vec{b}) \cdot \vec{c} &= (2i + 3j - k) \cdot (6i - 2k) \\
 &= 2(6) + 3(0) + (-1)(-2) \\
 &= 12 + 2 = 14
 \end{aligned}$$

**Answer: 14**

**Q#2:**

Let  $f(x, y, z) = xz^4 - 4y^3$ ; Find the gradient of  $f(x, y, z)$ .

**Solution:**

$$f(x, y, z) = xz^4 - 4y^3$$

$$\frac{\partial f}{\partial x} = z^4$$

$$\frac{\partial f}{\partial y} = -12y^2$$

$$\frac{\partial f}{\partial z} = 4xz^3$$

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$$\nabla f = z^4 \hat{i} - 12y^2 \hat{j} + 4xz^3 \hat{k}$$

$$\text{Answer: } \nabla f = z^4 \hat{i} - 12y^2 \hat{j} + 4xz^3 \hat{k}$$

**Q#3: Find Directional derivative of the function,  $f(x, y) = 3x^3y + 5y^2$ , at the point (3, 1) in the direction of vector,  $\vec{u} = 4\hat{i} + 7\hat{j}$ .**

**Solution:**

$$f(x, y) = 3x^3y + 5y^2$$

$$f_x = 9x^2y \quad f_y = 3x^3 + 10y$$

$$f_x(3,1) = 81 \quad f_y(3,1) = 91$$

$$|\vec{u}| = 1$$

$$\hat{u} = 4\hat{i} + 7\hat{j}$$

$$D_u f(3,1) = 81(4) + 91(7) = 961$$

$$\text{Answer: } 961$$

**Q#4: Find volume of the parallelepiped with three adjacent edges formed by the vectors**

$$\vec{a} = 5\hat{i} + 2\hat{j} + \hat{k}, \quad \vec{b} = 4\hat{i} + \hat{j} + 5\hat{k}, \quad \vec{c} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

**Solution:**

$$\vec{a} = 5\hat{i} + 2\hat{j} + \hat{k}, \quad \vec{b} = 4\hat{i} + \hat{j} + 5\hat{k}, \quad \vec{c} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\begin{aligned}\vec{a} \cdot \vec{b} \times \vec{c} &= \begin{vmatrix} 5 & 2 & 1 \\ 4 & 1 & 5 \\ 2 & -3 & 4 \end{vmatrix} \\ &= 5(4+15) - 2(16-10) + 1(-12-2) \\ &= 69\end{aligned}$$

$$\text{Volume of the Parallelepiped} = |\vec{a} \cdot \vec{b} \times \vec{c}| = |69| = 69$$

**Answer:** 69

**Q#5: Find Directional derivative of the function  $f(x, y) = xy$  at the point  $P(1, 2)$  in the direction of a vector making an angle  $\theta = \frac{\pi}{2}$  with the positive x-axis.**

**Solution:**

$$f(x, y) = xy$$

$$f_x(x, y) = y; \quad f_x(1, 2) = 2$$

$$f_y(x, y) = x; \quad f_y(1, 2) = 1$$

$$\hat{u} = \cos \frac{\pi}{2} \hat{i} + \sin \frac{\pi}{2} \hat{j}$$

$$\begin{aligned}D_u f(x, y) &= \nabla f(1, 2) \cdot \hat{u} = (2\hat{i} + \hat{j}) \cdot (\cos \frac{\pi}{2} \hat{i} + \sin \frac{\pi}{2} \hat{j}) \\ &= (2\hat{i} + \hat{j}) \cdot (0\hat{i} + \hat{j}) = 2(0) + 1(1) = 1\end{aligned}$$

**Answer:** 1

## Lecture#12

**Q#1: Write down the equation of a plane passing through the point  $(0, 2, -1)$  and perpendicular to the vector  $\vec{n} = 10\hat{i} + 7\hat{j} + 2\hat{k}$ .**

**Solution:**

$$10(x-0)+7(y-2)+2(z+1)=0$$

$$10x+7y-14+2z+2=0$$

$$10x+7y+2z-12=0$$

**Answer:** 0

**Q#2:**

A line in three dimensional space passes through the point  $(4, 2, 7)$  and parallel to the vector

$\vec{n} = 3\hat{i} + 8\hat{j} + 5\hat{k}$ . Write down the equation of this line in parametric and symmetric form.

**Solution:**

Equation of line in parametric form

$$x = 4 + 3t$$

$$y = 2 + 8t$$

$$z = 7 + 5t$$

Equation of line in symmetric form

$$\frac{x-4}{3} = \frac{y-2}{8} = \frac{z-7}{5}$$

**Q#3:** Find equation of tangent plane to the surface  $f(x, y, z) = x^2 + 3y^2 + 7z^2 + 18$  at a point  $(2, 5, 8)$ .

**Solution:**

$$f(x, y, z) = x^2 + 3y^2 + 7z^2 + 18$$

$$f_x = 2x$$

$$f_y = 6y$$

$$f_z = 14z$$

$$f_x(2, 5, 8) = 2(2) = 4$$

$$f_y(2, 5, 8) = 6(5) = 30$$

$$f_z(2, 5, 8) = 14(8) = 112$$

Equation of tangent is

$$\begin{aligned}
4(x-2) + 30(y-5) + 112(z-8) &= 0 \\
4x - 8 + 30y - 150 + 112z - 896 &= 0 \\
4x + 30y + 112z - 1054 &= 0
\end{aligned}$$

**Answer:**  $4(x-2) + 30(y-5) + 112(z-8) = 0$  (Simplify this equation.)

**Q#4:** Suppose that the surface  $f(x, y, z) = 0$  has continuous partial derivatives at the point  $(l, m, n)$ . Write down the equation of tangent plane at this point.

**Solution:**

**Answer:**  $(x-l)f_x(l, m, n) + (y-m)f_y(l, m, n) + (z-n)f_z(l, m, n) = 0$

**Q#5:** Write down the equation of a plane passing through the point  $(2, -1, 1)$  and perpendicular to the vector  $\vec{n} = 2\hat{i} + 3\hat{j} + \hat{k}$

**Solution:**

$$2(x-2) + 3(y+1) + (z-1)$$

**Answer:**  $2(x-2) + 3(y+1) + (z-1)$  (Simplify this equation.)



## Lecture 16

**Q1.** If  $f(x, y) = 2x - 3y^2 + xy$  and R is the triangular region with vertices O (0, 0), A (3, 0) and B (1, 4), then find the critical points along the line segment OB at which the absolute extreme values of  $f(x, y)$  can occur.

### Solution

As the equation of the line for two points is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

So the equation of line segment OB from (0,0) to (1,4) is  $y = 4x$  as compared to general equation of line. By putting  $y=4x$  in  $f(x, y) = 2x - 3y^2 + xy$ , we have

$$\begin{aligned} f(x, x) &= 2x - 3(4x)^2 + x(4x) \\ &= 2x - 48x^2 + 4x^2 \\ &= 2x - 44x^2 \end{aligned}$$

$$f_x(x, x) = 2 - 88x$$

Equating with zero, we get

$$2 - 88x = 0 \Rightarrow x = \frac{1}{44}$$

$$y = 4\left(\frac{1}{44}\right) = \frac{1}{11}$$

Thus, the critical point at which the function may assume the absolute extrema is  $\left(\frac{1}{44}, \frac{1}{11}\right)$ .

**Q 2.** If  $f(x, y) = xy + x + 1$  and R is the triangular region with vertices O (0, 0), A (6, 0) and B (0, 5), then find the critical points along the line segment AB at which the absolute extrema of  $f(x, y)$  can occur.

### Solution

Equation of the line segment from (6,0) to (0,5) is  $y = -\frac{5}{6}x + 5$  as compared to general

equation of straight line. By putting  $y = -\frac{5}{6}x + 5$  in  $f(x, y) = xy + x + 1$ , we have

$$f(x, -\frac{5}{6}x + 5) = x(-\frac{5}{6}x + 5) + x + 1$$

$$= -\frac{5}{6}x^2 + 6x + 1$$

$$f_x(x, -\frac{5}{6}x + 5) = -\frac{5}{3}x + 6$$

Equating with zero, we get

$$-\frac{5}{3}x + 6 = 0 \Rightarrow x = \frac{18}{5}$$

$$y = -\frac{5}{6}(\frac{18}{5}) + 5 = 2$$

Thus, the critical point at which  $f(x, y)$  may assume the absolute extrema is  $(\frac{18}{5}, 2)$ .

**Q3.** Find the critical point for the given function  $f(x, y) = x^2 - 2xy + y^2$  along the line  $y = 2x - 3$  at which the absolute extrema of the function can occur.

#### Solution

As  $f(x, y) = x^2 - 2xy + y^2$ , simplifies it to single variable  $x$

$$f(x, 2x - 3) = x^2 - 2x(2x - 3) + (2x - 3)^2$$

$$= x^2 - 6x + 9$$

As critical point exist when  $f'(x, 2x - 3) = 0$

So here  $2x - 6 = 0$

$$\Rightarrow x = 3$$

$$y = 2(3) - 3 = 3$$

Thus the critical point exists at  $(3, 3)$ .

**Q4.** Let  $f(x, y) = xy - x - y$  and  $R$  is the triangular region with vertices  $O(0, 0)$ ,  $A(0, 4)$  and  $B(5, 0)$ . Find the critical points along the line segment  $OA$  at which the absolute extrema of  $f(x, y)$  can occur.

#### Solution

Since along the line segment  $OA$   $x = 0$ , so the given function becomes

$$f(0, y) = -y$$

$$f_y(0, y) = -1$$

As the critical point exist where  $f_y(0, y) = 0$ .

Since  $-1 \neq 0$ , therefore, there does not exist any critical point along the line  $OA$  at which the function  $f(x, y)$  may assume the extreme values.

## Lecture#17

**Q1.** Consider three positive integers whose sum is 35 and the product of two integers with the cube of third integer will be maximum. Write down the function to be maximized.

### Solution

The function to be maximized is

$$f(x, y) = xy(35 - x - y)^3.$$

**Q2.** Show that  $f(x, y) = xy(11 - x - y)$  is maximum at  $x = 5, y = 5$ , by using second partial derivative test.

### Solution

$$\text{As } f(x, y) = 11xy - x^2y - xy^2$$

$$f_x = 11y - 2xy - y^2, \quad f_y = 11x - x^2 - 2xy$$

$$f_{xx} = -2y, \quad f_{yy} = -2x, \quad f_{xy} = 11 - 2x - 2y$$

$$\text{at } x = 5, y = 5$$

$$f_{xx} = -10 < 0, \quad f_{yy} = -10, \quad f_{xy} = -9$$

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2$$

$$= (-10)(-10) - (-9)^2 = 19 > 0$$

Hence, given function is maximum at the given points.

**Q3.** Using second partial derivative test show that the function  $f(x, y) = 2xy(5 - x - y)$  is maximum at  $x = 1, y = 1$ .

### Solution

$$\text{As } f(x, y) = 10xy - 2x^2y - 2xy^2$$

$$f_x = 10y - 4xy - 2y^2, \quad f_y = 10x - 2x^2 - 4xy$$

$$f_{xx} = -4y, \quad f_{yy} = -4x, \quad f_{xy} = 10 - 4x - 4y$$

$$\text{at } x = 1, y = 1$$

$$f_{xx} = -4 < 0, \quad f_{yy} = -4, \quad f_{xy} = 2$$

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2$$

$$= (-4)(-4) - 4 = 12 > 0$$

Hence, given function is maximum at given points.

## Lecture#18

**Q1.** Evaluate the following double integral.

$$\iint (x^3y - 6x^2) \, dy \, dx$$

**Solution**

$$\begin{aligned}\iint (x^3y - 6x^2) \, dy \, dx &= \int \left( \frac{x^3y^2}{2} - 6x^2y \right) dx \\ &= \frac{x^4y^2}{8} - 2x^3y\end{aligned}$$

**Q2.** Evaluate the double integral

$$\int_2^6 \int_{-1}^3 (7xy^2 + 3x^3y) \, dx \, dy.$$

**Solution**

$$\begin{aligned}\int_2^6 \int_{-1}^3 (7xy^2 + 3x^3y) \, dx \, dy &= \int_2^6 \left[ \frac{7x^2y^2}{2} + \frac{3x^4y}{4} \right]_{-1}^3 dy \\ &= \int_2^6 (28y^2 + 60y) dy \\ &= \left[ \frac{28y^3}{3} + 30y^2 \right]_2^6 = \frac{8704}{3}\end{aligned}$$

**Q3.** Evaluate  $\int_{-1}^3 \int_1^4 (3x^3y + x^2) dx \, dy$ .

**Solution**

$$\begin{aligned}\int_{-1}^3 \int_1^4 (3x^3y + x^2) dx \, dy &= \int_{-1}^3 \left[ \frac{3x^4y}{4} + \frac{x^3}{3} \right]_1^4 dy \\ &= \int_{-1}^3 \left( \frac{765y}{4} + 21 \right) dy \\ &= \left[ \frac{765y^2}{8} + 21y \right]_{-1}^3 \\ &= \frac{765}{8}[(3)^2 - (-1)^2] + 21[(3) - (-1)] \\ &= 765 + 84 = 849\end{aligned}$$

**Q4.** Evaluate each of the following integrals over the given region  $D$ .

$$\iint_D e^{\frac{x}{y}} dA, D = \{ \{x, y\} \mid 1 \leq y \leq 2, y \leq x \leq y^3 \}$$

**Solution**

$$\begin{aligned} \iint_D e^{\frac{x}{y}} dA &= \int_1^2 \int_y^{y^3} e^{\frac{x}{y}} dx dy = \int_1^2 y e^{\frac{x}{y}} \Big|_y^{y^3} dy \\ &= \int_1^2 y e^{y^2} - y e^1 dy \\ &= \left( \frac{1}{2} e^{y^2} - \frac{1}{2} y^2 e^1 \right) \Big|_1^2 = \frac{1}{2} e^4 - 2e^1 \end{aligned}$$

**Answer**  $\left( \frac{1}{2} e^4 - 2e^1 \right)$

Solution of Practice Question Lecture 19-22

Q#1: Evaluate  $\int_0^3 \int_{x^2}^{3x} (3x+1) dy dx$

Solution:

$$\begin{aligned} & \int_0^3 \int_{x^2}^{3x} (3x+1) dy dx \\ &= \int_0^3 [3x(y)_{x^2}^{3x} + (y)_{x^2}^{3x}] dx \\ &= \int_0^3 [3x(3x-x^2) + (3x-x^2)] dx \\ &= \int_0^3 [9x^2 - 3x^3 + 3x - x^2] dx = \frac{99}{4} \end{aligned}$$

Q#2:

If the order of integration for the integral  $\int_0^6 \int_{y/3}^2 e^{x^2} dx dy$  is changed. Find the change in the limits

of new integral.

Solution:

$$\begin{aligned} \frac{y}{3} \leq x \leq 2 &\Rightarrow y \leq 3x \leq 6 \\ 0 \leq y \leq 3x &\leq 6 \\ 0 \leq 3x \leq 6 \\ 0 \leq x \leq 2 \end{aligned}$$

Q#3: Evaluate the integral  $\int_0^3 \int_0^{2x} e^{x^2} dy dx$ .

Solution:

$$\begin{aligned}
& \int_0^3 \int_0^{2x} e^{x^2} dy dx \\
&= \int_0^3 e^{x^2} (y)_0^{2x} dx \\
&= \int_0^3 e^{x^2} 2x dx = (e^{x^2})_0^3 = e^{3^2} - 1 \\
&= e^9 - 1
\end{aligned}$$

Q#4: Evaluate the integral  $\int_0^3 \int_0^{y^2} \cos y^3 dx dy$ .

Solution:

$$\begin{aligned}
& \int_0^3 \int_0^{y^2} \cos y^3 dx dy \\
&= \int_0^3 (x)_0^{y^2} \cdot \cos y^3 dy \\
&= \int_0^3 y^2 \cdot \cos y^3 dy \\
&= \frac{1}{3} \int_0^3 3y^2 \cdot \cos y^3 dy \\
&= \frac{1}{3} (\sin y^3)_0^3 = \frac{1}{3} \sin 27
\end{aligned}$$