



Practice Question Mth301 23 to 45

calculus II (Virtual University of Pakistan)

Solutions of Practice Questions

Lecture No. 23 To 45

Q1.

Find all possible polar co-ordinates representations of the point P whose rectangular co-ordinates are $(3\sqrt{2}, -3\sqrt{2})$

Solution

We have

$$x=3\sqrt{2} \quad \text{and} \quad y=-3\sqrt{2}$$

For polar coordinate $P(r, \theta)$

$$r = \sqrt{x^2 + y^2} = \sqrt{(3\sqrt{2})^2 + (-3\sqrt{2})^2} = \sqrt{18+18} = \sqrt{36} = \pm 6$$

$$\tan \theta = \frac{y}{x} = \frac{-3\sqrt{2}}{3\sqrt{2}} = -1 \Rightarrow \theta = \tan^{-1}(-1)$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

The point $P(3\sqrt{2}, -3\sqrt{2})$ lies in the fourth quadrant

When

$$\theta = \frac{3\pi}{4} \quad r = -6$$

$$\text{When } \theta = \frac{7\pi}{4} \quad r = 6$$

Polar co-ordinates of P are $(-6, \frac{3\pi}{4})$ and $(6, \frac{7\pi}{4})$

Q2.

Find polar co-ordinates of the point whose rectangular co-ordinates are $(2, -2)$

Solution

As we know

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan \theta = \frac{y}{x}$$

So

$$r = \sqrt{(2)^2 + (-2)^2} = \sqrt{8} = \pm 2\sqrt{2}$$

$$\tan \theta = \frac{-2}{2} = -1$$

$$\theta = \tan^{-1}(-1)$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$\left(2\sqrt{2}, \frac{7\pi}{4} + 2n\pi \right), \left(-2\sqrt{2}, \frac{3\pi}{4} + 2n\pi \right)$$

$$n = 0, 1, 2, \dots$$

Q3.

Express the equation $x^2 = 8y$ in polar coordinates.

Solution

We use the formula $x=r\cos\theta$ and $y=r\sin\theta$

$$x^2 = 8y$$

$$(r \cos \theta)^2 = 8r \sin \theta$$

$$r^2 \cos^2 \theta = 8r \sin \theta$$

$$r \cos^2 \theta = 8 \sin \theta$$

$$r = \frac{8 \sin \theta}{\cos^2 \theta} = 8 \sec \theta \tan \theta$$

Q4.

Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Solution

$$(r, \theta) = (2, \frac{\pi}{3})$$

we have

$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in Cartesian co-ordinates.

Lecture#24

Q1.

Equation of a curve in polar co-ordinates is $r = 1 - \cos \theta$. Discuss the symmetry of graph of this curve about initial line.

Solution

Symmetry about initial line

As we know, $\cos(-\theta) = \cos \theta$ so equation of a curve remains same if θ is replaced by $-\theta$. So the curve is symmetrical about initial line.

Q2.

Equation of a curve in polar co-ordinates is $r^2 = a^2 \cos 2\theta$. Discuss the symmetry of the graph of this curve about y-axis.

Solution

Symmetry about y-axis

As we know

$$\begin{aligned}\cos(2(\pi - \theta)) &= \cos(2\pi - 2\theta) \\ &= \cos 2\pi \cos 2\theta + \sin 2\pi \sin 2\theta \\ &= \cos(2\theta)\end{aligned}$$

So equation of a curve remains same if θ is replaced by $\pi - \theta$. So the curve is symmetrical about y-axis.

Q3.

Equation of a curve in polar co-ordinates is $r^2 = 8 \sin 2\theta$. Discuss the symmetry of the graph of this curve about y-axis.

Solution

Symmetry about y-axis

As we know

$$\begin{aligned}\sin(2(\pi - \theta)) &= \sin(2\pi - 2\theta) \\ &= \sin 2\pi \cos 2\theta - \cos 2\pi \sin 2\theta \\ &= -\sin(2\theta)\end{aligned}$$

So equation of a curve does not remain same if θ is replaced by $\pi - \theta$. So the curve is not symmetrical about y-axis.

Lecture#25

Q1.

Use double integrals in polar co-ordinates to find area of the region bounded by the curve $r = 7 \cos \theta$.

Solution

Here $r = 7 \cos \theta$ is a circle of radius $\frac{7}{2}$ and center at $\left(\frac{7}{2}, 0\right)$

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{7\cos\theta} r \, dr \, d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \frac{r^2}{2} \right|_0^{7\cos\theta} d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (7\cos\theta)^2 d\theta = \frac{49}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \\
&= \frac{49}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta \\
&= \frac{49}{4} \int_0^{\pi} 1+\cos 2\theta \, d\theta \\
&= \frac{49}{4} \left| \theta + \frac{\sin 2\theta}{2} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \\
&= \frac{49}{4} \left(\frac{\pi}{2} - \frac{(-\pi)}{2} \right) = \frac{49}{4} \left(\frac{2\pi}{2} \right) = \frac{49\pi}{4}
\end{aligned}$$

Q2.

In each part, rewrite the double integral as an iterated integral in polar coordinates. (Do not evaluate.)

(a) $\iint_R \sqrt{1-x^2-y^2} \, dx \, dy$, where R is the left half of the unit disk.

(b) $\iint_R x^2 \, dx \, dy$ Where R is the right half of the ring $4 \leq x^2 + y^2 \leq 9$

Solution

(a) The region R is the polar rectangle $\pi/2 \leq \theta \leq 3\pi/2$, $0 \leq r \leq 1$. In polar coordinates, the integrand is $\sqrt{1-r^2}$. So, we can rewrite the double integral

as an iterated integral $\int_{\pi/2}^{3\pi/2} \int_0^1 (\sqrt{1-r^2}) r \, dr \, d\theta$.

(b) The region R is the polar rectangle $-\pi/2 \leq \theta \leq \pi/2$, $2 \leq r \leq 3$. In polar coordinates, the integrand is $(r \cos \theta)^2$. So, we can rewrite the double integral as an iterated integral

$$\int_{-\pi/2}^{\pi/2} \int_2^3 (r \cos \theta)^2 r dr d\theta = \int_{-\pi/2}^{\pi/2} \int_2^3 r^3 \cos^2 \theta dr d\theta$$

Q3. Evaluate the following double integral by changing to polar coordinates.

$$\int_0^1 \int_0^{\sqrt{2-x^2}} (x^2 + y^2) dy dx.$$

Solution

$$\int_0^1 \int_0^{\sqrt{2-x^2}} (x^2 + y^2) dy dx$$

Region of integration is the part of a circle center at origin and radius $\sqrt{2}$ in first quadrant

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{\sqrt{2}} r^2 r dr d\theta, \text{ since } x^2 + y^2 = r^2 \\ &= \int_0^{\pi/2} \int_0^{\sqrt{2}} r^3 dr d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^{\sqrt{2}} d\theta = \frac{1}{4} \int_0^{\pi/2} \left[r^4 \right]_0^{\sqrt{2}} d\theta = \frac{1}{4} \int_0^{\pi/2} 4 d\theta \\ &= \int_0^{\pi/2} d\theta = \left[\theta \right]_0^{\pi/2} = \frac{\pi}{2} \end{aligned}$$

Q4. Evaluate $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(x^2 + y^2) dy dx$ by converting to polar coordinates.

Solution

$$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(x^2 + y^2) dy dx$$

Region of integration is the part of a circle center at origin and radius 3 in upper half of plane

$$= \int_0^{\pi} \int_0^3 (\sin r^2) r dr d\theta, \text{ since } x^2 + y^2 = r^2$$

$$= \int_0^{\pi} \left. \frac{1}{2} \cos r^2 \right|_0^3 d\theta = -\frac{1}{2} \int_0^{\pi} (\cos 9 - 1) d\theta$$

$$= -\frac{1}{2} [\theta \cos 9 - \theta]_0^{\pi} = -\frac{\pi}{2} (\cos 9 - 1) = \frac{\pi}{2} (1 - \cos 9)$$

Solution of Practice Questions

Lecture#26

Q1. Given the integral $\iiint_G (x^3 + y + z^2) dz dy dx$ where $-2 \leq z \leq 2, 0 \leq y \leq 3, 0 \leq x \leq 2$.

Evaluate the integral with respect to the variable z only.

Solution

$$\begin{aligned} \int_0^2 \int_0^3 \int_{-2}^2 (x^3 + y + z^2) dz dy dx &= \int_0^2 \int_0^3 \left[x^3 z + yz + \frac{z^3}{3} \right]_{-2}^2 dy dx \\ &= \int_0^2 \int_0^3 \left[4x^3 + 4y + \frac{16}{3} \right] dy dx \end{aligned}$$

Q2. Evaluate the double integral $\int_{-1}^1 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2 + y^2 + 1) dy dx$ by changing to polar co-ordinates.

Solution

$$\int_{-1}^1 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2 + y^2 + 1) dy dx$$

$$-1 \leq x \leq 1, \quad -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}$$

$$y = \sqrt{9-x^2} \Rightarrow x^2 + y^2 = 9$$

Circle center at origin and radius 3. So r varies from 0 to 3 and θ varies from 0 to 2π

$$\text{Also } x^2 + y^2 + 1 = r^2 + 1$$

Putting values

$$\begin{aligned} \int_0^{2\pi} \int_0^3 (r^2 + 1) r dr d\theta &= \int_0^{2\pi} \int_0^3 (r^3 + r) dr d\theta \\ &= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^2}{2} \right]_0^3 d\theta \\ &= \int_0^{2\pi} \left(\frac{81}{4} + \frac{9}{2} \right) d\theta = \frac{99}{4} (2\pi - 0) = \frac{99\pi}{2} \end{aligned}$$

Q3. Evaluate the double integral $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2 + y^2} dy dx$ by changing to polar co-ordinates.

Solution

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2} dy dx$$

$$-1 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1-x^2}$$

$$y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1$$

Circle center at origin and radius 1. So r varies from 0 to 1 and θ varies from 0 to π

$$\text{Also } \sqrt{x^2+y^2} = r$$

Putting values

$$\int_0^\pi \int_0^1 r \, dr \, d\theta = \int_0^\pi \int_0^1 r^2 \, dr \, d\theta$$

$$= \int_0^\pi \left[\frac{r^3}{3} \right]_0^1 d\theta = \int_0^\pi \frac{1}{3} d\theta = \frac{1}{3} [\theta]_0^\pi = \frac{1}{3} (\pi) = \frac{\pi}{3}$$

Q4. Convert the following integral from Cartesian to polar coordinates. (Do not integrate it).

$$\int_0^1 \int_0^{\sqrt{a^2-x^2}} x^2 (x^2+y^2)^2 dy dx.$$

Solution

$$y = \sqrt{a^2 - x^2}$$

$$\Rightarrow y^2 = a^2 - x^2 \text{ or } x^2 + y^2 = a^2$$

Comparing it with the general equation of circle $x^2 + y^2 = r^2$

we get $r = a$, so we have $0 \leq r \leq a$ and θ varies from 0 to $\frac{\pi}{2}$,

$$\int_0^{\frac{\pi}{2}} \int_0^a r^2 \cos^2 \theta (r^2)^2 r dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^a r^7 \cos^2 \theta dr d\theta$$

Lecture#27

Q1. If $r(t) = 5\hat{i} + \sin t \hat{j} + \cos t \hat{k}$ then find $\|r(t)\|$.

Solution

$$\begin{aligned}\|r(t)\| &= \sqrt{(5)^2 + (\sin t)^2 + (\cos t)^2} \\ &= \sqrt{25 + \cos^2 t + \sin^2 t} \\ &= \sqrt{25 + 1} = \sqrt{26}\end{aligned}$$

Q2. If the vector valued function $r(t) = e^{2t}\hat{i} + \ln(t-1)\hat{j} + t\hat{k}$, then find the vector that $r(t)$ associates with $t=2$.

Solution

The vector associated with $t=2$ is

$$\begin{aligned}
 r(2) &= e^{2(2)}\hat{i} + \ln(2-1)\hat{j} + 2\hat{k} \\
 &= e^4\hat{i} + \ln(1)\hat{j} + 2\hat{k} \\
 &= e^4\hat{i} + 2\hat{k}
 \end{aligned}$$

Lecture#28

Q1. Check whether the given limit exists or not? Also justify your answer.

$$\lim_{t \rightarrow 0} \left[\left(\frac{1}{t} \right) \hat{i} + e^{2t} \hat{j} + \cos t \hat{k} \right].$$

Solution

$$\lim_{t \rightarrow 0} \left[\left(\frac{1}{t} \right) \hat{i} + e^{2t} \hat{j} + \cos t \hat{k} \right] = \lim_{t \rightarrow 0} \left(\frac{1}{t} \right) \hat{i} + \lim_{t \rightarrow 0} e^{2t} \hat{j} + \lim_{t \rightarrow 0} \cos t \hat{k}$$

Since the limit of first component, that is, $\lim_{t \rightarrow 0} \left(\frac{1}{t} \right)$ does not exist. So even though the limit of remaining two components exist, the limit of vector-valued function does not exist.

Q2. The vector equation for a given curve C is $\vec{r}(t) = \left(\frac{1}{t} \right) \hat{i} + t^2 \hat{j} + (2\sqrt{t}) \hat{k}$ and its derivative is $\vec{r}'(t) = -\frac{1}{t^2} \hat{i} + 2t \hat{j} + \frac{1}{\sqrt{t}} \hat{k}$. Find the tangent vector to C at the point $t_0 = 3$.

Solution

At $t_0 = 3$

$$\vec{r}(3) = \frac{1}{3}\hat{i} + 9\hat{j} + 2\sqrt{3}\hat{k} \quad \text{and} \quad \vec{r}'(3) = -\frac{1}{9}\hat{i} + 6\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

For vector equation of tangent line at the point where $t_0 = 3$, we have

$$\begin{aligned}\vec{r} &= \vec{r}(3) + t \vec{r}'(3) \\ &= \left(\frac{1}{3}\hat{i} + 9\hat{j} + 2\sqrt{3}\hat{k} \right) + t \left(-\frac{1}{9}\hat{i} + 6\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \right) \\ &= \left(\frac{1}{3} - \frac{1}{9}t \right)\hat{i} + (9 + 6t)\hat{j} + \left(2\sqrt{3} + \frac{1}{\sqrt{3}}t \right)\hat{k}\end{aligned}$$

Q3. Let $\vec{r}(t) = \sin t \hat{i} + 3 \cos t \hat{j}$. Find t , such that $\vec{r}(t)$ and $\vec{r}'(t)$ are perpendicular to each other.

Solution

$$\vec{r}(t) = \sin t \hat{i} + 3 \cos t \hat{j} \text{ and } \vec{r}'(t) = \cos t \hat{i} - 3 \sin t \hat{j}$$

Since $\vec{r}(t)$ and $\vec{r}'(t)$ are perpendicular so

$$\vec{r}(t) \cdot \vec{r}'(t) = 0$$

$$\left(\sin t \hat{i} + 3 \cos t \hat{j} \right) \cdot \left(\cos t \hat{i} - 3 \sin t \hat{j} \right) = 0$$

$$\sin t \cos t - 9 \sin t \cos t = 0$$

$$- 8 \sin t \cos t = 0$$

$$\sin t \cos t = 0$$

Multiply by 2 on both sides

$$2 \sin t \cos t = 0$$

$$\sin 2t = 0$$

$$2t = 0, \pi, 2\pi, 3\pi, \dots$$

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots$$

$$t = \frac{n\pi}{2}; n = 0, 1, 2, 3, \dots$$

Lecture#29

Q#1: Write down the expression for the arc-length of the curve represented by the vector valued function

$$\vec{r}(t) = (5t - \ln 3t)\hat{i} + (t^2 + 7)\hat{j} \text{ where } 0 \leq t \leq 4. \text{ (Do not evaluate the expression).}$$

Solution:

Here

$$x = 5t - \ln 3t, \quad y = t^2 + 7$$

$$\frac{dx}{dt} = 5 - \frac{3}{t}, \quad \frac{dy}{dt} = 2t$$

$$\begin{aligned}\text{Arc length} &= \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \\ &= \int_0^4 \sqrt{\left(5 - \frac{3}{3t}\right)^2 + (2t)^2}\end{aligned}$$

Q#2: What is the arc-length of the curve $\vec{r}(t) = 2 \sin t \hat{i} + 2 \cos t \hat{j}$; when $0 \leq t \leq 2\pi$?

Solution:

$$\vec{r}(t) = 2 \sin t \hat{i} + 2 \cos t \hat{j}$$

Parametric equations corresponding to the vector equation are

$$x = 2 \sin t, \quad y = 2 \cos t$$

$$\frac{dx}{dt} = 2 \cos t, \quad \frac{dy}{dt} = -2 \sin t$$

$$\begin{aligned}\text{Arc - length} &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{2\pi} \sqrt{(2 \cos t)^2 + (-2 \sin t)^2} dt \\ &= \int_0^{2\pi} \sqrt{4 \cos^2 t + 4 \sin^2 t} dt \\ &= \int_0^{2\pi} \sqrt{4(\sin^2 t + \cos^2 t)} dt \\ &= 2 \int_0^{2\pi} dt = 2 \left| t \right|_0^{2\pi} = 4\pi\end{aligned}$$

Q#3: What is the arc-length of the curve $\vec{r}(t) = (2 + 5t) \hat{i} + (3 - 6t) \hat{j}$ when $2 \leq t \leq 3$?

Solution:

$$\vec{r}(t) = (2 + 5t)\hat{i} + (3 - 6t)\hat{j}$$

Parametric equations corresponding to the vector equation are

$$x = 2 + 5t, \quad y = 3 - 6t$$

$$\frac{dx}{dt} = 5, \quad \frac{dy}{dt} = -6$$

$$\begin{aligned} \text{Arc-length} &= \int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_2^3 \sqrt{(5)^2 + (-6)^2} dt \\ &= \int_2^3 \sqrt{(25) + (36)} dt \\ &= \int_2^3 \sqrt{61} dt = \sqrt{61} \int_2^3 dt \\ &= \sqrt{61} \left| t \right|_2^3 = \sqrt{61}(3 - 2) = \sqrt{61} \end{aligned}$$

Q#4: Given the equations of two curves $y = x$ and $y = \sqrt{x}$. Find the intersecting points of these curves.

Solution:

$$x = \sqrt{x}$$

$$x^2 = x \quad \text{or} \quad x^2 - x = 0$$

$$x(x - 1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

$x = 0$ and $x = 1$ are the intersecting points.

Q#5: Given the equations of two curves $y = -6 - x^2$ and $y = 4 - 7x$. For which values of x these curves intersect each other?

Solution:

$$-6 - x^2 = 4 - 7x$$

$$x^2 - 7x + 10 = 0$$

$$x^2 - 5x - 2x + 10 = 0$$

$$x(x - 5) - 2(x - 5) = 0$$

$$(x - 2)(x - 5) = 0$$

$x = 2$ and $x = 5$ are the required intersecting points.

Q#6: Given the equations of two curves $y = 4 - x$ and $y = 4 - x^2$. For which values of x these curves intersect each other?

Solution:

$$4 - x = 4 - x^2$$

$$x^2 - x = 0$$

$$x(x - 1) = 0$$

$$x = 0 \text{ or } x - 1 = 0$$

$x = 0$ and $x = 1$ are the intersecting points.

Lecture#30

Q#1: Determine whether the following differential is exact or not.

$$dz = 4xy^3 dx + 6x^2y^2 dy$$

Solution:

$$P = 4xy^3$$

$$Q = 6x^2y^2$$

$$\frac{\partial P}{\partial y} = 12xy^2$$

$$\frac{\partial Q}{\partial x} = 12xy^2$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

So the given differential is exact.

Q#2: Determine whether the following differential is exact or not.

$$dz = (5x + 9xy) dx + 5x^3 dy$$

Solution:

Let $P = 5x + 9xy$ and $Q = 5x^3$

Then

$$\frac{\partial P}{\partial y} = 9x \quad \frac{\partial Q}{\partial x} = 15x^2$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ so given differential is not exact.

Q#3: Determine whether the following differential is exact? If so, find z.

$$dz = 3x(xy - 2) dx + (x^3 + 2y) dy$$

Solution:

Since $P = 3x(xy - 2)$ and $Q = (x^3 + 2y)$

Then

$$\frac{\partial P}{\partial y} = 3x^2 \quad \frac{\partial Q}{\partial x} = 3x^2$$

Since the given differential is exact, therefore, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Since dz is exact differential so

$$\begin{aligned} z &= \int P \, dx = \int 3x(xy - 2) \, dx = \int 3x^2y - 6x \, dx \\ &= x^3y - 3x + f(y) \text{----- (1)} \end{aligned}$$

$$\begin{aligned} z &= \int Q \, dy = \int (x^3 + 2y) \, dy \\ &= x^3y + y^2 + f(x) \text{----- (2)} \end{aligned}$$

Compare right hand side of both the expressions (1) and (2) we get

$$f(y) = y^2$$

and

$$f(x) = -3x$$

Thus

$$z = x^3y - 3x + y^2$$

Lecture#31

Q#1: Evaluate the line integral $\int_C y \, dx + x \, dy$ along $y^2 = 3x$ from the point (3, 4) to the point (1, 0).

Solution:

$$y^2 = 3x$$

$$x = \frac{y^2}{3}$$

$$dx = \frac{2}{3} y \, dy$$

Put values of C , x and dx in given line integral

$$\int_C y \, dx + x \, dy$$

$$\begin{aligned} \int_4^0 y \left(\frac{2}{3} y \, dy \right) + \frac{y^2}{3} dy &= \int_4^0 \frac{2}{3} y^2 \, dy + \frac{1}{3} y^2 \, dy \\ &= \int_4^0 \frac{1}{3} y^2 \, dy \\ &= \frac{-1}{3} \left| \frac{y^3}{3} \right|_4^0 \\ &= \frac{-1}{9} (0 - 64) \\ &= \frac{64}{9} \end{aligned}$$

Q#2: Evaluate the line integral $\int_C 3y^2 \, dy$ where C is the line segment from (2, 0) to (1, 5) .

Solution:

$$\begin{aligned} \int_C 3y^2 \, dy &= \int_0^5 3y^2 \, dy \\ &= \left| y^3 \right|_0^5 \\ &= 125 \end{aligned}$$

Q#3: Evaluate the line integral $\int_C 3x^2 y \, dx + xy \, dy$ along $y = x^3 + 1$ from the point (-1, 0) to the point (1, 2) .

Solution:

$$y = x^3 + 1$$

$$dy = 3x^2 dx$$

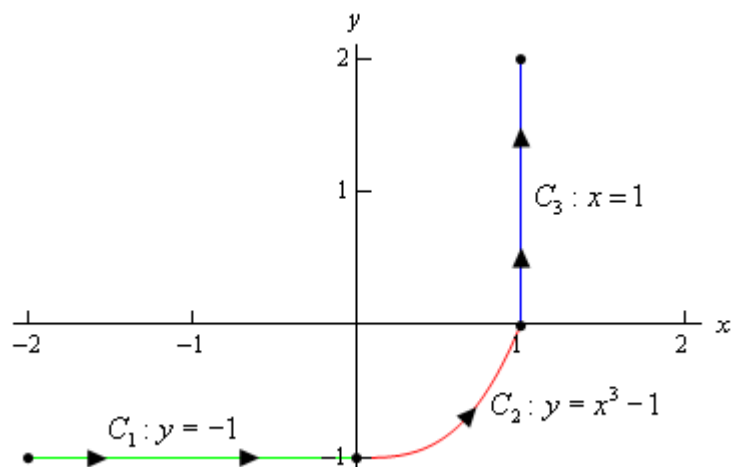
Put values of C, y and dy in given line integral

$$\int_C 3x^2 y dx + xy dy$$

$$\begin{aligned} \int_{-1}^1 3x^2 (x^3 + 1) dx + x(x^3 + 1)(3x^2) dx &= \int_{-1}^1 (3x^5 + 3x^2 + 3x^6 + 3x^3) dx \\ &= \left[\frac{x^6}{2} + x^3 + \frac{3}{7}x^7 + \frac{3}{4}x^4 \right]_{-1}^1 \\ &= \frac{1}{2} + 1 + \frac{3}{7} + \frac{3}{4} - \left(\frac{1}{2} + 1 + \frac{3}{7} - \frac{3}{4} \right) \\ &= 2 + \frac{6}{7} \\ &= \frac{14+6}{7} = \frac{20}{7} \end{aligned}$$

Lecture#32

Q1. Evaluate $\int_C 4x^3 ds$ where C is the curve shown below.



Solution:

So, first we need to parameterize each of the curves

$$C_1 : x = t, y = -1, \quad -2 \leq t \leq -1$$

$$C_2 : x = t, y = t^3 - 1, \quad -1 \leq t \leq 1$$

$$C_3 : x = 1, y = t, \quad 0 \leq t \leq 2$$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3 ds = \int_{-2}^{-1} 4t^3 \sqrt{(1)^2 + (0)^2} dt = \int_{-2}^{-1} 4t^3 dt = t^4 \Big|_{-2}^{-1} = -16$$

$$\begin{aligned}
 \int_{C_2} 4x^3 ds &= \int_0^1 4t^3 \sqrt{(1)^2 + (3t^2)^2} dt \\
 &= \int_0^1 4t^3 \sqrt{1+9t^4} dt \\
 &= \frac{1}{9} \left(\frac{2}{3} \right) (1+9t^4)^{\frac{3}{2}} \Big|_0^1 = \frac{2}{27} (10^{\frac{3}{2}} - 1) = 2.268
 \end{aligned}$$

$$\int_{C_3} 4x^3 ds = \int_0^2 4(1)^3 \sqrt{(0)^2 + (1)^2} dt = \int_0^2 4 dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\begin{aligned}
 \int_C 4x^3 ds &= \int_{C_1} 4x^3 ds + \int_{C_2} 4x^3 ds + \int_{C_3} 4x^3 ds \\
 &= -16 + 2.268 + 8 \\
 &= -5.732
 \end{aligned}$$

Answer (-5.732)

Q2. Evaluate the following line integral with respect to arc-lengths

$$\int_C 3xy^2z^2 ds, \text{ Where } C \text{ is the curve } x=t, \quad y=t^2, \quad z=\frac{2}{3}t^3 \text{ and } 0 \leq t \leq 1.$$

Solution:

$$x = t, \quad y = t^2, \quad z = \frac{2}{3}t^3$$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t, \quad \frac{dz}{dt} = 2t^2$$

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\ &= \sqrt{1 + 4t^2 + 4t^4} = \sqrt{(1 + 2t^2)^2} = 1 + 2t^2 \end{aligned}$$

$$3xy^2z^2 = 3t(t^2)^2\left(\frac{2}{3}t^3\right)^2 = \frac{4}{3}t^{11} \quad (\text{Putting values of } x, y, z)$$

$$\begin{aligned} \int_C (3xy^2z^2) ds &= \int_0^1 \left(\frac{4}{3}t^{11}\right) (1 + 2t^2) dt \\ &= \int_0^1 \left(\frac{4}{3}t^{11} + \frac{8}{3}t^{13}\right) dt \\ &= \frac{4}{3} \left| \frac{t^{12}}{12} \right|_0^1 + \frac{8}{3} \left| \frac{t^{14}}{14} \right|_0^1 \\ &= \frac{4}{3} \left(\frac{1}{12}\right) + \frac{8}{3} \left(\frac{1}{14}\right) = \frac{1}{9} + \frac{4}{21} = \frac{7+12}{63} = \frac{19}{63} \end{aligned}$$

Answer ($\frac{19}{63}$)

Q3.

Evaluate the following line integral which is independent of path.

$$\int_{(2,-2)}^{(-1,0)} (3x^2y^2) dx + (2x^3y) dy$$

Solution:

Let $dz = (3x^2y^2) dx + (2x^3y) dy$

As given line integral is independent of path so dz is an exact differential. z will be found as given below

$$P(x, y) = 3x^2y^2, \quad Q(x, y) = 2x^3y$$

$$\begin{aligned} z &= \int P dx = \int (3x^2y^2) dx \\ &= x^3y^2 + f(y) \quad \dots (1) \end{aligned}$$

$$\begin{aligned} z &= \int Q dy = \int (2x^3y) dy \\ &= x^3y^2 + f(x) \quad \dots (2) \end{aligned}$$

As there is no dissimilar term at the right side of equations (1) and (2), so

$$f(y) = 0 \quad \text{and} \quad f(x) = 0$$

$$\text{Thus } z = x^3y^2$$

So for $dz = (2xy^3) dx + (3y^2x^2) dy$, we have $z = x^3y^2$

Put this value in given integral, with limits $(2, -2)$ to $(-1, 0)$

$$\begin{aligned} \int_C (3x^2y^2) dx + (2x^3y) dy &= \int_{(2, -2)}^{(-1, 0)} d(x^3y^2) \\ &= \left[x^3y^2 \right]_{(2, -2)}^{(-1, 0)} \\ &= [8(4)] = 32 \end{aligned}$$

Answer (32)

Q4.

Evaluate the following line integral which is independent of path.

$$\int_{(0,0)}^{(1, \frac{\pi}{2})} (2x \sin y) dx - (x^2 \cos y) dy$$

Solution:

$$\text{Let } dz = (2x \sin y) dx - (x^2 \cos y) dy$$

Since given line integral is independent of path so dz is an exact differential.

z will be found as given below.

$$P(x, y) = 2x \sin y, \quad Q(x, y) = -x^2 \cos y$$

$$\begin{aligned} z &= \int P dx = \int 2x \sin y dx \\ &= x^2 \sin y + f(y) \text{ --- (1)} \end{aligned}$$

$$\begin{aligned} z &= \int Q dy = - \int x^2 \cos y dy \\ &= x^2 \sin y + f(x) \text{ --- (2)} \end{aligned}$$

As there is no dissimilar term at the right side of equations (1) and (2) so

$$f(y) = 0 \text{ and } f(x) = 0$$

$$\text{Thus } z = x^2 \sin y$$

So for $dz = (2x \sin y) dx - (x^2 \cos y) dy$, we have $z = x^2 \sin y$

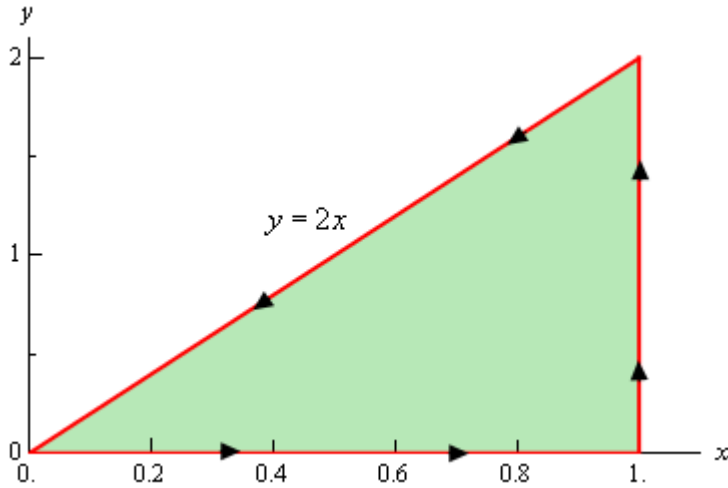
Put this value in given integral, with limits $(0, 0)$ to $\left(1, \frac{\pi}{2}\right)$

$$\begin{aligned} \int_C (2x \sin y) dx - (x^2 \cos y) dy &= \int_{(0,0)}^{\left(1, \frac{\pi}{2}\right)} d(x^2 \sin y) \\ &= \left[x^2 \sin y \right]_{(0,0)}^{\left(1, \frac{\pi}{2}\right)} \\ &= 1 \end{aligned}$$

Answer (1)

Lecture#33

Q1. Use Green's Theorem to evaluate $\oint_C xy dx + x^2 y^3 dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.



Solution:

So, the curve does satisfy the conditions of Green's Theorem and we can see that the following inequalities will define the region enclosed.

$$0 \leq x \leq 1 \qquad 0 \leq y \leq 2x$$

We can identify P and Q from the line integral. Here they are.

$$P = xy \qquad Q = x^2 y^3$$

So, using Green's Theorem the line integral becomes,

$$\begin{aligned}
\oint_C xy \, dx + x^2 y^3 \, dy &= \iint_D (2xy^3 - x) \, dA \\
&= \int_0^1 \int_0^{2x} (2xy^3 - x) \, dy \, dx \\
&= \int_0^1 \left(\frac{1}{2} xy^4 - xy \right) \bigg|_0^{2x} \, dx \\
&= \int_0^1 (8x^5 - 2x^2) \, dx \\
&= \left(\frac{4}{3} x^6 - \frac{2}{3} x^3 \right) \bigg|_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

Answer ($\frac{2}{3}$)

Q2.

Use Green's Theorem to evaluate $\oint_C (x^2 - 4xy) \, dx + (y^2 - xy) \, dy$ where C is a square with vertices at $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$

Solution:

Here

$$P = (x^2 - 4xy)$$

and

$$Q = (y^2 - xy)$$

$$\frac{\partial P}{\partial y} = -4x$$

$$\frac{\partial Q}{\partial x} = -y$$

Limits for x is from 0 to 2 and for y is 0 to 2 .

Thus by Green's Theorem

$$\begin{aligned}\oint_C (x^2 - 4xy) dx + (y^2 - xy) dy &= - \int_0^2 \int_0^2 (-4x - y) dy dx \\ &= \int_0^2 \int_0^2 (4x + y) dy dx \\ &= \int_0^2 \left[4xy + \frac{y^2}{2} \right]_0^2 dx \\ &= \int_0^2 (8x + 2) dx \\ &= \left[4x^2 + 2x \right]_0^2 = 16 + 4 = 20\end{aligned}$$

Answer (20)

Q3. Use Green's Theorem to evaluate $\oint_C y^3 dx - x^3 dy$ where C is the positively oriented circle of radius 2 centered at the origin.

Solution:

Let's first identify P and Q from the line integral.

$$P = y^3 \quad Q = -x^3$$

Now, using Green's theorem on the line integral gives,

$$\oint_C y^3 dx - x^3 dy = \iint_D -3x^2 - 3y^2 dA$$

Where D is a disk of radius 2 centered at the origin.

Since D is a disk it seems like the best way to do this integral is to use polar coordinates. Here is the evaluation of the integral.

$$\begin{aligned}
 \oint_C y^3 dx - x^3 dy &= -3 \iint_D (x^2 + y^2) dA \\
 &= -3 \int_0^{2\pi} \int_0^2 r^3 dr d\theta \\
 &= -3 \int_0^{2\pi} \left. \frac{1}{4} r^4 \right|_0^2 d\theta \\
 &= -3 \int_0^{2\pi} 4 d\theta \\
 &= -24\pi
 \end{aligned}$$

Answer (-24π)

Lecture#34

Q1. Find $\text{curl } \vec{F}$, if $\vec{F} = x^2 y \hat{i} + xyz \hat{j} - x^2 y^2 \hat{k}$.

Solution:

$$\begin{aligned}\vec{F} &= x^2 y \hat{i} + xyz \hat{j} - x^2 y^2 \hat{k} \\ \text{curl } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xyz & -x^2 y^2 \end{vmatrix} \\ &= \left(\frac{\partial(-x^2 y^2)}{\partial y} - \frac{\partial(xyz)}{\partial z} \right) \hat{i} + \left(\frac{\partial(x^2 y)}{\partial z} - \frac{\partial(-x^2 y^2)}{\partial x} \right) \hat{j} + \left(\frac{\partial(xyz)}{\partial x} - \frac{\partial(x^2 y)}{\partial y} \right) \hat{k} \\ &= (-2x^2 y - xy) \hat{i} - 2xy^2 \hat{j} + (yz - x^2) \hat{k}\end{aligned}$$

$$\text{Answer } ((-2x^2 y - xy) \hat{i} - 2xy^2 \hat{j} + (yz - x^2) \hat{k})$$

Q2. Find $\text{div } \vec{F}$, if $\vec{F} = yz^2 \hat{i} + xy \hat{j} + yz \hat{k}$

Solution:

$$\begin{aligned}\vec{F} &= yz^2 \hat{i} + xy \hat{j} + yz \hat{k} \\ \text{div } \vec{F} &= \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\vec{F} = yz^2 \hat{i} + xy \hat{j} + yz \hat{k} \right) \\ &= \frac{\partial(yz^2)}{\partial x} + \frac{\partial(xy)}{\partial y} + \frac{\partial(yz)}{\partial z} \\ &= 0 + x + y = x + y\end{aligned}$$

Answer ($x + y$)

Q3. If $A = x^3 y \hat{i} + y^2 z^3 \hat{j} - z^3 x^2 \hat{k}$ then find $\text{grad div } A$.

Solution:

$$\begin{aligned}A &= x^3 y \hat{i} + y^2 z^3 \hat{j} - z^3 x^2 \hat{k} \\ \text{div } A &= \nabla \cdot A = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x^3 y \hat{i} + y^2 z^3 \hat{j} - z^3 x^2 \hat{k}) \\ &= 3x^2 y + 2yz^3 - 3z^2 x^2 = \phi \text{ (say)} \\ \text{grad div } A &= \text{grad } \phi \\ &= \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \\ &= (6xy - 6xz^2) \hat{i} + (3x^2 + 2z^3) \hat{j} - (6yz^2 - 6zx^2) \hat{k} \\ &= 6x(y - z^2) \hat{i} + (3x^2 + 2z^3) \hat{j} - 6z(yz - x^2) \hat{k}\end{aligned}$$

Answer ($6x(y - z^2) \hat{i} + (3x^2 + 2z^3) \hat{j} - 6z(yz - x^2) \hat{k}$)

Q4: Given , $f(x,y)=\sin xy+\cos xy$ find $\text{div}(\nabla F)$.

Solution:

We are given that

$$f(x, y) = \sin xy + \cos xy$$

$$\begin{aligned}\nabla f &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} \right) (\sin xy + \cos xy) \\ &= (y \cos xy - y \sin xy) \hat{i} + (x \cos xy - x \sin xy) \hat{j} \\ \text{grad}(\nabla f) &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} \right) \cdot ((y \cos xy - y \sin xy) \hat{i} + (x \cos xy - x \sin xy) \hat{j}) \\ &= -y^2 \sin xy - y^2 \cos xy - x^2 \sin xy - x^2 \cos xy \\ &= -\sin xy(x^2 + y^2) - \cos xy(x^2 + y^2)\end{aligned}$$

$$\text{Answer } (-\sin xy(x^2 + y^2) - \cos xy(x^2 + y^2))$$

Question 1

Evaluate $\int x^3 \ln x dx$ by parts.

Ans

$$\frac{x^4}{4} \ln x - \frac{x^4}{16} + c$$

Question 2

Evaluate $\int x \cos x dx$ by parts.

Ans

$$x \sin x + \cos x + c$$

Question 3

Evaluate $\int_0^{\frac{\pi}{2}} \sin^{11} x dx$

$$\frac{10}{11} \cdot \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

Lecture#35

1. Use Wallis cosine formula to evaluate $\int_0^{\frac{\pi}{2}} (\cos^3 x + \cos^5 x) dx$.

Solution

$$\int_0^{\frac{\pi}{2}} (\cos^3 x + \cos^5 x) dx = \int_0^{\frac{\pi}{2}} \cos^3 x dx + \int_0^{\frac{\pi}{2}} \cos^5 x dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n} \text{ if } n \text{ is odd and } \geq 3$$

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x dx = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$$

$$\int_0^{\frac{\pi}{2}} (\cos^3 x + \cos^5 x) dx = \frac{2}{3} + \frac{8}{15} = \frac{18}{15}$$

2. Use Wallis sine formula to evaluate $\int_0^{\frac{\pi}{2}} (\sin^4 x + \sin^6 x) dx$.

Solution:

$$\int_0^{\frac{\pi}{2}} (\sin^4 x + \sin^6 x) dx = \int_0^{\frac{\pi}{2}} \sin^4 x dx + \int_0^{\frac{\pi}{2}} \sin^6 x dx$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{\pi}{2} \frac{1.3.5 \dots (n-1)}{2.4.6 \dots n} \text{ if } n \text{ is even and } \geq 2$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{\pi}{2} \left(\frac{3}{8} \right) = \frac{3\pi}{16}$$

$$\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{\pi}{2} \left(\frac{3 \times 5}{2 \times 4 \times 6} \right) = \frac{15\pi}{96}$$

$$\int_0^{\frac{\pi}{2}} (\sin^4 x + \sin^6 x) dx = \frac{3\pi}{16} + \frac{15\pi}{96} = \frac{33\pi}{96}$$

Lecture#36

1. Evaluate the line integral $\int_c \vec{F} \cdot d\vec{r}$ where $F(x, y, z) = xy\hat{i} + x^2z\hat{j}$ and c is given by the curve $r(t) = t\hat{i} + 2t\hat{j} + (1-t)\hat{k}$, $0 \leq t \leq 1$.

Solution:

We are given that $F(x, y, z) = xy\hat{i} + x^2z\hat{j}$ and $r(t) = t\hat{i} + 2t\hat{j} + (1-t)\hat{k}$.

This implies

$$d\vec{r} = (\hat{i} + 2\hat{j} - \hat{k})dt$$

Also,

$$F(r(t)) = (2t^2)\hat{i} + t^2(1-t)\hat{j}$$

$$= 2t^2\hat{i} + (t^2 - t^3)\hat{j}$$

$$\vec{F} \cdot d\vec{r} = (2t^2\hat{i} + (t^2 - t^3)\hat{j}) \cdot (\hat{i} + 2\hat{j} - \hat{k}) dt$$

$$\int_0^1 \vec{F} \cdot d\vec{r} = \int_0^1 (2t^2 + 2t^2 - 2t^3) dt = \int_0^1 (4t^2 - 2t^3) dt$$

$$= \left(\frac{4}{3}t^3 - \frac{1}{2}t^4 \right) \Big|_0^1 = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$$

which is the required result.

2. Evaluate the line integral $\int_c \vec{F} \cdot d\vec{r}$ where $F(x, y, z) = yz\hat{i} + 3xyz\hat{j} + 2x^2y\hat{k}$ and c is the curve given by $r(t) = \hat{i} + t\hat{j} + t^2\hat{k}$, $0 \leq t \leq 1$.

Solution:

We are given that $F(x, y, z) = yz\hat{i} + 3xyz\hat{j} + 2x^2y\hat{k}$ and $r(t) = \hat{i} + t\hat{j} + t^2\hat{k}$.
 $d\vec{r} = (\hat{j} + 2t\hat{k})dt$.

Also,

$$\vec{F}(r(t)) = t^3\hat{i} + 3t^3\hat{j} + 2t\hat{k}$$

$$\vec{F} \cdot d\vec{r} = (t^3\hat{i} + 3t^3\hat{j} + 2t\hat{k}) \cdot (\hat{j} + 2t\hat{k})$$

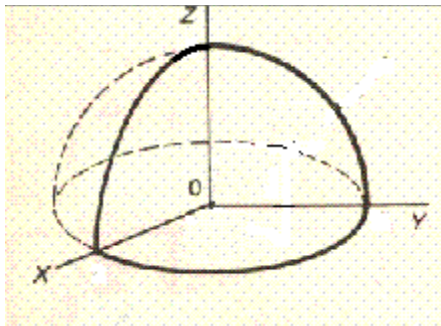
$$\int_0^1 \vec{F} \cdot d\vec{r} = \int_0^1 (3t^3 + 4t^2) dt$$

$$= \left(\frac{3t^4}{4} + \frac{4t^3}{3} \right) \Big|_0^1 = \frac{3}{4} + \frac{4}{3} = \frac{25}{12}$$

Lecture#37

Consider the vector field $\vec{F} = 3xz\hat{i} + 2y\hat{j}$ over the surface S which is the portion of the sphere $x^2 + y^2 + z^2 = 9$ bounded by the planes $x=0$, $y=0$, $z=0$ (as shown in

the figure below). If $\hat{n} = \frac{x}{3}\hat{i} + \frac{y}{3}\hat{j} + \frac{z}{3}\hat{k}$ is a unit upper normal to S . express $\int_S \vec{F} \cdot \hat{n} dS$ in spherical coordinates.



Note: Do not evaluate the integral.

Solution:

Put values of $\vec{F}$ and $\hat{n}$ in given integral

$$\begin{aligned}\int_S \vec{F} \cdot \hat{n} \, dS &= \int_S (3xz \hat{i} + 2y \hat{j}) \cdot \left(\frac{x}{3} \hat{i} + \frac{y}{3} \hat{j} + \frac{z}{3} \hat{k} \right) dS \\ &= \int_S \left(x^2 z + \frac{2}{3} y^2 \right) dS\end{aligned}$$

To convert it to spherical polar coordinates, we have

$$x = 3 \sin \phi \cos \theta$$

$$y = 3 \sin \phi \sin \theta$$

$$z = 3 \cos \phi$$

$$dS = 9 \sin \phi \, d\phi \, d\theta$$

Limits are:

$$\phi \text{ ranges from } 0 \text{ to } \frac{\pi}{2}$$

$$\theta \text{ ranges from } 0 \text{ to } \frac{\pi}{2}$$

Put these values

Practice Questions

Lecture#38

Q1. Determine whether the following vector field $\vec{F}$ is conservative or not.

$$a) \vec{F}(x, y, z) = (2z^4 - y^3) \hat{i} + (z - 3xy^2) \hat{j} + (y + 8xz^3) \hat{k}$$

$$b) \vec{F}(x, y, z) = -2x \cos z \hat{i} + y^3 \hat{j} + x^2 \sin z \hat{k}$$

Q2. Let S is the surface bounded by the plane $x=0$ $x=1$, $y=0$ $y=2$, $z=0$ $z=3$. If

$\vec{F} = (2x^2y + z)\hat{i} + xz^2\hat{j} + (e^y - yz)\hat{k}$, use Divergence Theorem to find $\int_S \vec{F} \cdot \hat{n} \, dS$, where $\hat{n}$ denote the unit outer normal to S .

Q3. Use Stokes' Theorem to evaluate the integral $\int_S \text{curl } \vec{F} \cdot \hat{n} \, dS$

If $\vec{F} = z^2\hat{i} - 3xy\hat{j} + x^3y^3\hat{k}$ and surface S is the portion of paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that S is oriented upward.

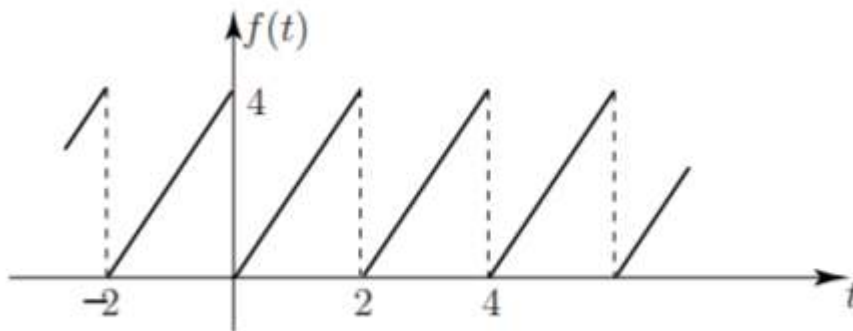
Lecture#39

Q1. Write amplitude, period and frequency of the following:

a) $y = 2\sin 3x$

- b) $y = 3\sin \frac{x}{2}$
 c) $y = 5\sin 8x$

Q2. Define the periodic function whose graph is shown below.



Q3. Sketch the graph of the following periodic functions showing all relevant

values:
$$f(t) = \begin{cases} t^2 / 2 & 0 < t < 4 \\ 8 & 4 < t < 6 \\ 0 & 6 < t < 8 \end{cases}$$

Q4. Evaluate

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad (n \neq m)$$

Lecture#40

Q1. Determine the Fourier co-efficient a_0 of the following function.

$$f(x) = \frac{x}{2} \quad 0 < x < 2\pi$$

Q2. Determine the Fourier co-efficient a_n of the following function.

$$f(x) = \frac{x}{2} \quad 0 < x < 2\pi$$

Q3. Determine the Fourier co-efficient b_n of the following function.

$$f(x) = \frac{x}{2} \quad 0 < x < 2\pi$$

Q4. Determine the Fourier co-efficient a_0 , a_n and b_n of the following function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases} \quad \text{and has a period } 2\pi$$

$$\int_S \vec{F} \cdot \hat{n} \, dS = \int_S \left(x^2 z + \frac{2}{3} y^2 \right) dS$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left((3 \sin \phi \cos \theta)^2 (3 \cos \phi) + \frac{2}{3} (3 \sin \phi \sin \theta)^2 \right) 9 \sin \phi \, d\phi \, d\theta$$

Practice Questions Lecture#41-43

Q1. Check whether the given function is even, odd or neither.

$$f(x) = x^2 e^x.$$

Solution

$$\text{Let } g(x) = (x^2)$$

$$h(x) = e^x.$$

$$\text{As } g(-x) = (-x)^2$$

$$= x^2$$

$$= g(x)$$

So $g(x)$ is an even function.

$h(x)$ is neither odd nor even function. Since product of even function and

"neither even nor odd function" is "neither even nor odd function"

so $f(x)$ is neither even nor odd function.

Q2. Check whether the given function is even, odd or neither.

$$f(x) = \cos^2 x \sin 5x$$

Solution

$$\text{Let } g(x) = \cos^2 x$$

$$h(x) = \sin 5x$$

$g(x)$ is an even function as

$$g(-x) = (\cos(-x))^2$$

$$= (\cos x)^2$$

$$= \cos^2 x = g(x)$$

$h(x)$ is also an odd function as

$$h(-x) = \sin 5(-x)$$

$$= \sin(-5x)$$

$$= -\sin 5x = -h(x)$$

Since product of an even and an odd function is an odd function

so $f(x)$ is an odd function.

Q3. Consider a periodic function defined by

$$f(x) = x; \quad -\pi \leq x \leq \pi$$

Determine Fourier Co-efficient a_0 , a_n and b_n .

Solution

$$f(x) = x; \quad -\pi \leq x \leq \pi$$

Since $f(-x) = -x = -f(x)$ so f is an odd function.

If $f(x)$ is an odd function defined over the interval $-\pi < x < \pi$, then the Fourier series for $f(x)$ contains sine terms only. So a_0 and a_n are zero.

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\
 &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx \\
 &= \frac{2}{\pi} \left(\left[-\frac{x \cos nx}{n} \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right) \\
 &= \frac{2}{n\pi} \left[-\pi \cos n\pi + \frac{1}{n} \sin nx \Big|_0^{\pi} \right] \\
 &= \frac{-2 \cos n\pi}{n} + 0 = \frac{-2 \cos n\pi}{n} \\
 &= \frac{-2(-1)^n}{n}
 \end{aligned}$$

Q4. Determine the Fourier coefficient a_0 for a periodic function $f(t)$ of period 4 defined by

$$f(t) = \begin{cases} (1+t) & -2 < t < 0 \\ 0 & 0 < t < 2 \end{cases}$$

Solution

Here the given function is of period 4, so $T = 4$ and

$$\begin{aligned}
 a_0 &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \, dt = \frac{2}{4} \int_{-2}^2 f(t) \, dt = \frac{1}{2} \left[\int_{-2}^0 f(t) \, dt + \int_0^2 f(t) \, dt \right] \\
 &= \frac{1}{2} \left[\int_{-2}^0 (1+t) \, dt + \int_0^2 (0) \, dt \right] = \frac{1}{2} \left(t + \frac{t^2}{2} \right)_{-2}^0 = \frac{1}{2} (2 - 2) = 0
 \end{aligned}$$

Q5. Determine the Fourier co-efficient b_n of the periodic function

$$f(x) = 2x + 2, 0 < x < 2$$

Solution

Here period = $T = 2$

$$\omega = \frac{2\pi}{T} = \pi$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin n\omega x \, dx$$

$$b_n = \int_0^2 (2x + 2) \sin \pi n x \, dx$$

$$= \left[(2x + 2) \frac{(-\cos \pi n x)}{\pi n} - 2 \int \frac{(-\cos \pi n x)}{\pi n} dx \right]_0^2$$

$$= \left[- (2x + 2) \frac{\cos \pi n x}{\pi n} + \frac{2(\sin \pi n x)}{\pi^2 n^2} \right]_0^2$$

$$= \left[- \frac{6}{\pi n} + \frac{2}{\pi n} \right] = \frac{-4}{\pi n}$$

Q6. Determine the Fourier co-efficient b_n of the following periodic function.

$$f(x) = x + 1, 0 < x < 1$$

Solution

Here period= $T=1$

$$\omega = \frac{2\pi}{T} = 2\pi$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin n\omega x \, dx$$

$$b_n = 2 \int_0^1 (x+1) \sin 2\pi n x \, dx$$

$$= 2 \left[(x+1) \frac{(-\cos 2\pi n x)}{2\pi n} - \int \frac{(-\cos 2\pi n x)}{2\pi n} dx \right]_0^1$$

$$= 2 \left[-(x+1) \frac{\cos 2\pi n x}{2\pi n} + \frac{(\sin 2\pi n x)}{4\pi^2 n^2} \right]_0^1$$

$$= 2 \left(\frac{-2}{2\pi n} + \frac{1}{2\pi n} \right) = \frac{-1}{\pi n}$$

Practice Questions with Solution

Lecture#44 &45

Q1. Find Laplace transform of the functions $F(t)$ if

a) $F(t) = e^{5t}$

Solution:

$$F(t) = e^{5t}$$

$$\text{As } L\{e^{at}\} = \frac{1}{s-a} \text{ where } s \text{ and } a \text{ are constants}$$

$$L\{e^{5t}\} = \frac{1}{s-5}$$

b) $F(t) = t^5$

Solution:

$$F(t) = t^5$$

$$\text{As } L\{t^n\} = \frac{n!}{s^{n+1}} \text{ where } s \text{ is a constant}$$

$$L\{t^5\} = \frac{5!}{s^{5+1}} = \frac{5 \times 4 \times 3 \times 2 \times 1}{s^6} = \frac{120}{s^6}$$

c) $F(t) = \sin 5t$

Solution:

$$F(t) = \sin 5t$$

$$\text{As } L\{\sin at\} = \frac{a}{s^2 + a^2} \text{ where } s \text{ and } a \text{ are constants}$$

$$L\{\sin 5t\} = \frac{5}{s^2 + 25}$$

d) $F(t) = e^{2t} \cos 3t$

Solution:

$$F(t) = e^{2t} \cos 3t$$

$$\text{As } L\{e^{at} \cos bt\} = \frac{s - a}{(s - a)^2 + b^2} \text{ where } s, a \text{ and } b \text{ are constants}$$

$$L\{e^{2t} \cos 3t\} = \frac{s - 2}{(s - 2)^2 + 9}$$

e) $F(t) = t^3 + 3t - 2$

Solution:

$$F(t) = t^3 + 3t - 2$$

$$\text{As } L\{t^3 + 3t - 2\} = L\{t^3\} + 3L\{t\} - 2L\{1\} = \frac{3!}{s^4} + 3\frac{1}{s^2} - 2\frac{1}{s}$$

Q2. Show that Laplace transform of the function

$$f(t) = t \text{ is } \frac{1}{s^2} \text{ where } s \text{ is a constant for the integration and } s > 0.$$

Solution:

As we know

$$L[f(t)] = \int_0^{+\infty} e^{-st} f(t) dt$$

So for $f(t) = t$

$$\begin{aligned} L[t] &= \int_0^{+\infty} e^{-st} t dt \\ &= \lim_{l \rightarrow +\infty} \int_0^l e^{-st} t dt \\ &= \lim_{l \rightarrow +\infty} \left\{ \left[t \frac{e^{-st}}{s} \right]_0^l + \int_0^l 1 \cdot \frac{e^{-st}}{s} dt \right\} \\ &= \lim_{l \rightarrow +\infty} \left\{ \left[t \frac{e^{-st}}{s} \right]_0^l - \left[\frac{e^{-st}}{s^2} \right]_0^l \right\} \end{aligned}$$

In order for this limit to exist, we again must insist that $s \neq 0$ and that $s > 0$ so that e^{-sl} has a limit (of zero). We obtain

$$\begin{aligned} &= -\frac{1}{s} \lim_{l \rightarrow +\infty} (l e^{-sl} - 0) - \frac{1}{s^2} \lim_{l \rightarrow +\infty} (e^{-sl} - 1) \\ &= \frac{1}{s^2} ; \quad s > 0 \end{aligned}$$

Q3. Find Laplace transform of the function $F(t)$ if

$$F(t) = e^{2t} \cos t$$

Solution:

Since

$$L\{\cos t\} = \frac{s}{s^2 + 1}$$

So by First Shift Theorem

$$\begin{aligned} L\{e^{-(2t)} \cos 2t\} &= \frac{s - 2}{(s - 2)^2 + 1} \\ &= \frac{s - 2}{(s^2 + 4 - 4s) + 1} \\ &= \frac{s - 2}{s^2 - 4s + 5} \end{aligned}$$

Q4. Determined:

$$a) \quad L^{-1} \left\{ \frac{3}{s^2 + 4} \right\}$$

Solution:

$$L^{-1} \left\{ \frac{3}{s^2 + 4} \right\} = L^{-1} \left\{ \frac{3}{2} \frac{3}{s^2 + 2^2} \right\} = \frac{3}{2} L^{-1} \left\{ \frac{3}{2} \frac{3}{s^2 + 2^2} \right\} = \frac{3}{2} \sin 2t$$

$$b) \quad L^{-1} \left\{ \frac{2}{(s+5)^4} \right\}$$

Solution:

$$L^{-1} \left\{ \frac{2}{(s+5)^4} \right\} = L^{-1} \left\{ \frac{1}{3} \frac{6}{(s+5)^4} \right\} = \frac{1}{3} L^{-1} \left\{ \frac{3!}{(s+5)^4} \right\} = \frac{1}{3} e^{-5t} L^{-1} \left\{ \frac{3!}{s^4} \right\} = \frac{1}{3} e^{-5t} t^3$$

Q5. Find Laplace transform of the function $F(t)$ if

$$F(t) = e^{3t} \sin 4t$$

Solution:

Since

$$L\{\sin 4t\} = \frac{4}{s^2 + 16}$$

So by First Shift Theorem

$$\begin{aligned} L\{e^{-(3t)} \sin 4t\} &= \frac{4}{(s - 3)^2 + 16} \\ &= \frac{4}{(s^2 + 9 - 6s) + 16} \\ &= \frac{4}{s^2 - 6s + 25} \end{aligned}$$