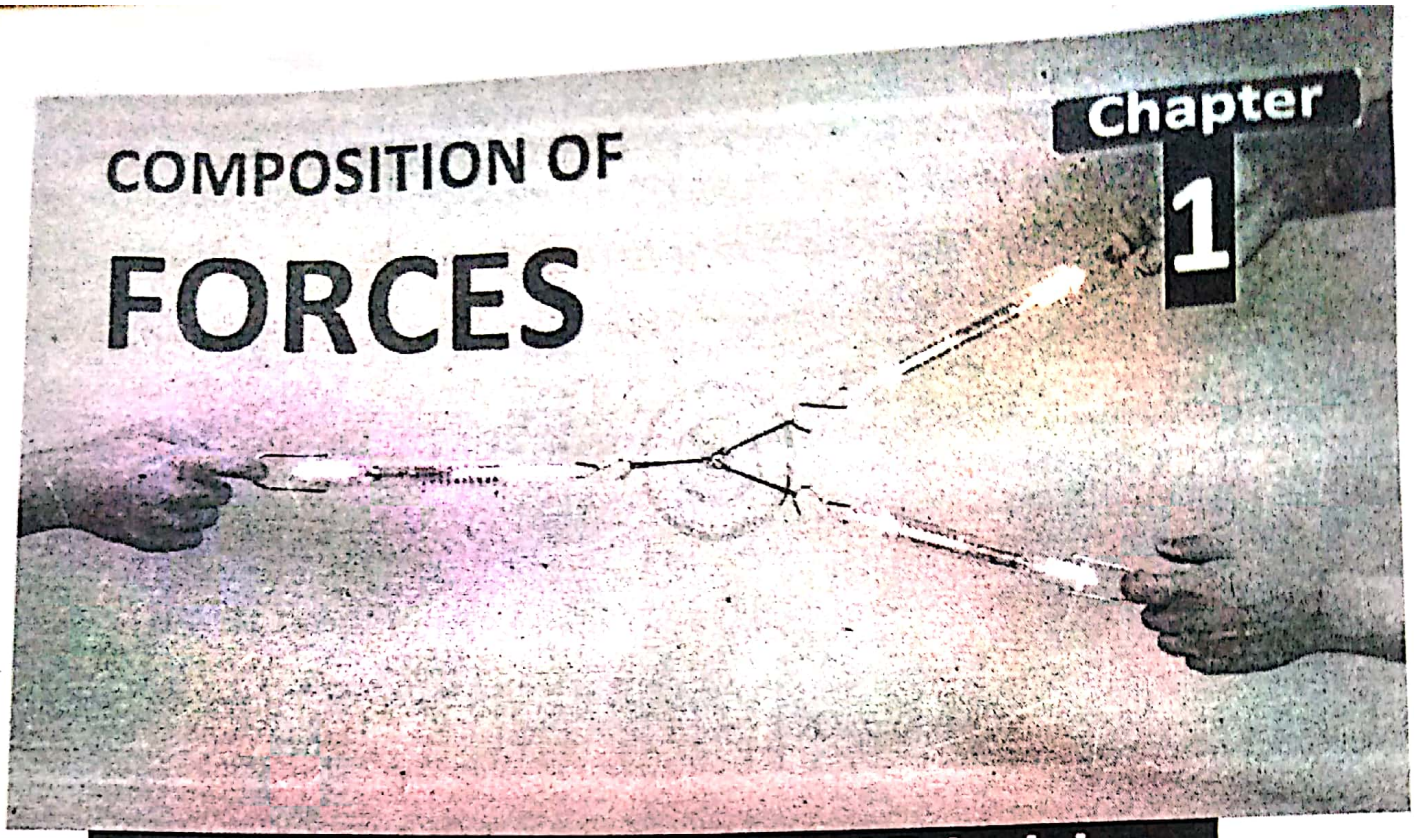


1-1 Fundamental Concepts and Principles

In this section, we shall discuss fundamental concepts, definitions, principles and laws. In fact, each definition of an entity involves other entities; therefore certain particular entities must remain, undefined; these are the *fundamental concepts.* Similarly, the proof of each proposition uses other propositions; therefore certain particular propositions must remain unproved; these are the *fundamental principles* or *axioms.*

Fundamental Concepts: *Space, time* and *matter* are the fundamental concepts.

Body: Matter which occupies a limited portion of space is called a *body.*

Mass: The quantity of matter in a body is called its *mass.*

Particle: A body of negligibly small dimensions is termed a *particle.* The concept of a particle is an ideal one and represents a geometrical point which possesses mass. A body may be regarded as an aggregate of particles.

Rigid body: A body is said to be a *rigid body* if the distance between any two particles of the body remains unchanged so that that the body does not expand or contract or change its shape in any way. The concept of a rigid body like that of a particle is also an ideal one and actual bodies are not rigid in the true sense of the word. However, such idealisations are very convenient for constructing a mathematical theory to describe physical phenomena. In this book whenever we speak of a body, a rigid body will be implied unless otherwise stated.

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Rest and Motion: A body is said to be in motion if, in the course of time, its position changes with respect to its surroundings, otherwise the body is said to be at rest. Rest and motion of a body are relative concepts and are meaningful only when referred to surrounding bodies.

Reference System: In order to locate a moving body (or particle) we select a coordinate system which is fixed in the rigid body with respect to which the motion is being considered. This coordinate system is called a *frame of reference* or *reference system*.

It is implied that some time measuring device is also present in the reference system. We usually employ a rectangular Cartesian coordinate system $Oxyz$ (Fig.1.1) as a reference system in which the position of any particle P of the body is indicated by its position vector $\vec{r}$, or by the components x,y,z of $\vec{r}$, i.e., by the coordinates of the particle P. If the particle P moves in a plane (Fig.1.2), only x, y coordinates are needed.

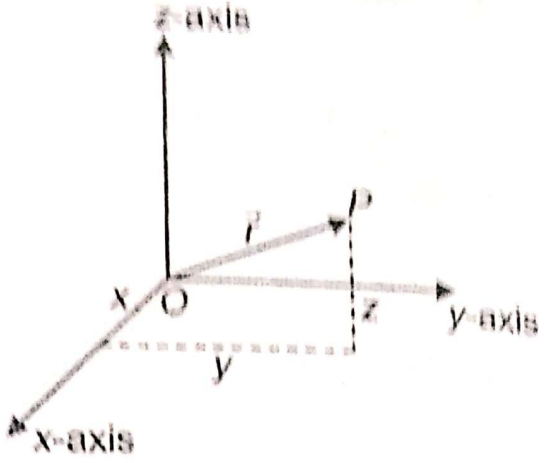


Fig.1.1

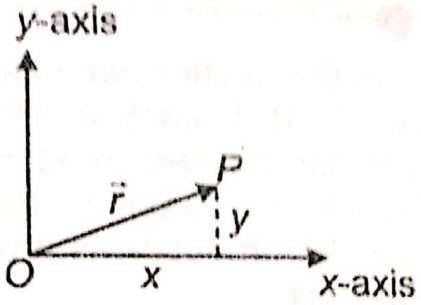


Fig.1.2

When the position vector $\vec{r}$ of every particle of a body remains constant, the body is at rest relative to the given frame of reference.

On the other hand, if the position vector $\vec{r}$ of any point of the body changes with the time t, the body is in motion relative to the reference system (or relative to the body in which the reference system is fixed). Thus, for a moving body the position vector $\vec{r}$ of any particle P depends on the time t and we write

$$\vec{r} = \vec{r}(t)$$

Velocity Vector: If $\vec{r} = \vec{r}(t)$ is the position vector of the particle P, then the *velocity vector of P* is the vector $\frac{d\vec{r}}{dt}$ or $\dot{\vec{r}}$, where the dot denotes differentiation with respect to t.

Acceleration Vector: If $\vec{r} = \vec{r}(t)$ is the position vector of the particle P, then the *acceleration vector of P* is the vector $\frac{d^2\vec{r}}{dt^2}$ or $\ddot{\vec{r}}$, where the dots denote differentiations with respect to t.

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Speed of Particle: The magnitude of the velocity vector is called the speed of the particle.

Force: *Force* is defined as a measure of push or pull on an object.
In other words, we define *force* as that which changes or tends to change the state of rest or motion of a body.

The state of rest or motion of a given body implies the existence of other bodies. We assume that it is the influence of other bodies that causes the given body to be at rest or in motion. We describe this process by saying that bodies are exerting forces on one another. In general, forces are exerted whenever a body comes into contact with other bodies. In some situations forces may act at a distance, as in the case of attraction which the earth has for everybody or electric and magnetic forces.

A force is a vector quantity. It is characterised by its *magnitude*, its *direction*, and its *point of application*.

Point of Application of a Force: The point of body where the force acts is called its *point of application*. In Fig.1.3, A is the point of application of the force $\vec{F}$, because the force $\vec{F}$ acts at A.

Line of Action of a Force: The line passing through the point of application in the direction of the force is called its *line of action*. In Fig.1.3, the doted line is the line of action of the force as it passes through the point of application in the direction of the force.

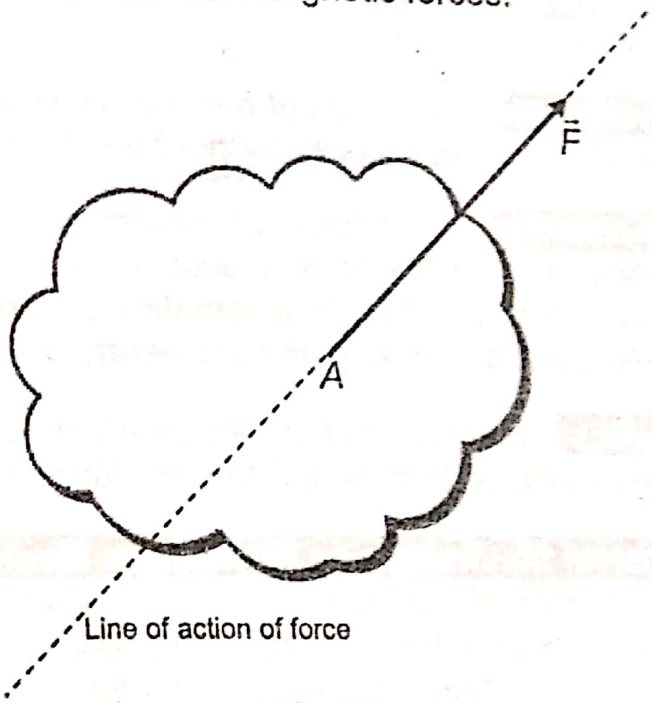


Fig.1.3

Force System: A set of forces acting on a particle or rigid body is called a *force system* or *system of forces*.

In Fig.1.4, four forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \vec{F}_4$ act on a rigid body, this set of forces is a force system.

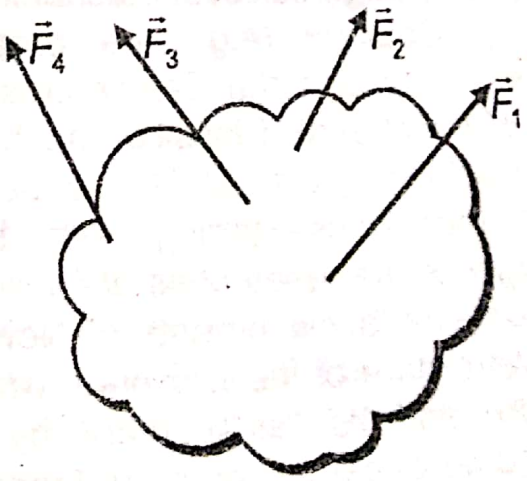


Fig.1.4

Equivalent Force Systems: Two force systems are said to be *equivalent* if one system can be replaced by the other system without disturbing the instantaneous state of rest or motion of a body.

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Resultant of a Force System: If a force system is equivalent to a single force $\vec{R}$, then it is called the *resultant* of the system. Thus, the resultant is a single force whose effect on a body or a particle is the same as the combined effect of all the forces of the system.

Mechanics: *Mechanics* is the branch of science concerned with the behaviour of physical bodies when subjected to forces or displacements, and the subsequent effects of the bodies on their environment.

In other words, the science of *mechanics* deals with the action of forces on bodies which we encounter in nature and technology. It is divided into *statics* and *dynamics.*

Statics: The branch of mechanics which deals with bodies at rest is known as *statics.*

Dynamics: The branch of mechanics which deals with bodies in motion is known as *dynamics.* Dynamics can further be subdivided into *kinematics* and *kinetics.*

Kinematics: The branch of dynamics which treats of the geometry of motion of bodies, without taking into account their masses or the forces acting on them is known as *kinematics.* In kinematics we are interested in calculating such quantities as the position, velocity and acceleration vectors of a particle.

Kinetics: The branch of dynamics which is concerned with the relations between the forces acting on bodies and the resulting motion is known as *kinetics.*

Classical Mechanics or Newtonian Mechanics: *Classical mechanics* is the study of the *motion* of bodies (including the special case in which bodies remain at rest) in accordance with the general principles first enunciated by Sir Isaac Newton in his *Philosophiae Naturalis Principia Mathematica* (1687), commonly known as the *Principia.* Classical mechanics was the first branch of Physics to be discovered, and is the foundation upon which all other branches of Physics are built. Moreover, classical mechanics has many important applications in other areas of science, such as Astronomy (e.g., celestial mechanics), Chemistry (e.g., the dynamics of molecular collisions), Geology (e.g., the propagation of seismic waves, generated by earthquakes, through the Earth's crust), and Engineering (e.g., the equilibrium and stability of structures). Classical mechanics is also of great significance outside the realm of science. After all, the sequence of events leading to the discovery of classical mechanics--starting with the ground-breaking work of Copernicus, continuing with the researches of Galileo, Kepler, and Descartes, and culminating in the monumental achievements of Newton--involved the complete overthrow of the Aristotelian picture of the Universe, which had previously prevailed for more than a millennium, and its replacement by a recognizably modern picture in which humankind no longer played a privileged role.

Classical mechanics is also known as *Newtonian mechanics* because it is based on Newton's laws of motion. In Newtonian mechanics, space, time and mass are

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absolute concepts, independent of each other. However, the concept of force is not independent of the other three.

Newton's First Law of Motion:

Newton's First Law of Motion is stated as:

> An object at rest tends to stay at rest and an object in motion tends to stay in motion with the same speed and in the same direction unless acted upon by an unbalanced external force.

Inertia is a property of matter that resists changes in motion. If a mass is not moving, it will stay that way until an unbalanced external force starts to move it; if a mass is in motion, it will stay in motion with the same speed and direction until an unbalanced external force changes its motion characteristics (friction could slow it down, or a force could accelerate its motion).

For example, let us consider a hockey puck on the ice (assume the ice is perfectly level and frictionless). If the puck is placed down on the ice, it will stay motionless until someone hits it with a stick or skate because of its inertia. Also due to inertia, when slapped, the puck will tend to move in a straight line with constant speed until an external force (such as another player, or the goalie, or the net) changes its motion.

As a second example of Newton's First Law, consider a car accelerating from a stoplight. As the car accelerates from zero motion, your body tends to push back into the seat due to its inertia (trying to remain at rest). Also, as the car is braked from a high speed back to stopping, your body is flung forward due to its inertia in motion. Hopefully you have your seatbelt on, or else Newton's First Law could have bad consequences.

Linear Momentum: The *linear momentum* of a particle is the vector $\vec{p} = m\vec{v}$, where m is the mass of the particle and v its velocity relative to a reference system.

Newton's Second Law of Motion:

Newton's Second Law of Motion is stated as:

> The time rate of change of linear momentum of a particle is equal to the force producing it, and takes place in the direction in which the force acts.

If a particle of mass m, subject to a force $\vec{F}$, moves with velocity $\vec{v}$ at time t, the second law, in mathematical form, states that

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \qquad \ldots(1.1)$$

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If the mass of the particle is constant, the last equation gives

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

or

$$\vec{F} = m\vec{a} \qquad \ldots(1.2)$$

where $\vec{a}$ is the acceleration vector. The vector equation (1.1) or (1.2) is called the *equation of motion.*

Forces In Equilibrium: The forces acting on a particle are said to be in *equilibrium* if these do not change the state of rest or character of motion of the particle. In this case the state of the particle is referred to as an *equilibrium state.*

In the equilibrium state of particle, the rate of change of its velocity is zero, i.e. its acceleration is zero.

Inertial or Newtonian Reference System: A reference system relative to which equation of motion $\vec{F} = m\vec{a}$ holds is called an *inertial* or *Newtonian system.*

Newton's Third Law of Motion:

Newton's Third Law of Motion is stated as:

> For every action, there is an equal and opposite reaction.

This law is familiar in everyday situations; a force cannot be applied to an object unless something resists the reaction of that force.

In order to walk across the floor, you must push back on the floor with your foot; then, according to Newton's Third Law, the floor pushes forward on your foot, which propels you forward. This, of course, requires friction to work.

If a free-floating astronaut were to throw a baseball, there is nothing to resist the throwing force, so as the baseball accelerates in the direction of throwing, the astronaut would accelerate backwards, with a force equal and opposite to the throwing force. The astronaut would accelerate at a much smaller level (by Newton's Second Law) since her mass is much greater than the baseball's mass.

The recoil of a gun during firing is another example of Newton's Third Law.

Parallelogram Law: This law sates as:

> If two forces act on a body at a point *A* along the adjacent sides of a parallelogram, then their resultant is the diagonal of the parallelogram with *A* as initial point.

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Let the forces $\vec{P}$ and $\vec{Q}$ act at a point A of the body along the adjacent sides AB and AC respectively as shown in the Fig.1.5.

According to the parallelogram law of forces, the resultant of forces $\vec{P}$ and $\vec{Q}$ is along the diagonal AD of the parallelogram. In fact, $\vec{AD}$ is the resultant of forces $\vec{P}$ and $\vec{Q}$, i.e. $\vec{AD} = \vec{P} + \vec{Q}$.

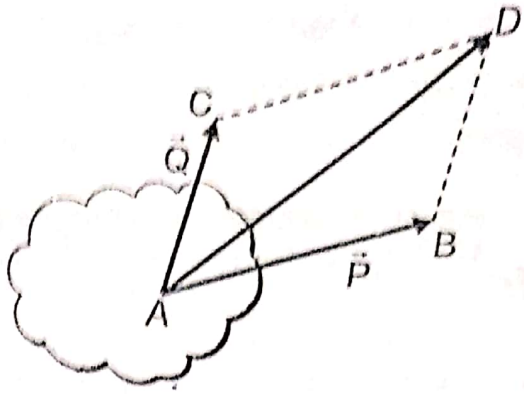


Fig.1.5

Equilibrium Law: This law sates as:

> Two forces can be in equilibrium if and only if they are equal in magnitude, opposite in direction, and act in the same straight line.

In Fig.1.6, the forces $\vec{F}_1$ and $\vec{F}_2$, acting, at points A and B of a body have same line of action, equal magnitude and opposite directions, so $\vec{F}_1$ and $\vec{F}_2$ are in equilibrium.

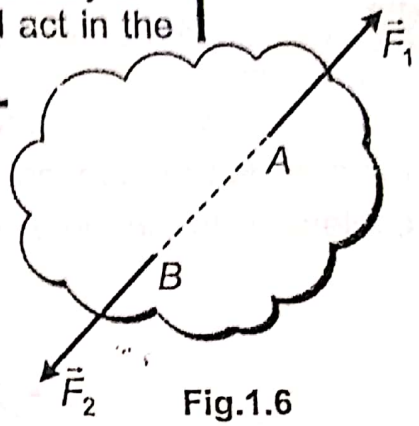


Fig.1.6

Law of Superposition: This law states as:

> The action of a given system of forces on a body remains unchanged if we add to or subtract from them another system of forces in equilibrium.

Principle of Transmissibility of Forces: This law sates as:

> The point of application of a force, acting on a rigid body can be transferred to any other point of the body lying on the line of action of the force without changing its effect.

Let a force $\vec{F}$ act at a point A of a rigid body (Fig.1.7). Take any point B on the line of action of the force and apply at B two equal and opposite forces $\vec{F}_1$ and $\vec{F}_2$ such that $\vec{F}_1 = \vec{F}$ and $\vec{F}_2 = -\vec{F}$. These forces are in equilibrium, so these forces will have no effect on the body. From the equilibrium law it follows that the forces $\vec{F}$ and $\vec{F}_2$ are in equilibrium and can be removed.

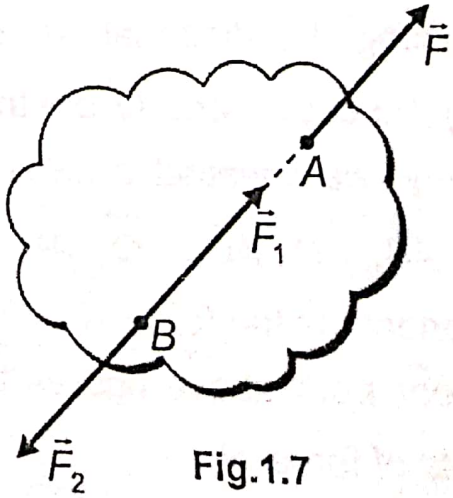


Fig.1.7

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We are then left with the force $\vec{F}_1$, equal to $\vec{F}$ in magnitude and direction, with the point of application transferred to the point B. The effect of the force $\vec{F}_1$ at B is the same as that of the original force $\vec{F}$ at A.

1-2 Components and Resolved Parts of a Force

In this section, we shall be concerned with the decomposition of a force into two forces whose effect will be same as that of the original force.

Components of a Force: If $\vec{R}$ is the resultant of two forces $\vec{P}$ and $\vec{Q}$ acting on a particle of a body, then $\vec{P}$ and $\vec{Q}$ are said to be *components* of $\vec{R}$.

Theorem-1: *A given force $\vec{R}$ can be replaced by an infinite number of pairs of possible components.*

Proof: In order to prove this theorem we need to construct a parallelogram of which the given force $\vec{R}$ is the diagonal and whose sides are along any two distinct lines which are coplanar with the force $\vec{R}$ and pass through its point of application.

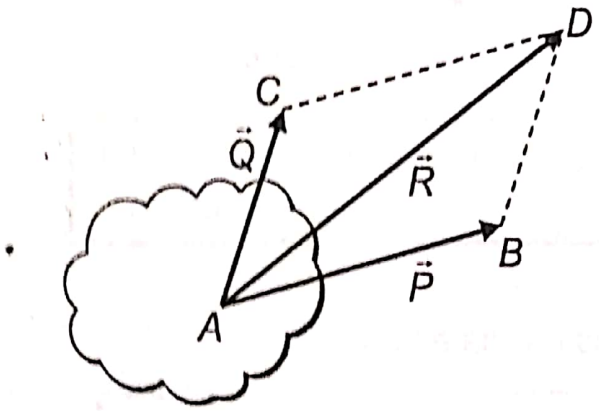


Fig.1.8(a)

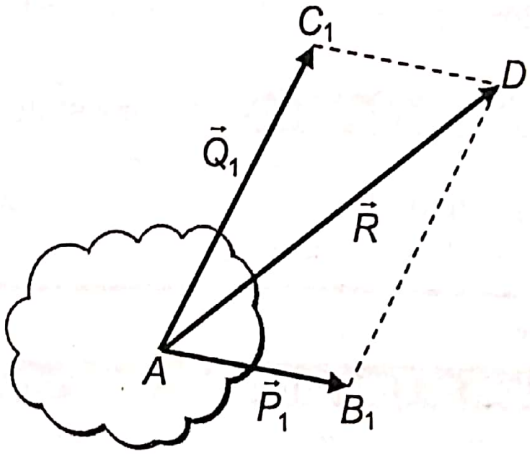


Fig.1.8(b)

The force $\vec{R}$ acts at the point A of the body. We constrict a parallelogram $ABCD$, with $\vec{R}$ along the diagonal AD such that $\vec{AB} = \vec{P}$, $\vec{AC} = \vec{Q}$, $\vec{AD} = \vec{R}$ as shown in Fig.1.8(a). Forces $\vec{P}$ and $\vec{Q}$ are the components of the force $\vec{R}$.

Similarly, we constrict a parallelogram AB_1C_1D, with $\vec{R}$ along the diagonal AD such that $\vec{AB_1} = \vec{P_1}$, $\vec{AC_1} = \vec{Q_1}$, $\vec{AD} = \vec{R}$ as shown in Fig.1.8(b). Forces $\vec{P}_1$ and $\vec{Q}_1$ are the components of the force $\vec{R}$.

It is clear from above figures that $\vec{P}$, $\vec{Q}$ and $\vec{P}_1$, $\vec{Q}_1$ are two different pairs of components of force $\vec{R}$.

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From this illustration, we conclude that a given force $\vec{R}$ can be replaced by an infinite number of pairs of possible components.

The forces $\vec{P}$ and $\vec{Q}$ are called the *vector components* of $\vec{R}$ and their magnitudes P and Q are termed the *scalar* components of $\vec{R}$.

Resolved Parts of a Force: If the two components of a force are mutually perpendicular, then such components are called the *resolved parts* of the given force.

In other words, if the forces $\vec{P}$ and $\vec{Q}$ are components of a force $\vec{R}$ such that $\vec{P}$ and $\vec{Q}$ are perpendicular to each other, then $\vec{P}$ and $\vec{Q}$ are said to be the *resolved parts* of $\vec{R}$.

Let the force $\vec{R}$ make an angle θ with a given direction Ox as shown in Fig.1.9. If $\vec{R}_x$ and $\vec{R}_y$ are resolved parts of $\vec{R}$ in the directions Ox and Oy respectively, then using parallelogram law of forces, we have

$$\vec{R} = \vec{R}_x + \vec{R}_y \qquad \ldots(1)$$

If $\hat{i}$ and $\hat{j}$ are unit vectors along Ox and Oy respectively, then from (1), we have

$$\vec{R}\cdot\hat{i} = \vec{R}_x\cdot\hat{i} + \vec{R}_y\cdot\hat{i}$$

$$R\left|\hat{i}\right|\cos\theta = R_x\left|\hat{i}\right|\cos 0 + R_y\left|\hat{i}\right|\cos 90°$$

$$R\cos\theta = R_x(1) + 0 \quad \because \left|\hat{i}\right| = 1,\ \cos 90° = 0$$

$$R_x = R\cos\theta \qquad \ldots(2)$$

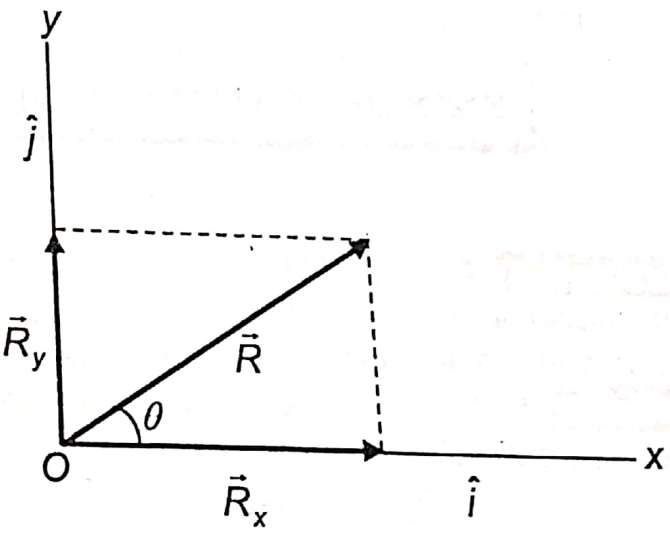


Fig.1.9

Similarly,

$$\vec{R}\cdot\hat{j} = \vec{R}_x\cdot\hat{j} + \vec{R}_y\cdot\hat{j}$$

$$R\left|\hat{j}\right|\cos(90°-\theta) = R_x\left|\hat{j}\right|\cos 90° + R_y\left|\hat{j}\right|\cos 0°$$

$$R\sin\theta = 0 + R_y(1) \quad \because \cos(90°-\theta) = \sin\theta, \left|\hat{j}\right| = 1,\ \cos 90° = 0$$

$$R_y = R\sin\theta \qquad \ldots(3)$$

This shows that the resolved parts of $\vec{R}$ in the direction Ox and in the perpendicular direction Oy are $R_x = R\cos\theta$ and $R_y = R\sin\theta$ respectively.

We conclude that:

> The resolved part of a given force in a given direction is obtained by multiplying the magnitude of the given force by the cosine of the angle between the given force and the given direction.

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Since R_x and R_y are also scalar components of $\vec{R}$, so above result can also be written as:

> The scalar component of a given force in a given direction is obtained by multiplying the magnitude of the given force by the cosine of the angle between the given force and the given direction.

Since $\vec{R}\cdot\hat{i} = R\cos\theta$ and $\vec{R}\cdot\hat{j} = R\sin\theta$, so $\vec{R}\cdot\hat{i}$ and $\vec{R}\cdot\hat{j}$ are the resolved parts of $\vec{R}$ in the directions $\hat{i}$ and $\hat{j}$ respectively.

We conclude that:

> The resolved part or scalar component of a given force in a given direction is obtained by taking the dot product of the force and the unit vector along the given direction.

Thus,

> Scalar component of $\vec{R}$ along vector $\vec{u}$ is the dot product $\vec{R}\cdot\hat{u}$.

Example-1: *A force of 20 N is acting on a block as shown in Fig.1.10. Find its resolved parts along coordinate axes (x-axis and y-axis).*

Solution: Here $F = 20$ N, $\theta = 30°$, so the resolved parts are:

$$F_x = F\cos\theta$$
$$= 20\cos 30°$$
$$= 20\left(\frac{\sqrt{3}}{2}\right)$$
$$= 10\sqrt{3}\ \text{N}$$
$$F_y = F\sin\theta$$
$$= 20\sin 30°$$
$$= 20\left(\frac{1}{2}\right)$$
$$= 10\ \text{N}$$

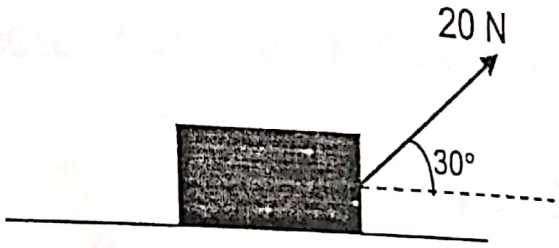
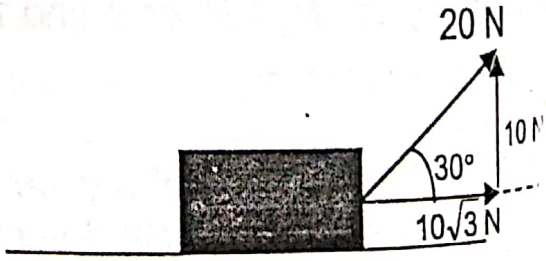


Fig.1.10

The resolved parts of given force are shown in Fig.1.11.



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1-3 Composition of Concurrent Forces

Concurrent Forces:
Forces whose lines of action intersect at one point are called *concurrent forces*. Such forces always have a resultant through the point of concurrence. In Fig.1.12(a), the lines of action of forces pass through the point *A*, so these are concurrent forces and *A* is the point of concurrence.

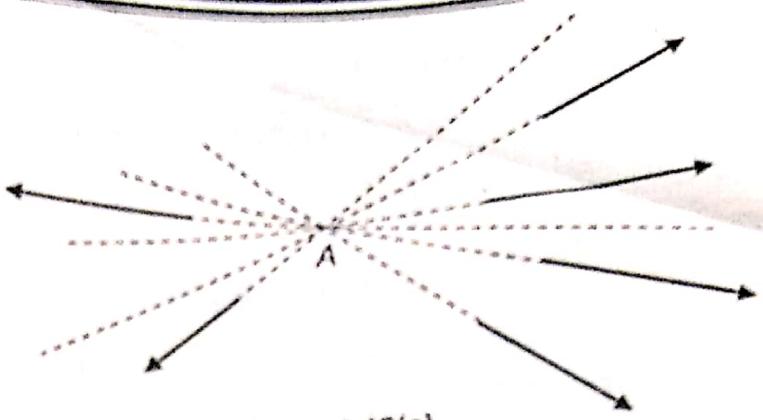


Fig.1.12(a)

Using the principle of transmissibility of forces, we can shift all the forces at point the concurrence point *A* of their lines of action as shown in Fig.1.12(b).

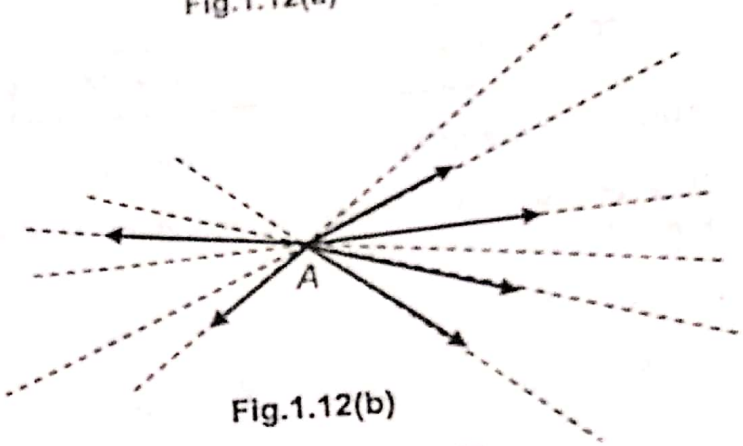


Fig.1.12(b)

This shows that the concurrent forces always pass through a common point, i.e. the concurrence point.

In Fig.1.12(c), the concurrent forces are shown without their lines of actions.

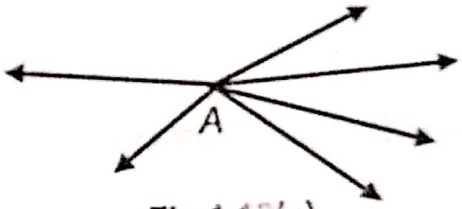


Fig.1.12(c)

Composition of Forces: If two or more forces are acting simultaneously at a point, the same effect can be produced by a single force of the proper size and direction. This single force, which is equivalent to the action of two or more forces, is called the resultant. Putting component forces together to find the resultant force is called *composition of forces*.

Resultant of Two Concurrent Forces:

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Let the forces $\vec{P}$ and $\vec{Q}$ act at a point *A* such that α is the angle between them as shown in the Fig.1.13. Further assume that force $\vec{P}$ is acting along *x*-axis. Let $\vec{R}$ be their resultant, then

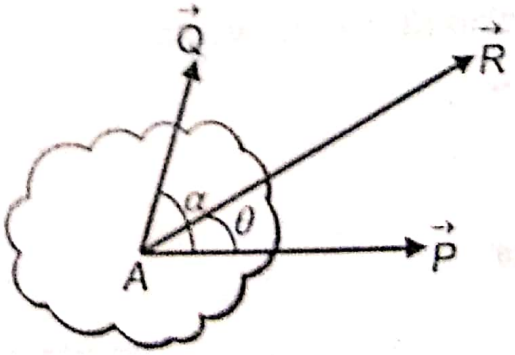


Fig.1.13

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$$\vec{R} = \vec{P} + \vec{Q} \qquad \ldots(1)$$

Taking dot product, we have

$$\vec{R}\cdot\vec{R} = (\vec{P}+\vec{Q})\cdot(\vec{P}+\vec{Q})$$

$$|\vec{R}|^2 = \vec{P}\cdot\vec{P} + \vec{P}\cdot\vec{Q} + \vec{Q}\cdot\vec{P} + \vec{Q}\cdot\vec{Q} \quad \because \vec{R}\cdot\vec{R} = |\vec{R}|^2$$

$$|\vec{R}|^2 = |\vec{P}|^2 + \vec{P}\cdot\vec{Q} + \vec{P}\cdot\vec{Q} + |\vec{Q}|^2 \quad \because \vec{P}\cdot\vec{P} = |\vec{P}|^2,\ \vec{Q}\cdot\vec{Q} = |\vec{Q}|^2,\ \vec{Q}\cdot\vec{P} = \vec{P}\cdot\vec{Q}$$

$$R^2 = P^2 + 2\vec{P}\cdot\vec{Q} + Q^2 \quad \because |\vec{P}| = P,\ |\vec{Q}| = Q,\ |\vec{R}| = R$$

$$R^2 = P^2 + 2PQ\cos\alpha + Q^2 \quad \because \vec{P}\cdot\vec{Q} = PQ\cos\alpha$$

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

or $$\boxed{R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}} \qquad \ldots(1.3)$$

This gives the magnitude of the resultant $\vec{R}$ of the forces $\vec{P}$ and $\vec{Q}$.

If θ is the angle which the resultant $\vec{R}$ makes with the force $\vec{P}$ as shown in the Fig.1.13, then from (1), we have

$$\vec{R}\cdot\hat{i} = \vec{P}\cdot\hat{i} + \vec{Q}\cdot\hat{i}$$ (taking dot product with unit vector along x-axis)

$$|\vec{R}||\hat{i}|\cos\theta = |\vec{P}||\hat{i}|\cos 0 + |\vec{Q}||\hat{i}|\cos\alpha$$ (using dot product formula)

$$R\cos\theta = P + Q\cos\alpha \qquad \ldots(2) \quad \because |\vec{P}| = P,\ |\vec{Q}| = Q,\ |\vec{R}| = R,\ |\hat{i}| = 1$$

Similarly, taking dot product of (1) with unit vector perpendicular to x-axis, we have

$$\vec{R}\cdot\hat{j} = \vec{P}\cdot\hat{j} + \vec{Q}\cdot\hat{j}$$

$$|\vec{R}||\hat{j}|\sin\theta = |\vec{P}||\hat{j}|\cos 90° + |\vec{Q}||\hat{j}|\sin\alpha$$

$$R\sin\theta = Q\sin\alpha \qquad \ldots(3) \quad \because |\vec{P}| = P,\ |\vec{Q}| = Q,\ |\vec{R}| = R,\ |\hat{j}| = 1,\ \cos 90° = 0$$

Dividing (3) by (2), we have

$$\frac{R\sin\theta}{R\cos\theta} = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

or $$\boxed{\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}} \qquad \ldots(1.4)$$

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This gives the direction of resultant, i.e. the angle θ which the resultant $\vec{R}$ makes with the force $\vec{P}$.

Note: The angle θ which the resultant $\vec{R}$ makes with the force $\vec{Q}$ is given by

$$\tan\theta = \left(\frac{P\sin\alpha}{Q + P\cos\alpha}\right) \qquad \ldots(1.5)$$

Note: If the forces $\vec{P}$ and $\vec{Q}$ have same magnitude, i.e. $Q = P$, then using Eq.(1.4), we have

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} = \frac{P\sin\alpha}{P + P\cos\alpha} = \frac{P\sin\alpha}{P(1+\cos\alpha)} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$= \frac{2\sin\alpha/2\cos\alpha/2}{2\cos^2\alpha/2} = \frac{\sin\alpha/2}{\cos\alpha/2} = \tan\frac{\alpha}{2}$$

$$\theta = \frac{\alpha}{2}$$

This shows that:

> The resultant of two forces bisects the angle between them if the forces have equal magnitudes.

Greatest Resultant:

The magnitude R of the resultant of two forces $\vec{P}$ and $\vec{Q}$ acting at an angle α is given by

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} \qquad \ldots(2)$$

It is clear that R will have greatest value for maximum value of $\cos\alpha$. The maximum value of $\cos\alpha$ is 1 at $\alpha = 0$. In this case the forces $\vec{P}$ and $\vec{Q}$ act along same direction. Putting $\cos\alpha = 1$ in (1), we get the greatest magnitude of resultant, i.e.

$$R_{Greatest} = \sqrt{P^2 + Q^2 + 2PQ(1)} = \sqrt{P^2 + Q^2 + 2PQ} = \sqrt{(P+Q)^2}$$

or

$$R_{Greatest} = P + Q \qquad \ldots(1.6)$$

This shows that the greatest magnitude of the resultant is obtained when the forces act along the same direction and is equal to the sum of the magnitudes of the forces. We conclude that:

> The resultant of two forces is greatest when the forces act along the same direction and is equal to the sum of the magnitudes of the forces.

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Least Resultant:

The magnitude R of the resultant of two forces $\vec{P}$ and $\vec{Q}$ acting at an angle α is given by

$$R=\sqrt{P^2+Q^2+2PQ\cos\alpha} \qquad ...(3)$$

It is clear that R will have least value for least value of $\cos\alpha$. The least value of $\cos\alpha$ is -1 at $\alpha=90°$. In this case the forces $\vec{P}$ and $\vec{Q}$ are perpendicular to each other. Putting $\cos\alpha=-1$ in (1), we get the least magnitude of resultant, i.e.

$$R_{Least}=\sqrt{P^2+Q^2+2PQ(-1)}=\sqrt{P^2+Q^2-2PQ}=\sqrt{(P-Q)^2}$$

or

$$\boxed{R_{Least}=P-Q} \qquad ...(1.7)$$

This shows that:

> The least magnitude of the resultant is obtained when the forces act opposite to each other and is equal to the difference of the magnitudes of the forces. (Difference is taken so that the magnitude of the small force is subtracted from the magnitude of the large force).

Resultant of System of Concurrent Coplanar Forces:

Let us consider a system of concurrent forces $\vec{F}_1,\vec{F}_2,\vec{F}_3,...,\vec{F}_n$ in y-plane passing through the point O as shown in Fig.1.14.

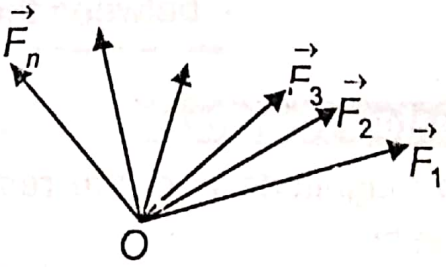


Fig.1.14

If $\vec{R}$ is the resultant of the forces, then

$$\vec{R}=\vec{F}_1+\vec{F}_2+\vec{F}_3+...+\vec{F}_n \qquad ...(1)$$

Taking dot product with unit vector $\hat{i}$ and using its distributive property, we have

$$\vec{R}\cdot\hat{i}=(\vec{F}_1+\vec{F}_2+\vec{F}_3+...+\vec{F}_n)\cdot\hat{i}$$

$$\vec{R}\cdot\hat{i}=\vec{F}_1\cdot\hat{i}+\vec{F}_2\cdot\hat{i}+\vec{F}_3\cdot\hat{i}+...+\vec{F}_n\cdot\hat{i}$$

$$X=X_1+X_2+X_3+...+X_n$$

or

$$X=\sum_{i=1}^{n}X_i \qquad ...(2)$$

where $X,X_1,X_2,X_3,...,X_n$ are resolved parts of $\vec{R},\vec{F}_1,\vec{F}_2,\vec{F}_3,...,\vec{F}_n$ along x-axis respectively.

Similarly, taking dot product of (1) with unit vector $\hat{j}$ and using its distributive property, we have

$$\vec{R}\cdot\hat{j}=(\vec{F}_1+\vec{F}_2+\vec{F}_3+...+\vec{F}_n)\cdot\hat{j}$$

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$$\vec{R}\cdot\hat{j} = \vec{F}_1\cdot\hat{j} + \vec{F}_2\cdot\hat{j} + \vec{F}_3\cdot\hat{j} + \ldots + \vec{F}_n\cdot\hat{j}$$

$$Y = Y_1 + Y_2 + Y_3 + \ldots + Y_n$$

or
$$Y = \sum_{i=1}^{n} Y_i \qquad \ldots(3)$$

where $Y, Y_1, Y_2, Y_3, \ldots, Y_n$ are resolved parts of $\vec{R}, \vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_n$ along y-axis respectively.

The magnitude of the resultant is given by

$$\boxed{R = \sqrt{X^2 + Y^2}} \qquad \ldots(1.8)$$

If θ is the angle which the $\vec{R}$ resultant makes with x-axis, then

$$\boxed{\tan\theta = \frac{Y}{X}} \qquad \ldots(1.9)$$

or
$$\boxed{\theta = \tan^{-1}\left(\frac{Y}{X}\right)} \qquad \ldots(1.10)$$

Example-2: *Find the resolved parts of the forces shown in Fig.1.15 in the directions Ox and Oy. Also find the magnitude and direction of their resultant.*

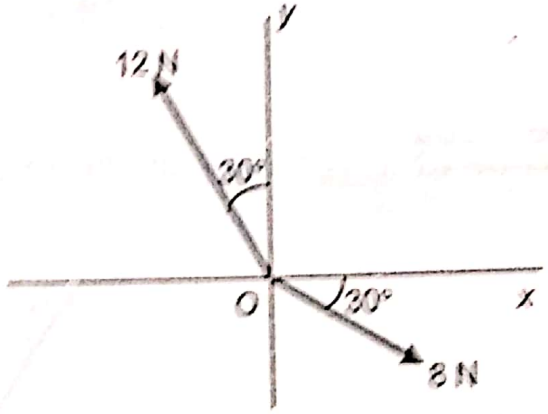


Fig.1.15

Solution: Here $F_1 = 12\,\text{N}$, $\theta_1 = 30^\circ + 90^\circ = 120^\circ$ and $F_2 = 8\,\text{N}$, $\theta_2 = 360^\circ - 30^\circ = 330^\circ$

The resolved parts of the forces $\vec{F}_1$ and $\vec{F}_2$ in the direction Ox are

$$X_1 = F_1\cos\theta = 12\cos 120^\circ = 12(-0.5) = -6\,\text{N}$$

$$X_2 = F_2\cos\theta = 8\cos 330^\circ = 8(0.866) = 6.928\,\text{N}$$

The resolved parts of the forces $\vec{F}_1$ and $\vec{F}_2$ in the direction Oy are

$$Y_1 = F_1\sin\theta = 12\sin 120^\circ = 12(0.866) = 10.392\,\text{N}$$

$$Y_2 = F_2\sin\theta = 8\sin 330^\circ = 8(-0.5) = -4\,\text{N}$$

The resolved part of resultant in the direction Ox is

$$X = X_1 + X_2 = -6\,\text{N} + 6.928\,\text{N} = 0.928\,\text{N}$$

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The resolved part of resultant in the direction Oy is

$$Y = Y_1 + Y_2 = 10.392\,\text{N} - 4\,\text{N} = 6.392\,\text{N}$$

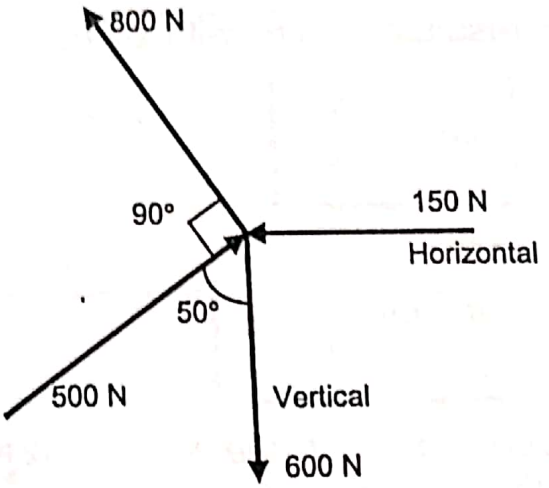
The magnitude of the resultant is

$$R = \sqrt{X^2 + Y^2} = \sqrt{(0.928)^2 + (6.392)^2}$$
$$= \sqrt{0.86 + 40.86} = \sqrt{41.72} = 6.46\,\text{N}$$

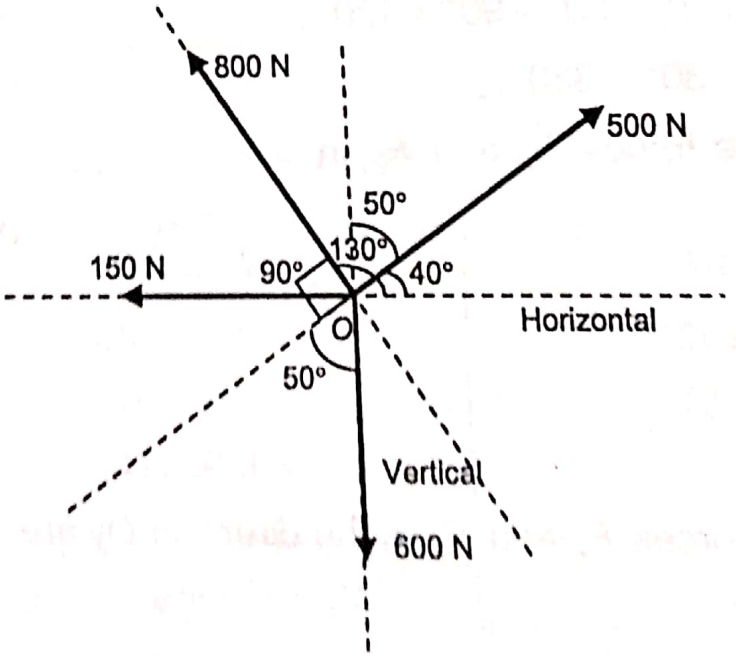
If θ is the angle which the resultant makes with x-axis, then

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right) = \tan^{-1}\left(\frac{6.392}{0.928}\right) = \tan^{-1}(6.89) = 82°$$

Example-3: *Find the resultant of all the forces shown in figure below:*



Solution: We resolve all the forces at O along coordinate axes as shown in the figure below:



It is clear from above figure that the forces $F_1 = 500\,\text{N}, F_2 = 800\,\text{N}, F_3 = 150\,\text{N}$ a
$F_4 = 600\,\text{N}$ make angle 40°, 130°, 180° and 270° respectively with positive x-axis.

If R is the magnitude of the resultant, then its resolved part along x-axis is

$$X = X_1 + X_2 + X_3 + X_4$$

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$$X = F_1\cos 40° + F_2\cos 130° + F_3\cos 180° + F_4\cos 270°$$
$$X = 500(0.766) + 800(-0.64) + 150(-1) + 600(0)$$
$$X = 383 - 512 - 150$$
$$X = -279$$

Similarly, resolved part of resultant R along y-axis is

$$Y = Y_1 + Y_2 + Y_3 + Y_4$$
$$Y = F_1\sin 40° + F_2\sin 130° + F_3\sin 180° + F_4\sin 270°$$
$$Y = 500(0.64) + 800(0.766) + 150(0) + 600(-1)$$
$$Y = 320 + 612.8 + 0 - 600 = 332.8$$

Therefore

$$R = \sqrt{X^2 + Y^2} = \sqrt{(-279)^2 + (332.8)^2} = \sqrt{77841 + 110755.84}$$
$$= \sqrt{188596.84} = 434.3\,\text{N}$$

Next consider $\tan^{-1}\left(\frac{|Y|}{|X|}\right) = \tan^{-1}\left(\frac{|332.8|}{|-279|}\right) = \tan^{-1}\left(\frac{332.8}{279}\right) = \tan^{-1}(1.19) = 50°$

Since $X < 0, Y > 0$, so R lies in the II quadrant, so the angle θ which the resultant makes with positive x-axis is given by

$$\theta = 180° - 50° = 130°$$

Example-4: *The greatest resultant that two forces can have is of magnitude P and the least is of magnitude Q. Show that, when they act at an angle α, their resultant is of magnitude*

$$\sqrt{P^2\cos^2\frac{\alpha}{2} + Q^2\sin^2\frac{\alpha}{2}}$$

AJK, 2011 (B.A./B.Sc.)

Solution: Let the forces P_1, P_2 act at an angle α. If R is the magnitude of their resultant, then

$$R^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\alpha \quad \ldots(1$$

Since P is the magnitude of the greatest resultant of forces P_1, P_2, so

$$P = P_1 + P_2 \quad \ldots($$

Since Q is the magnitude of the least resultant of forces P_1, P_2, so

$$Q = P_1 - P_2 \quad \ldots$$

Adding (1) and (2), we have

$$P + Q = (P_1 + P_2) + (P_1 - P_2)$$
$$P + Q = 2P_1$$
$$P_1 = \frac{P+Q}{2} \quad \ldots(4)$$

Subtracting (2) from (1), we have

$$P - Q = (P_1 + P_2) - (P_1 - P_2)$$
$$P - Q = P_1 + P_2 - P_1 + P_2 = 2P_2$$
$$P_2 = \frac{P-Q}{2} \quad \ldots(5)$$

Putting these values of P_1 and P_2 from (4) and (5) in (1), we have

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$$R^2 = \left(\frac{P+Q}{2}\right)^2 + \left(\frac{P-Q}{2}\right)^2 + 2\left(\frac{P+Q}{2}\right)\left(\frac{P-Q}{2}\right)\cos\alpha$$

$$= \frac{P^2+Q^2+2PQ}{4} + \frac{P^2+Q^2-2PQ}{4} + \left(\frac{P^2-Q^2}{2}\right)\cos\alpha$$

$$= \frac{P^2+Q^2+2PQ+P^2+Q^2-2PQ}{4} + \frac{P^2}{2}\cos\alpha - \frac{Q^2}{2}\cos\alpha$$

$$= \frac{2P^2+2Q^2}{4} + \frac{P^2}{2}\cos\alpha - \frac{Q^2}{2}\cos\alpha$$

$$= \frac{P^2}{2} + \frac{Q^2}{2} + \frac{P^2}{2}\cos\alpha - \frac{Q^2}{2}\cos\alpha$$

$$= \frac{P^2}{2} + \frac{P^2}{2}\cos\alpha + \frac{Q^2}{2} - \frac{Q^2}{2}\cos\alpha$$

$$= \frac{P^2}{2}(1+\cos\alpha) + \frac{Q^2}{2}(1-\cos\alpha)$$

$$= \frac{P^2}{2}\left(2\cos^2\frac{\alpha}{2}\right) + \frac{Q^2}{2}\left(2\sin^2\frac{\alpha}{2}\right)$$

$$= P^2\cos^2\frac{\alpha}{2} + Q^2\sin^2\frac{\alpha}{2}$$

$$R = \sqrt{P^2\cos^2\frac{\alpha}{2} + Q^2\sin^2\frac{\alpha}{2}}$$

Example-5: *Forces P, Q, R act at a point parallel to the sides of a triangle ABC taken in the same order. Show that the magnitude of the resultant is*

$$(P^2+Q^2+R^2-2QR\cos A-2RP\cos B-2PQ\cos C)^{\frac{1}{2}}$$

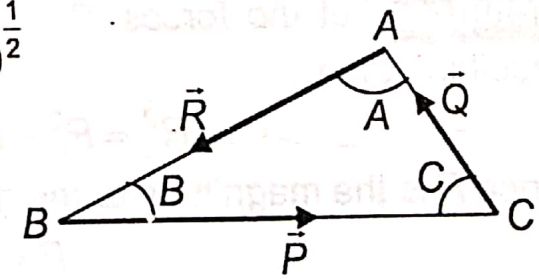


Fig.1.16(a)

Solution: Let the forces P, Q, R act along the sides BC, CA, AB respectively as shown in the Fig.1.16(a). Let the side BC be parallel to x-axis with B as origin. Resoiving all the forces at B in Fig.1.16(b), we note that forces $\vec{P},\vec{Q},\vec{R}$ make angles 0°, 180° – C, 180° + B respectively with positive x-axis. Therefore, the resolved part of resultant along x-axis is

$$X = P\cos 0° + Q\cos(180° - C) + R\cos(180° + B)$$

$$X = P - Q\cos C - R\cos B \qquad ...(1)$$

Similarly, the resolved part of resultant along y-axis is

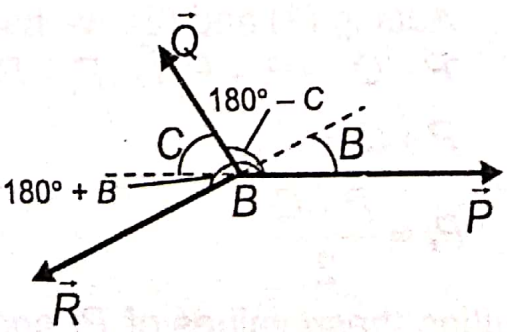


Fig.1.16(b)

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$$Y = P\sin 0° + Q\sin(180° - C) + R\sin(180° + B)$$
$$Y = Q\sin C - R\sin B \quad ...(2)$$

Squaring and adding (1) and (2), we have

$$X^2 + Y^2 = (P - Q\cos C - R\cos B)^2 + (Q\sin C - R\sin B)^2$$
$$= (P^2 + Q^2\cos^2 C + R^2\cos^2 B^2 - 2PQ\cos C + 2QR\cos B\cos C$$
$$- 2RP\cos B) + (Q^2\sin^2 C + R^2\sin^2 B - 2QR\sin B\sin C)$$
$$= P^2 + Q^2(\cos^2 C + \sin^2 C) + R^2(\cos^2 B^2 + \sin^2 B) - 2RP\cos B$$
$$- 2PQ\cos C + 2QR(\cos B\cos C - \sin B\sin C)$$
$$= P^2 + Q^2(1) + R^2(1) + 2QR\cos(B + C) - 2RP\cos B - 2PQ\cos C$$
$$X^2 + Y^2 = P^2 + Q^2 + R^2 + 2QR\cos(180° - A) - 2RP\cos B - 2PQ\cos C$$
$$= P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C$$

The magnitude of the resultant is

$$\sqrt{X^2 + Y^2} = \sqrt{P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C}$$

Thus, the magnitude of the resultant is

$$(P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C)^{\frac{1}{2}}$$

Example-6: *Forces X, P + X, Q + X act at a point in the directions of the sides of an equilateral triangle, taken one way round. Show that they are equivalent to two forces P and Q acting at an angle of 120°.*

PU, 2013; 2009; 2006; 2004 (B.A./B.Sc.)

Solution: Let the forces X, $P + X$, $Q + X$ act along the sides BC, CA, AB respectively as shown in Fig.1.17(a). Since all the three angles of an equilateral triangle are of measure 60°, so resolving the forces at point O in Fig.1.17(b), we see that the forces $\vec{X}$, $\vec{P} + \vec{X}$, $\vec{Q} + \vec{X}$ make angles 0°, 120°, 240° with positive x-axis.

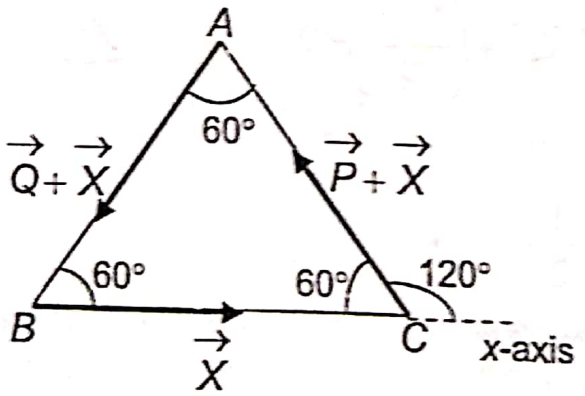


Fig.1.17(a)

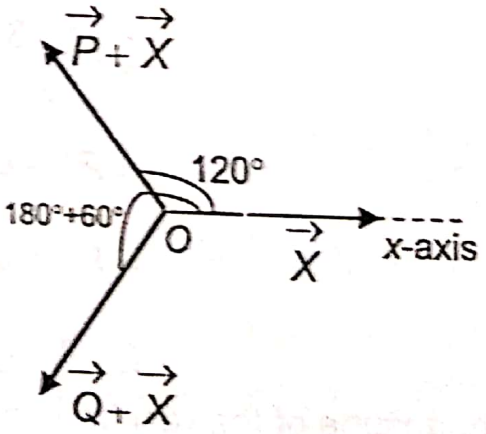


Fig.1.17(b)

If R is the magnitude of the resultant of the given forces, then its resolved part along x-axis is

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$$R_x = X\cos 0° + (P+X)\cos 120° + (Q+X)\cos 240°$$

$$R_x = X + (P+X)\left(-\frac{1}{2}\right) + (Q+X)\left(-\frac{1}{2}\right)$$

$$R_x = X - \frac{1}{2}P - \frac{1}{2}X - \frac{1}{2}Q - \frac{1}{2}X$$

$$R_x = -\frac{1}{2}(P+Q) \qquad ...(1)$$

Similarly, the resolved part of the resultant along y-axis is

$$R_y = X\sin 0° + (P+X)\sin 120° + (Q+X)\sin 240°$$

$$R_y = (P+X)\left(\frac{\sqrt{3}}{2}\right) + (Q+X)\left(-\frac{\sqrt{3}}{2}\right)$$

$$R_y = \frac{\sqrt{3}}{2}P + \frac{\sqrt{3}}{2}X - \frac{\sqrt{3}}{2}Q - \frac{\sqrt{3}}{2}X$$

$$R_y = \frac{\sqrt{3}}{2}(P-Q) \qquad ...(2)$$

Squaring and adding (1) and (2), we have

$$R_x^2 + R_y^2 = \left(-\frac{1}{2}(P+Q)\right)^2 + \left(\frac{\sqrt{3}}{2}(P-Q)\right)^2$$

$$= \frac{1}{4}(P^2+Q^2+2PQ) + \frac{3}{4}(P^2+Q^2-2PQ)$$

$$= \frac{1}{4}P^2 + \frac{1}{4}Q^2 + \frac{1}{2}PQ + \frac{3}{4}P^2 + \frac{3}{4}Q^2 - \frac{3}{2}PQ$$

$$= \frac{1}{4}P^2 + \frac{3}{4}P^2 + \frac{1}{4}Q^2 + \frac{3}{4}Q^2 + \frac{1}{2}PQ - \frac{3}{2}PQ$$

$$= \left(\frac{1+3}{4}\right)P^2 + \left(\frac{1+3}{4}\right)Q^2 + \left(\frac{1-3}{2}\right)PQ$$

$$= P^2 + Q^2 - PQ$$

$$= P^2 + Q^2 + 2PQ\left(-\frac{1}{2}\right)$$

$$= P^2 + Q^2 + 2PQ\cos 120°$$

The magnitude of the resultant is given by

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{P^2 + Q^2 + 2PQ\cos 120°}$$

This is equal to the magnitude of the resultant of the two forces P and Q acting at an angle of 120°. This shows that the given forces are equivalent to two forces P and Q acting at an angle of 120°.

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Example-7: *Forces X, Y, Z, $P+X$, $Q+Y$, $P+Z$ act at a point in the directions of the sides of a regular hexagon, taken one way round. Show that their resultant is a force $P+Q$ in the direction of the force $Q+Y$.*

Solution: Let the forces X, Y, Z, $P+X$, $Q+Y$, $P+Z$ act at a point in the directions of the sides of a regular hexagon, taken one way round as shown in Fig.1.18(a). Resolving the forces at point O in Fig.1.18(b), we see that the forces $\vec{X}, \vec{Y}, \vec{Z}$ $\vec{P}+\vec{X}, \vec{Q}+\vec{Y}, \vec{P}+\vec{Z}$ make angles 0°, 60°, 120°, 180°, 240°, 300° with positive x-axis. If R is the magnitude of the resultant of the given forces, then its resolved part along x-axis is

$$R_x = X\cos 0° + Y\cos 60° + Z\cos 120° + (P+X)\cos 180°$$
$$+(Q+Y)\cos 240° + (P+Z)\cos 300°$$

$$R_x = X + \frac{1}{2}Y - \frac{1}{2}Z - (P+X) - \frac{1}{2}(Q+Y) + \frac{1}{2}(P+Z)$$

$$R_x = X + \frac{1}{2}Y - \frac{1}{2}Z - P - X - \frac{1}{2}Q - \frac{1}{2}Y + \frac{1}{2}P + \frac{1}{2}Z$$

$$R_x = -\frac{1}{2}P - \frac{1}{2}Q$$

$$R_x = -\frac{1}{2}(P+Q) \qquad \ldots(1)$$

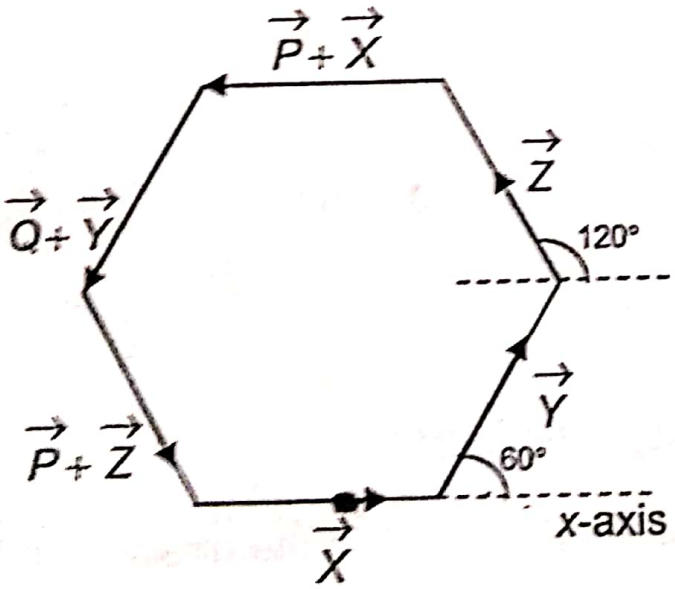


Fig.1.18(a)

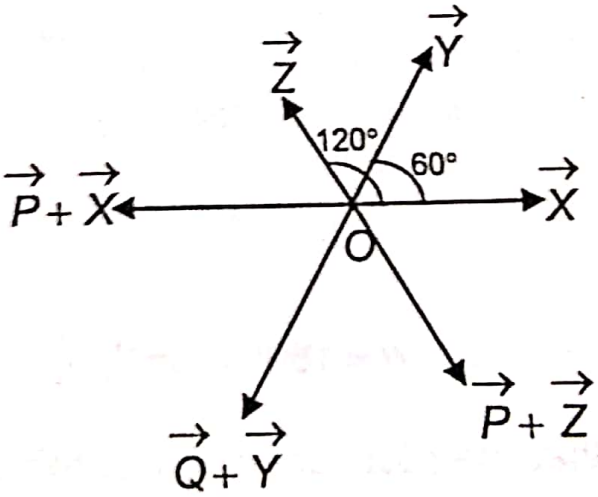


Fig.1.18(b)

Similarly, the resolved part of the resultant along y-axis is

$$R_y = X\sin 0° + Y\sin 60° + Z\sin 120° + (P+X)\sin 180°$$
$$+(Q+Y)\sin 240° + (P+Z)\sin 300°$$

$$R_y = 0 + \frac{\sqrt{3}}{2}Y + \frac{\sqrt{3}}{2}Z + 0 - \frac{\sqrt{3}}{2}(Q+Y) - \frac{\sqrt{3}}{2}(P+Z)$$

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$$R_y = \frac{\sqrt{3}}{2}Y + \frac{\sqrt{3}}{2}Z - \frac{\sqrt{3}}{2}Q - \frac{\sqrt{3}}{2}Y - \frac{\sqrt{3}}{2}P - \frac{\sqrt{3}}{2}Z$$

$$R_y = -\frac{\sqrt{3}}{2}P - \frac{\sqrt{3}}{2}Q$$

$$R_y = -\frac{\sqrt{3}}{2}(P+Q) \qquad \ldots(2)$$

Squaring and adding (1) and (2), we have

$$R_x^2 + R_y^2 = \left(-\frac{1}{2}(P+Q)\right)^2 + \left(-\frac{\sqrt{3}}{2}(P+Q)\right)^2$$

$$R^2 = \frac{1}{4}(P^2+Q^2+2PQ) + \frac{3}{4}(P^2+Q^2+2PQ)$$

$$= \left(\frac{1+3}{4}\right)(P^2+Q^2+2PQ)$$

$$= P^2+Q^2+2PQ$$

$$= (P+Q)^2$$

$$R = P+Q$$

Let R make an angle θ with positive x-axis. Since both R_x and R_y are negative, so θ must lie in third quadrant, i.e. we should add 180° in the angle $\tan^{-1}\left(\frac{R_y}{R_x}\right)$.

Now

$$\tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{-\frac{\sqrt{3}}{2}(P+Q)}{-\frac{1}{2}(P+Q)}\right) = \tan^{-1}(\sqrt{3}) = 60°$$

Therefore

$$\theta = 180° + \tan^{-1}\left(\frac{R_y}{R_x}\right) = 180° + 60° = 240°$$

This shows that the resultant of given forces is a force $P + Q$ in the direction of the force $Q + Y$.

Example-8: *If two forces P and Q act at such an angle that their resultant R = P, show that if P is doubled, the new resultant is at right angles to Q.*

PU, 2012, 1999 (B.A./B.Sc.); PU, 2010 (BS Math); AJK, S2011(B.A./B.Sc.)

Solution: Let the forces P and Q act at an angle α, then

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$$

$$P^2 = P^2 + Q^2 + 2PQ\cos\alpha \qquad \because R = P$$

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$$Q^2 + 2PQ\cos\alpha = 0$$
$$Q(Q + 2P\cos\alpha) = 0$$
$$Q + 2P\cos\alpha = 0 \quad \ldots(1) \quad \because Q \neq 0$$

If P is doubled, then the angle θ which the new resultant makes with Q is given by

$$\tan\theta = \frac{2P\sin\alpha}{Q + 2P\cos\alpha}$$
$$= \frac{2P\sin\alpha}{0} \quad \because Q + 2P\cos\alpha = 0$$
$$\Rightarrow \theta = \tan^{-1}(\infty)$$
$$\Rightarrow \theta = 90^\circ$$

Example-9:

The resolved part of the resultant $\vec{R}$ of two coplanar forces $\vec{P}$ and $\vec{Q}$ acting at a point O in the direction of $\vec{P}$ is of magnitude Q. Show that the angle between the forces is $2\sin^{-1}\sqrt{\frac{P}{2Q}}$.

PU, 2010, 2003 (B.A./B.Sc.)

Solution:

Let α be the angle between the forces $\vec{P}$ and $\vec{Q}$. Since $\vec{R}$ is the resultant of $\vec{P}$ and $\vec{Q}$, so

$$\vec{R} = \vec{P} + \vec{Q} \quad \ldots(1)$$

If $\hat{u}$ is the unit vector along $\vec{P}$, then taking its dot product with (1), we have

$$\vec{R}\cdot\hat{u} = \vec{P}\cdot\hat{u} + \vec{Q}\cdot\hat{u} \quad \ldots(2)$$

Since $\vec{R}\cdot\hat{u}$, being the resolved part of $\vec{R}$ along $\vec{P}$, is Q, i.e.

$$\vec{R}\cdot\hat{u} = Q \quad \ldots(3)$$

Since $\hat{u}$ is the unit vector along $\vec{P}$, so

$$\vec{P}\cdot\hat{u} = P \quad \ldots(4)$$

Since α is the angle between the forces $\vec{P}$ and $\vec{Q}$, so α is also the angle between the unit vector $\hat{u}$ and $\vec{Q}$, so

$$\vec{Q}\cdot\hat{u} = Q\cos\alpha \quad \ldots(5)$$

Putting these values of $\vec{R}\cdot\hat{u}, \vec{P}\cdot\hat{u}$ and $\vec{Q}\cdot\hat{u}$ from (3), (4) and (5) in (2), we have

$$Q = P + Q\cos\alpha$$

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$$Q - Q\cos\alpha = P$$
$$Q(1-\cos\alpha) = P$$
$$Q\left(2\sin^2\frac{\alpha}{2}\right) = P$$
$$\sin^2\frac{\alpha}{2} = \frac{P}{2Q}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{P}{2Q}}$$
$$\frac{\alpha}{2} = \sin^{-1}\sqrt{\frac{P}{2Q}}$$
$$\alpha = 2\sin^{-1}\sqrt{\frac{P}{2Q}}$$

Example-10: *The resultant of two forces P & Q is R, if Q is double R is double, if Q is reversed R is again double, show that* $P^2 : Q^2 : R^2 :: 2:3:2$.

Solution: If α is the angle between the forces P and Q, then the magnitude of their resultant is

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \quad \ldots(1)$$

If Q is double, R is double, so

$$(2R)^2 = P^2 + (2Q)^2 + 2P(2Q)\cos\alpha$$
$$4R^2 = P^2 + 4Q^2 + 4PQ\cos\alpha \quad \ldots(2)$$

If Q is reversed, R is again double, so

$$(2R)^2 = P^2 + (-Q)^2 + 2P(-Q)\cos\alpha$$
$$4R^2 = P^2 + Q^2 - 2PQ\cos\alpha \quad \ldots(3)$$

Adding (1) and (3), (eliminating $\cos\alpha$ from (1) and (3)), we have

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$$
$$4R^2 = P^2 + Q^2 - 2PQ\cos\alpha$$
$$5R^2 = 2P^2 + 2Q^2$$
$$2P^2 + 2Q^2 - 5R^2 = 0 \quad \ldots(4)$$

Multiplying (1) by 2, we have

$$2R^2 = 2P^2 + 2Q^2 + 4PQ\cos\alpha \quad \ldots(5)$$

Subtracting (5) from (2), (eliminating $\cos\alpha$ from (2) and (5)), we have

$$4R^2 = P^2 + 4Q^2 + 4PQ\cos\alpha$$
$$\underline{2}R^2 = \underline{2}P^2 \pm 2Q^2 \pm 4PQ\cos\alpha$$
$$2R^2 = -P^2 + 2Q^2$$
$$P^2 - 2Q^2 + 2R^2 = 0 \quad \ldots(6)$$

Solving equations (4) and (6), we have

$$2P^2 + 2Q^2 - 5R^2 = 0$$
$$P^2 - 2Q^2 + 2R^2 = 0$$
$$\frac{P^2}{2\times 2-(-5)\times(-2)} = -\frac{Q^2}{2\times 2-(-5)\times 1} = \frac{R^2}{2\times(-2)-2\times 1}$$

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$$\Rightarrow \frac{P^2}{4-10} = \frac{Q^2}{4+5} = \frac{R^2}{-4-2}$$

$$\Rightarrow \frac{P^2}{-6} = \frac{Q^2}{-9} = \frac{R^2}{-6}$$

$$\Rightarrow \frac{P^2}{2} = \frac{Q^2}{3} = \frac{R^2}{2}$$

$$\Rightarrow P^2 : Q^2 : R^2 :: 2 : 3 : 2$$

Example-11: *The resultant of two forces P and Q acting at an angle θ is equal to $(2m+1)\sqrt{P^2+Q^2}$; when they act at an angle $90°-\theta$, the resultant is $(2m-1)\sqrt{P^2+Q^2}$, prove that $\tan\theta = \frac{m-1}{m+1}$.*

Solution: Since $(2m+1)\sqrt{P^2+Q^2}$ is the resultant when P and Q at an angle θ, so

$$\left((2m+1)\sqrt{P^2+Q^2}\right)^2 = P^2+Q^2+2PQ\cos\theta$$

$$(2m+1)^2(P^2+Q^2) = P^2+Q^2+2PQ\cos\theta$$

$$2PQ\cos\theta = (2m+1)^2(P^2+Q^2)-(P^2+Q^2)$$

$$2PQ\cos\theta = [(2m+1)^2-1](P^2+Q^2)$$

$$2PQ\cos\theta = (4m^2+4m+1-1)(P^2+Q^2)$$

$$2PQ\cos\theta = (4m^2+4m)(P^2+Q^2)$$

$$2PQ\cos\theta = 4m(m+1)(P^2+Q^2)$$

$$PQ\cos\theta = 2m(m+1)(P^2+Q^2) \quad ...(1)$$

Since $(2m-1)\sqrt{P^2+Q^2}$ is the resultant when P and Q at an angle $90°-\theta$, so

$$\left((2m-1)\sqrt{P^2+Q^2}\right)^2 = P^2+Q^2+2PQ\cos(90°-\theta)$$

$$(2m-1)^2(P^2+Q^2) = P^2+Q^2+2PQ\sin\theta$$

$$2PQ\sin\theta = (2m-1)^2(P^2+Q^2)-(P^2+Q^2)$$

$$2PQ\sin\theta = [(2m-1)^2-1](P^2+Q^2)$$

$$2PQ\sin\theta = (4m^2-4m+1-1)(P^2+Q^2)$$

$$2PQ\sin\theta = (4m^2-4m)(P^2+Q^2)$$

$$2PQ\sin\theta = 4m(m-1)(P^2+Q^2)$$

$$PQ\sin\theta = 2m(m-1)(P^2+Q^2) \quad ...(2)$$

Dividing (2) by (1), we have

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$$\frac{PQ\sin\phi}{PQ\cos\phi} = \frac{2m(m-1)(P^2+Q^2)}{2m(m+1)(P^2+Q^2)}$$

$$\tan\phi = \frac{m-1}{m+1}$$

Example-12:

Two forces $\vec{P}+\vec{Q}$ and $\vec{P}-\vec{Q}$ make an angle 2α with one another and their resultant makes an angle θ with the bisector of the angle between them. Show that $P\tan\theta = Q\tan\alpha$.

Solution:

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent forces $\vec{P}+\vec{Q}$ and $\vec{P}-\vec{Q}$ respectively as shown in Fig.1.19. If $\overline{OD}$ is the angle bisector of $\angle BOA$, then $\overline{OC}$ makes an angle θ with $\overline{OD}$. Hence $\angle OCA = \alpha+\theta$ and $\angle AOC = \alpha-\theta$.

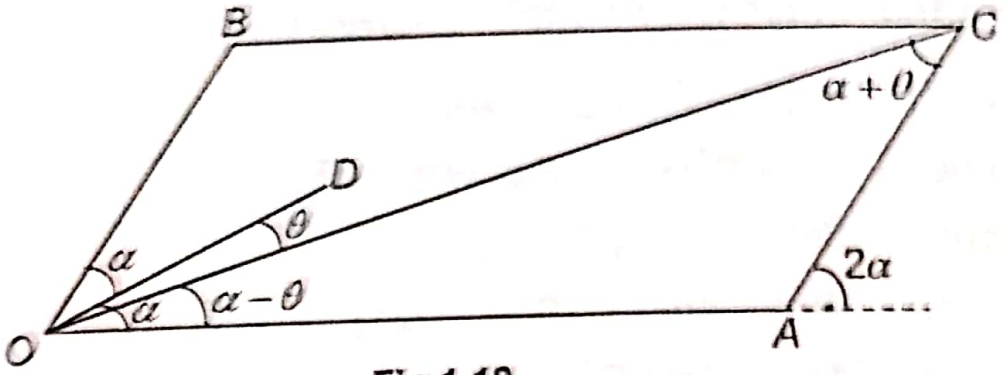


Fig.1.19

Using law of sine in triangle OAC, we have

$$\frac{OA}{\sin(\alpha+\theta)} = \frac{AC}{\sin(\alpha-\theta)}$$

$$\frac{OA}{AC} = \frac{\sin(\alpha+\theta)}{\sin(\alpha-\theta)}$$

$$\frac{P+Q}{P-Q} = \frac{\sin(\alpha+\theta)}{\sin(\alpha-\theta)}$$

$$\frac{(P+Q)+(P-Q)}{(P+Q)-(P-Q)} = \frac{\sin(\alpha+\theta)+\sin(\alpha-\theta)}{\sin(\alpha+\theta)-\sin(\alpha-\theta)}$$

$$\frac{P+Q+P-Q}{P+Q-P+Q} = \frac{2\sin\alpha\cos\theta}{2\cos\alpha\sin\theta}$$

$$\frac{2P}{2Q} = \frac{\tan\alpha}{\tan\theta}$$

$$P\tan\theta = Q\tan\alpha$$

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1-4 The (λ, μ) Theorem

In this section we shall state and prove (λ, μ) theorem. We shall also give some applications of (λ, μ) theorem.

(λ, μ) Theorem:

If two concurrent forces are represented by $\lambda\vec{OA}$ and $\mu\vec{OB}$, their resultant is given by $(\lambda+\mu)\vec{OC}$, where C divides AB so that $AC : CB = \mu : \lambda$.

PU, 2011 (B.A./B.Sc.); PU, 2011 (BS Math)

Proof: It is clear from Fig.1.20 that

$$\vec{OA} = \vec{OC} + \vec{CA} \qquad ...(1)$$

and

$$\vec{OB} = \vec{OC} + \vec{CB} \qquad ...(2)$$

Multiplying (1) by λ and (2) by μ and adding, we get

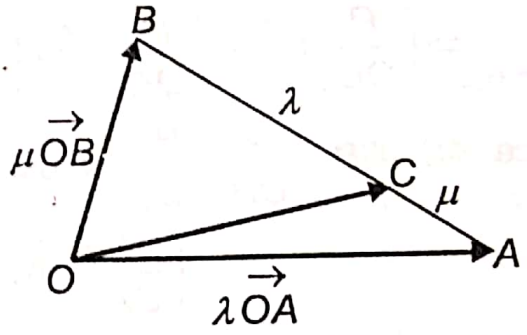


Fig.1.20

$$\lambda\vec{OA} + \mu\vec{OB} = (\lambda\vec{OC} + \lambda\vec{CA}) + (\mu\vec{OC} + \mu\vec{CB})$$

$$= (\lambda+\mu)\vec{OC} + \lambda\vec{CA} + \mu\vec{CB}$$

$$= (\lambda+\mu)\vec{OC} - \lambda\vec{AC} + \mu\vec{CB} \qquad ...(1) \quad (\because \lambda\vec{CA} = -\lambda\vec{AC})$$

It is given that $AC : CB = \mu : \lambda$, so $\lambda\vec{AC} = \mu\vec{CB}$. Using this in (1), we have

$$\lambda\vec{OA} + \mu\vec{OB} = (\lambda+\mu)\vec{OC}$$

This completes the proof.

Example-13:

Forces $\vec{P}, \vec{Q}$ act at a point and their resultant is $\vec{R}$. If any transversal cuts three lines of action of the forces in the points A, B, C respectively, prove that $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$.

PU, 2009 (B.A./B.Sc.)

Solution:

Forces $\vec{P}, \vec{Q}$ and their resultant $\vec{R}$ are shown in Fig.1.20.

We can write the force $\vec{P}$ as follows:

$$\vec{P} = (\text{Magnitude of } \vec{P})(\text{Unit vector along } \vec{P})$$

$$= P\left(\frac{\vec{OA}}{OA}\right)$$

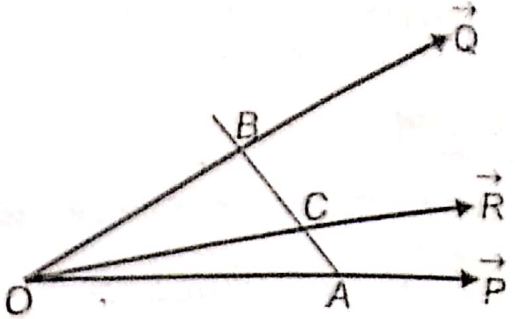
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$$\vec{P}=\left(\frac{P}{OA}\right)\vec{OA} \qquad ...(1)$$

Similarly,

$$\vec{Q}=\left(\frac{Q}{OB}\right)\vec{OB} \qquad ...(2)$$

and $$\vec{R}=\left(\frac{R}{OC}\right)\vec{OC} \qquad ...(3)$$



Since $\vec{R}$ is the resultant of forces $\vec{P},\vec{Q}$, so $\vec{R}=\vec{P}+\vec{Q}$

$$\Rightarrow\left(\frac{R}{OC}\right)\vec{OC}=\left(\frac{P}{OA}\right)\vec{OA}+\left(\frac{Q}{OB}\right)\vec{OB} \qquad ...(4)$$

Since $AC:CB=\frac{Q}{OB}:\frac{P}{OA}$, so using (λ,μ) theorem, we have

$$\left(\frac{P}{OA}\right)\vec{OA}+\left(\frac{Q}{OB}\right)\vec{OB}=\left(\frac{P}{OA}+\frac{Q}{OB}\right)\vec{OC}$$

Using this in (4), we have

$$\left(\frac{P}{OA}+\frac{Q}{OB}\right)\vec{OC}=\left(\frac{R}{OC}\right)\vec{OC}$$

This gives

$$\frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}$$

Example-14:

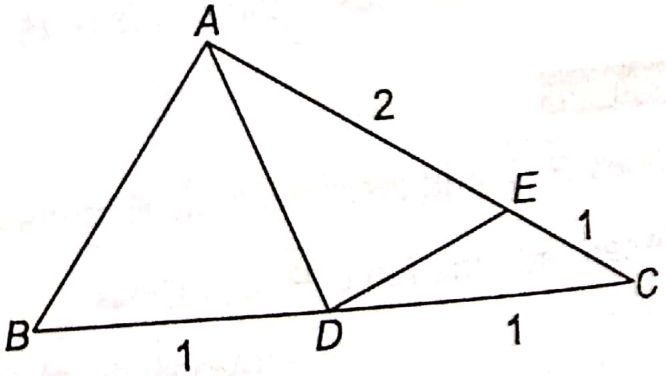
Forces $2\vec{BC},\vec{CA},\vec{BA}$ act along the sides of a triangle ABC. Show that their resultant is $6\vec{DE}$, where D bisects BC and E is a point on CA such that $CE=\frac{1}{3}CA$.

PU, 2009 (B.A./B.Sc.); University of Sargodha, 2014 (B.A./B.Sc.)

Solution: Consider

$$2\vec{BC}+\vec{CA}+\vec{BA}=2\vec{BC}-\vec{AC}-\vec{AB}$$

$$=2\vec{BC}-(\vec{AC}+\vec{AB}) \qquad ...(1)$$



Using (λ,μ) theorem, we have

$$1\vec{AC}+1\vec{AB}=(1+1)\vec{AD}$$

$$\vec{AC}+\vec{AB}=2\vec{AD} \qquad ...(2)$$

It is also given that

$$\vec{\quad}=2\vec{DC} \qquad ...(3)$$

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Using (2) and (3) in (1), we have

$$2\vec{BC}+\vec{CA}+\vec{BA}=2(2\vec{DC})-(2\vec{AD})$$

$$=2(2\vec{DC}+\vec{DA})$$

$$2\vec{BC}+\vec{CA}+\vec{BA}=2(2+1)\vec{DE} \text{ (again using } (\lambda, \mu) \text{ theorem)}$$

$$2\vec{BC}+\vec{CA}+\vec{BA}=6\vec{DE}$$

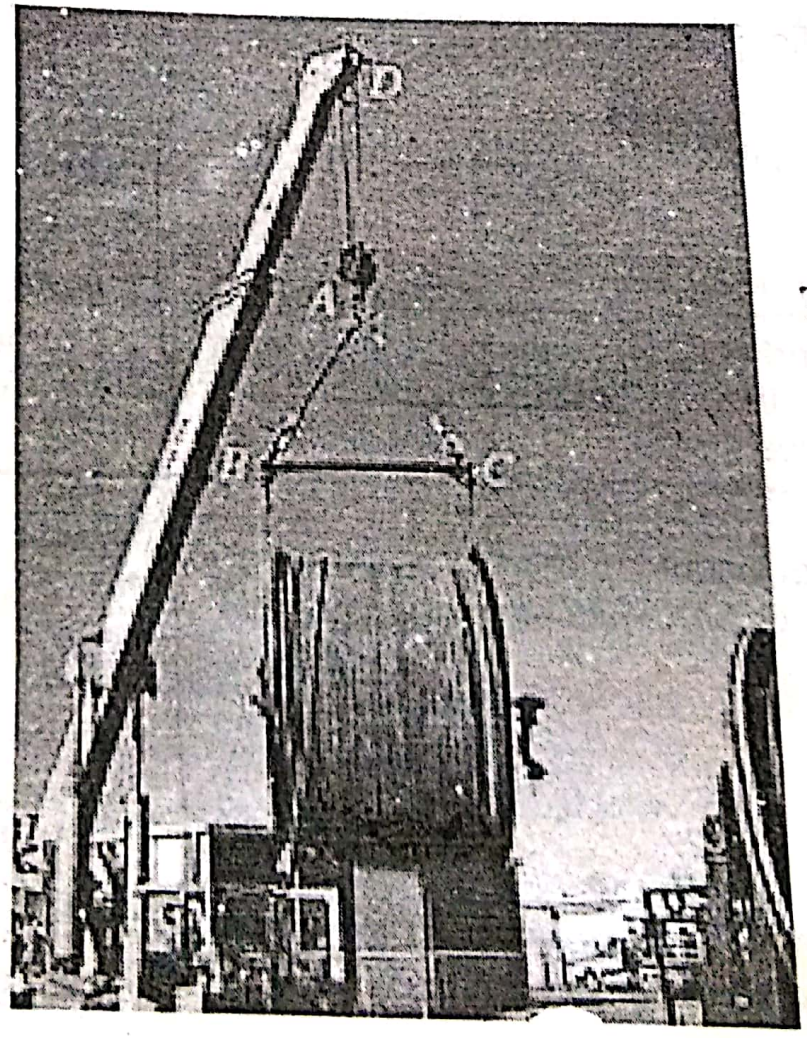
This shows that the resultant of the forces $2\vec{BC},\vec{CA},\vec{BA}$ is $6\vec{DE}$.

1-5 Equilibrium of a Particle

In this section we shall define a particle in equilibrium, state and prove Lamy's Theorem, and solve some relevant examples.

Equilibrium of a Particle: A particle is in *equilibrium* if the resultant of all forces acting on the particle is equal to zero.

In the Fig.1.21, three forces, acting at point A, weight $\vec{W}$, and tensions $\vec{T_B},\vec{T_C}$ are in equilibrium.



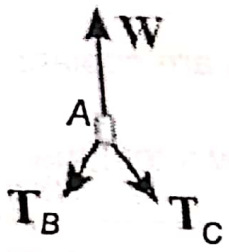


Fig.1.21

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The Polygon of Forces: If a particle is in equilibrium under the action of several forces, these forces may be represented in magnitude and direction (but not in position) by the sides of a closed polygon, taken in order. Such a polygon is known as *polygon of forces*.

The Triangle of Forces: If a particle is in equilibrium under the action of three forces, these forces may be represented in magnitude and direction (but not in position) by the sides of a triangle, taken in order (Fig.1.22). Such a triangle is known as *triangle of forces*.

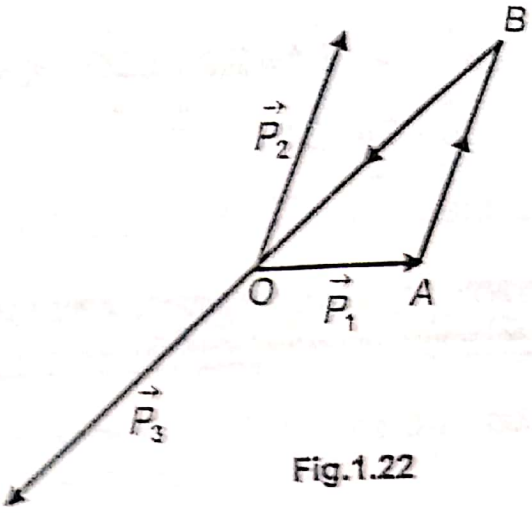


Fig.1.22

If $\vec{P}_1, \vec{P}_2, \vec{P}_3, \ldots, \vec{P}_n$ are the forces acting on the particle at the point O and $\vec{R}$ is their resultant, so that

$$\vec{R} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \ldots + \vec{P}_n = \sum_{i=1}^{n} \vec{P}_i \qquad \ldots(1.11)$$

The necessary and sufficient condition for equilibrium is that

$$\vec{R} = 0 \qquad \ldots(1.12)$$

If the forces are coplanar, then from (1.12), we have

$$X = 0, \quad Y = 0 \qquad \ldots(1.13)$$

This shows that the

> Necessary and sufficient condition for coplanar forces to be in equilibrium is that the resolved parts of their resultant along coordinate axes are zero.

Note: If a particle is in equilibrium under the action of several forces, then the sums of their resolved parts along coordinate axes are zero, i.e.

$$X_1 + X_2 + \ldots + X_n = 0, \qquad Y_1 + Y_2 + \ldots + Y_n = 0$$

or

$$\sum_{i=1}^{n} X_i = 0, \qquad \sum_{i=1}^{n} Y_i = 0$$

Lami's Theorem: *If a particle is in equilibrium under the action of three coplanar forces, each force has a magnitude proportional to the sine of the angle between the other two.*

PU. 2009 (B.A./B.Sc.); AJK, 2012, 2010 (B.A./B.Sc.)

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Proof: Let a particle be in equilibrium under the action of three coplanar concurrent forces $\vec{P},\vec{Q},\vec{R}$ as shown in Fig.1.23. Let α be the angle between $\vec{Q}$ and $\vec{R}$; β be the angle between $\vec{R}$ and $\vec{P}$; γ be the angle between $\vec{P}$ and $\vec{Q}$.

Since forces are in equilibrium, so

$$\vec{P}+\vec{Q}+\vec{R}=0 \qquad ...(1)$$

Let $\hat{a},\hat{b},\hat{c}$ be unit vectors along $\vec{P},\vec{Q},\vec{R}$ respectively, then (1) takes the form

$$P\hat{a}+Q\hat{b}+R\hat{c}=0 \qquad ...(2)$$

Taking cross product of (2) with $\hat{a}$, we have

$$\hat{a}\times(P\hat{a}+Q\hat{b}+R\hat{c})=0$$

$$P\hat{a}\times\hat{a}+Q\hat{a}\times\hat{b}+R\hat{a}\times\hat{c}=0$$

$$0+Q\hat{a}\times\hat{b}+R\hat{a}\times\hat{c}=0 \qquad \because \hat{a}\times\hat{a}=0$$

$$Q\hat{a}\times\hat{b}-R\hat{c}\times\hat{a}=0 \qquad \because \hat{a}\times\hat{c}=-\hat{c}\times\hat{a}$$

$$Q\hat{a}\times\hat{b}=R\hat{c}\times\hat{a}$$

$$Q|\hat{a}\times\hat{b}|=R|\hat{c}\times\hat{a}| \qquad \text{(Taking magnitudes on both sides)}$$

$$Q|\hat{a}||\hat{b}|\sin\gamma=R|\hat{c}||\hat{a}|\cos\beta$$

$$Q\sin\gamma=R\cos\beta \qquad \because |\hat{a}|=|\hat{b}|=|\hat{c}|=1$$

$$\frac{Q}{\cos\beta}=\frac{R}{\sin\gamma} \qquad ...(3)$$

Similarly, taking the cross product of (2) with $\hat{b}$ and simplifying, we have

$$\frac{P}{\cos\alpha}=\frac{R}{\sin\gamma} \qquad ...(4)$$

Equating (3) and (4), we have

$$\frac{P}{\cos\alpha}=\frac{Q}{\cos\beta}=\frac{R}{\sin\gamma}$$

This shows that each force has a magnitude proportional to the sine of the angle between the other two. This completes the proof.

Example-15:

Three forces $\vec{P},\vec{Q},\vec{R}$ acting at a point are in equilibrium and the angle between $\vec{P}$ and $\vec{Q}$ is double of the angle between $\vec{P}$ and $\vec{R}$. Prove that $R^2 = Q(Q-P)$.

PU, 2000 (B.A./B.Sc.); PU, 2011; 2010 (BS Math)

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Solution:

If α is the angle between $\vec{P}$ and $\vec{R}$ then 2α is the angle between $\vec{P}$ and $\vec{Q}$ as shown in Fig.1.24. If $\vec{P}$ acts along x-axis, then $\vec{R}$ and $\vec{Q}$ make angles α and $360^\circ - 2\alpha$ with positive x-axis. Since the forces are in equilibrium, so

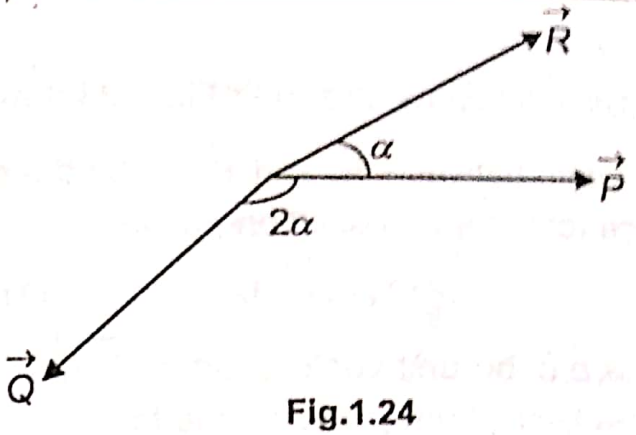


Fig.1.24

$$X = 0, \quad Y = 0$$

Now $X = 0$

$$\Rightarrow P\cos 0^\circ + R\cos\alpha + Q\cos(360^\circ - 2\alpha) = 0$$

$$\Rightarrow P + R\cos\alpha + Q\cos 2\alpha = 0$$

$$\Rightarrow P + R\cos\alpha + Q(2\cos^2\alpha - 1) = 0 \quad ...(1) \quad \because \cos 2\alpha = 2\cos^2\alpha - 1$$

Similarly, $Y = 0$

$$\Rightarrow P\sin 0^\circ + R\sin\alpha + Q\sin(360^\circ - 2\alpha) = 0$$

$$\Rightarrow 0 + R\sin\alpha - Q\sin 2\alpha = 0 \qquad \because \sin(360^\circ - 2\alpha) = -\sin 2\alpha$$

$$\Rightarrow R\sin\alpha - Q(2\sin\alpha\cos\alpha) = 0$$

$$\Rightarrow \sin\alpha(R - 2Q\cos\alpha) = 0$$

$$\Rightarrow R - 2Q\cos\alpha = 0 \qquad \because \sin\alpha \neq 0$$

$$\Rightarrow 2Q\cos\alpha = R$$

$$\Rightarrow \cos\alpha = \frac{R}{2Q} \qquad ...(2)$$

Putting this value of $\cos\alpha$ from (2) in (1), we have

$$P + R\left(\frac{R}{2Q}\right) + Q\left(2\left(\frac{R}{2Q}\right)^2 - 1\right) = 0$$

$$P + \frac{R^2}{2Q} + Q\left(2\left(\frac{R^2}{4Q^2}\right) - 1\right) = 0$$

$$P + \frac{R^2}{2Q} + \frac{R^2}{2Q} - Q = 0$$

$$P + \frac{R^2 + R^2}{2Q} - Q = 0$$

$$\frac{2R^2}{2Q} = Q - P$$

$$\frac{R^2}{Q} = Q - P$$

$$R^2 = Q(Q - P)$$

Example-16: *P is any point in the plane of a triangle ABC and D, E and F are the middle points of its sides, prove that the forces* $\vec{AP}, \vec{BP}, \vec{CP}, \vec{PD}, \vec{PE}$ *and* $\vec{PF}$ *are in equilibrium.*

Solution: All the forces are shown in Fig.1.25. Using (λ, μ) theorem, we have **PU, 2010 (BS Math)**

$$1\vec{PB} + 1\vec{PC} = (1+1)\vec{PD}$$

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$$\Rightarrow \vec{PB}+\vec{PC}=2\vec{PD} \quad ...(1)$$

Again using (λ, μ) theorem, we have

$$1\vec{PC}+1\vec{PA}=(1+1)\vec{PE}$$

$$\Rightarrow \vec{PC}+\vec{PA}=2\vec{PE} \quad ...(2)$$

Once again using (λ, μ) theorem, we have

$$1\vec{PA}+1\vec{PB}=(1+1)\vec{PF}$$

$$\Rightarrow \vec{PA}+\vec{PB}=2\vec{PF} \quad ...(3)$$

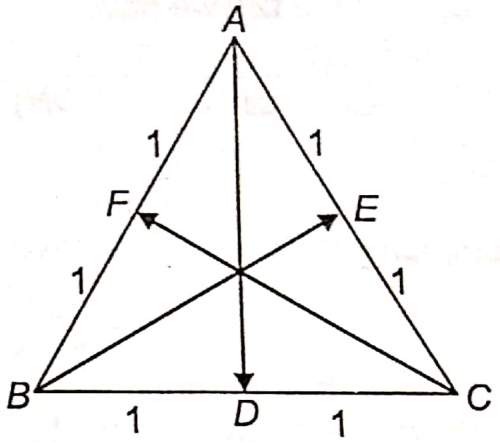


Fig.1.25

Adding (1), (2) and (3), we have

$$(\vec{PB}+\vec{PC})+(\vec{PC}+\vec{PA})+(\vec{PA}+\vec{PB})=2\vec{PD}+2\vec{PE}+2\vec{PF}$$

$$2\vec{PB}+2\vec{PC}+2\vec{PA}=2\vec{PD}+2\vec{PE}+2\vec{PF}$$

$$2(\vec{PA}+\vec{PB}+\vec{PC})=2(\vec{PD}+\vec{PE}+\vec{PF})$$

$$\vec{PA}+\vec{PB}+\vec{PC}=\vec{PD}+\vec{PE}+\vec{PF}$$

$$-\vec{AP}-\vec{BP}-\vec{CP}=\vec{PD}+\vec{PE}+\vec{PF} \qquad \because \vec{PA}=-\vec{AP}, \vec{PB}=-\vec{BP}, \vec{PC}=-\vec{CP}$$

$$\vec{AP}+\vec{BP}+\vec{CP}+\vec{PD}+\vec{PE}+\vec{PF}=0$$

This shows that the forces $\vec{AP}, \vec{BP}, \vec{CP}, \vec{PD}, \vec{PE}$ and $\vec{PF}$ are in equilibrium.

Example-17: *Three forces act perpendicularly to the sides of a triangle at their middle points and are proportional to the sides. Prove that they are in equilibrium.*

AJK, 2011 (B.A./B.Sc.)

Solution: Let ABC be the triangle, and let the forces $\vec{P}, \vec{Q}, \vec{R}$ act along the perpendicular bisectors of the sides as shown in Fig.1.26.

Suppose that the lines of action of the forces meet in a point O.

Let a, b, c be the lengths of the sides of the triangle ABC. Since the forces are proportional to sides of the triangle, so

$$\frac{P}{a}=\frac{Q}{b}=\frac{R}{c} \quad ...(1)$$

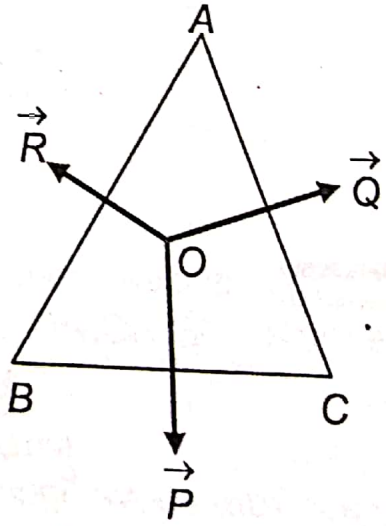


Fig.1.26

Using law of sine, we have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad ...(2)$$

Also $\angle QOR=180^\circ-A, \angle ROP=180^\circ-B, \angle POQ=180^\circ-C$

So $A=180^\circ-\angle QOR, B=180^\circ-\angle ROP, C=180^\circ-\angle POQ$

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Using these in (2), we have

$$\frac{a}{\sin(180^\circ - QOR)} = \frac{b}{\sin(180^\circ - ROP)} = \frac{c}{\sin(180^\circ - POQ)}$$

$$\frac{a}{\sin QOR} = \frac{b}{\sin ROP} = \frac{c}{\sin POQ} \quad \ldots(3)$$

Comparing (1) and (3), we have

$$\frac{P}{\sin QOR} = \frac{Q}{\sin ROP} = \frac{R}{\sin POQ}$$

This shows that the magnitude of each force is proportional to the sine of the angle between other two, so by Lami's theorem, we conclude that the forces are in equilibrium.

Example-18: *In the Fig.1.27, find the forces in the ropes AB and AD.*

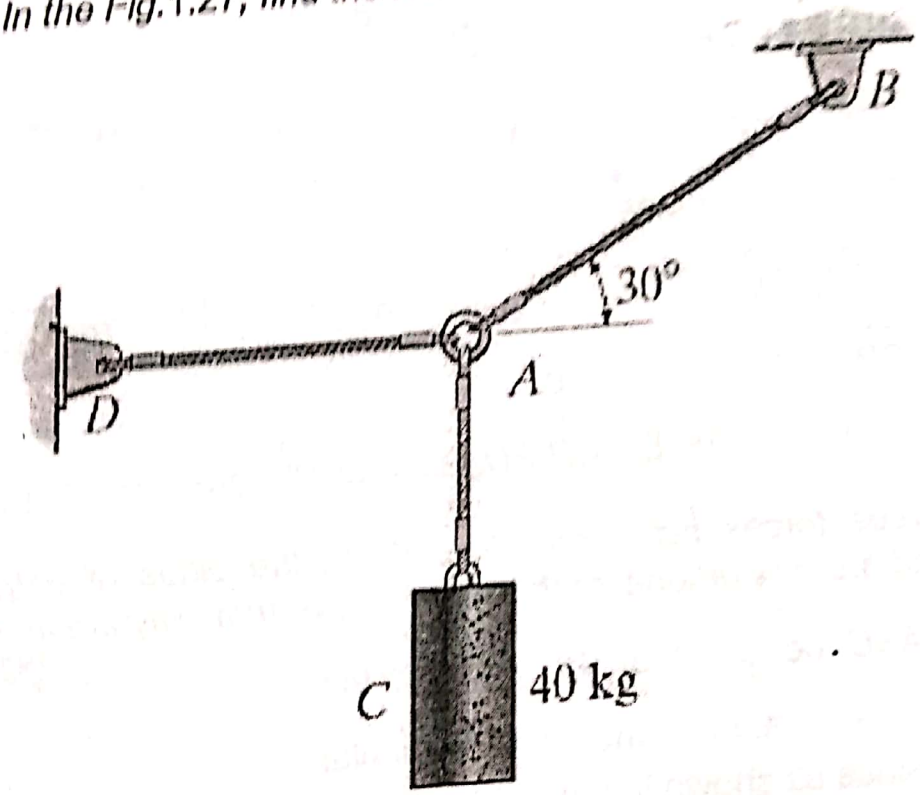


Fig.1.27

Solution: It is clear from the Fig.1.27 that ring A is in equilibrium under the action of three forces AB, AC, AD, so using Lami's theorem, we have

$$\frac{AB}{\sin DAC} = \frac{AC}{\sin DAB} = \frac{AD}{\sin CAB} \quad \ldots(1)$$

It is clear from above figure that

$$\angle DAC = 90^\circ,\ \angle DAB = 180^\circ - 30^\circ,\ \angle CAB = 90^\circ + 30^\circ$$

Using these values in (1), we have

$$\frac{AB}{\sin 90^\circ} = \frac{AC}{\sin(180^\circ - 30^\circ)} = \frac{AD}{\sin(90^\circ + 30^\circ)}$$

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$$\frac{AB}{1}=\frac{40}{\sin 30^\circ}=\frac{AD}{\cos 30^\circ} \qquad \because AC=40,\ \sin 90^\circ=1$$

$$\frac{AB}{1}=\frac{40}{\sin 30^\circ},$$

$$AB=\frac{40}{\sin 30^\circ},$$

$$AB=\frac{40}{0.5},$$

$$AB=80 \text{ kg},$$

$$\frac{AD}{\cos 30^\circ}=\frac{40}{\sin 30^\circ}$$

$$AD=40\frac{\cos 30^\circ}{\sin 30^\circ}$$

$$AD=40\frac{0.866}{0.5}$$

$$AD=69.28 \text{ kg}$$

This shows that the forces in the ropes AB and AD are 80 kg and 69.28 kg respectively.

Example-19: *In the Fig.*1.28, *find the forces in the ropes AB and AC.*

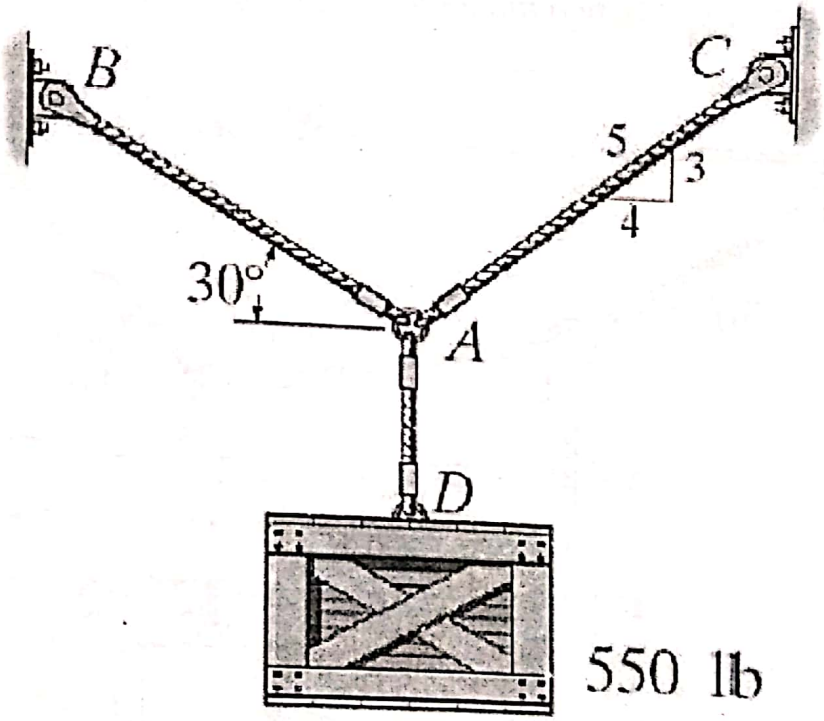


Fig.1.28

Solution: It is clear from the Fig.1.28 that box D is in equilibrium under the action of three forces AB, AC, AD, so using Lami's theorem, we have

$$\frac{AB}{\sin DAC}=\frac{AC}{\sin DAB}=\frac{AD}{\sin CAB} \qquad \ldots(1)$$

It is clear from above figure that

$$\angle DAC=90^\circ+\tan^{-1}\left(\tfrac{3}{4}\right)=90^\circ+36.87^\circ=126.87^\circ$$

$$\angle DAB=90^\circ+30^\circ=120^\circ$$

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$$\angle CAB = 180° - 30° - \tan^{-1}\left(\frac{3}{4}\right) = 180° - 30° - 36.87° = 113.13°$$

Using these values in (1), we have

$$\frac{AB}{\sin 120.87°} = \frac{AC}{\sin 120°} = \frac{AD}{\sin 113.13°}$$

$$\frac{AB}{0.8} = \frac{AC}{0.866} = \frac{550}{0.92} \quad \because AD = 550$$

$$\frac{AB}{0.8} = 597.83, \qquad \frac{AC}{0.866} = 597.83$$

$$AB = 0.8 \times 597.83, \qquad AC = 0.866 \times 597.83$$

$$AB = 478 \text{ lb}, \qquad AC = 518 \text{ lb}$$

This shows that the forces in the ropes AB and AC are 478 lb and 518 lb respectively.

Example-20: *In the Fig.1.29, find the forces in the ropes CD, CB, BA and the weight on the cylinder F.*

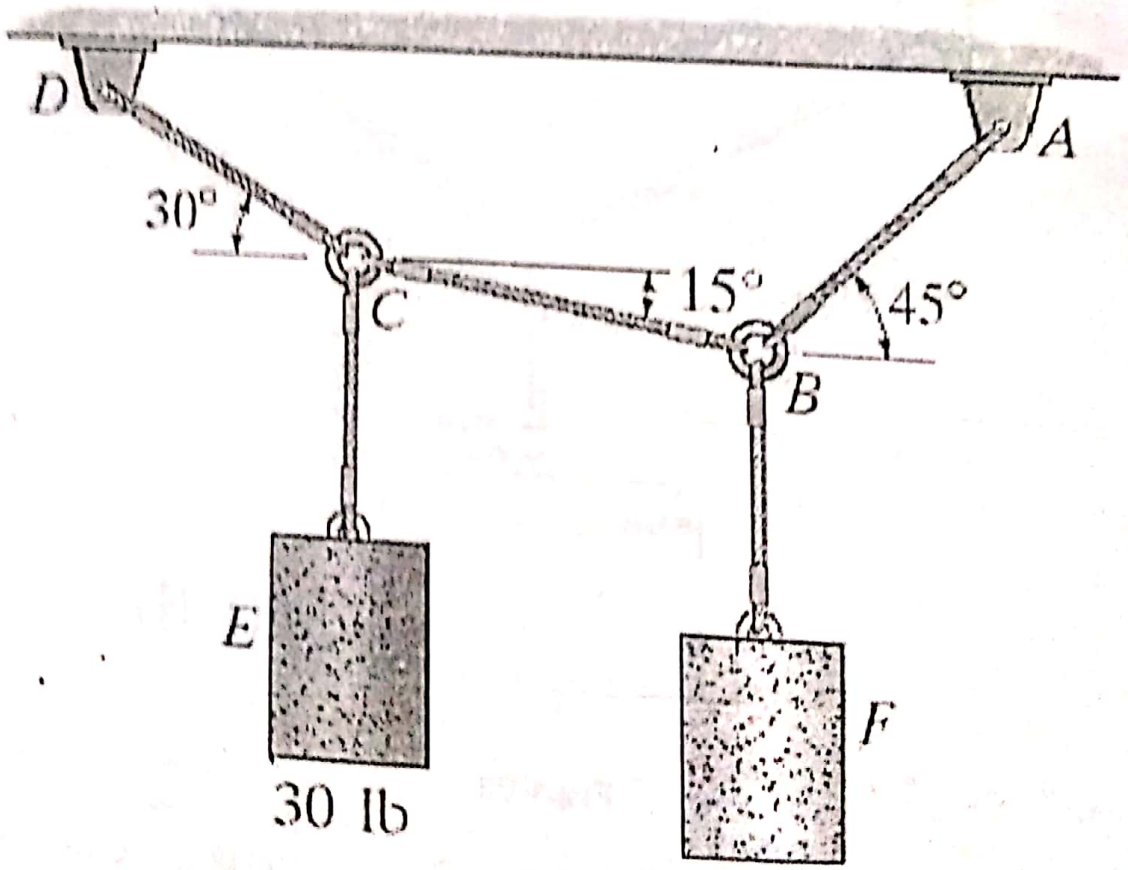


Fig.1.29

Solution: It is clear from the Fig.1.29 that ring C is in equilibrium under the action of three forces CB, CD, CE, so using Lami's theorem, we have

$$\frac{CB}{\sin DCE} = \frac{CD}{\sin ECB} = \frac{CE}{\sin DCB} \qquad \ldots(1)$$

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It is clear from above figure that

$$\angle DCE = 90° + 30° = 120°$$
$$\angle ECB = 90° - 15° = 75°$$
$$\angle DCB = 180° - 30° + 15° = 165°$$

Using these values in (1), we have

$$\frac{CB}{\sin 120°} = \frac{CD}{\sin 75°} = \frac{CE}{\sin 165°}$$

$$\frac{CB}{0.866} = \frac{CD}{0.97} = \frac{30}{0.26} \qquad \because CE = 30$$

$$\frac{CB}{0.866} = \frac{30}{0.26}, \qquad \frac{CD}{0.97} = \frac{30}{0.26}$$

$$CB = 0.866 \times 115.38, \qquad CD = 0.97 \times 115.38$$

$$CB = 100 \text{ lb}, \qquad CD = 112 \text{ lb}$$

This shows that the forces in the ropes CB and CD are 100 lb and 112 lb respectively.

It is clear from the Fig.1.29 that ring B is in equilibrium under the action of three forces BA, BC, BF, so using Lami's theorem, we have

$$\frac{BA}{\sin CBF} = \frac{BC}{\sin FBA} = \frac{BF}{\sin CBA} \qquad ...(2)$$

It is clear from above figure that

$$\angle CBF = 90° + 15° = 105°$$
$$\angle FBA = 90° + 45° = 135°$$
$$\angle CBA = 180° - 15° - 45° = 120°$$

Using these values in (2), we have

$$\frac{BA}{\sin 105°} = \frac{BC}{\sin 135°} = \frac{BF}{\sin 120°}$$

$$\frac{BA}{0.97} = \frac{100}{0.707} = \frac{BF}{0.866} \qquad \because CB = BC = 100$$

$$\frac{BA}{0.97} = \frac{100}{0.707}, \qquad \frac{BF}{0.866} = \frac{100}{0.707}$$

$$BA = \frac{0.97 \times 100}{0.707}, \qquad BF = \frac{0.866 \times 100}{0.707}$$

$$BA = \frac{0.97 \times 100}{0.707}, \qquad BF = \frac{0.866 \times 100}{0.707}$$

$$BA = 137 \text{ lb}, \qquad BF = 122.5 \text{ lb}$$

This shows that the force in the rope BA is 137 lb and the weight on the cylinder F is 122.5 lb.

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1-6 Moment of a Force about a Point

Moment: The tendency of a force to turn a body about a point is called the *moment* of that force.

Let a force $\vec{F}$ acting at a point A of a body tend to turn the body about the point O as shown in Fig.1.30.

Let $\vec{r}$ be the position vector of A with respect to the fixed reference point O.

The moment of the force $\vec{F}$ about the point O is defined as the vector

$$\vec{M} = \vec{r} \times \vec{F} \qquad ...(1.14)$$

The moment $\vec{M}$ is perpendicular to the plane containing vectors $\vec{r}$ and $\vec{F}$ and its sense, determined by the right-hand rule, characterises the sense of rotation that $\vec{F}$ tends to impart to the body.

If θ is the angle between $\vec{r}$ and $\vec{F}$, then from (1.14), we have

$$|\vec{M}| = |\vec{r} \times \vec{F}|$$

$$M = |\vec{r}||\vec{F}| \sin\theta$$

$$M = Fr\sin\theta \qquad ...(1)$$

Let d be the perpendicular from O to the line of action of force as shown in Fig.1.30. OAA' is a right triangle with $OA' = d$ perpendicular and $OA = r$ hypotenuse, so

$$\frac{d}{r} = \sin\theta$$

$$\Rightarrow d = r\sin\theta$$

Putting this value of $r\sin\theta$ in (1), we have

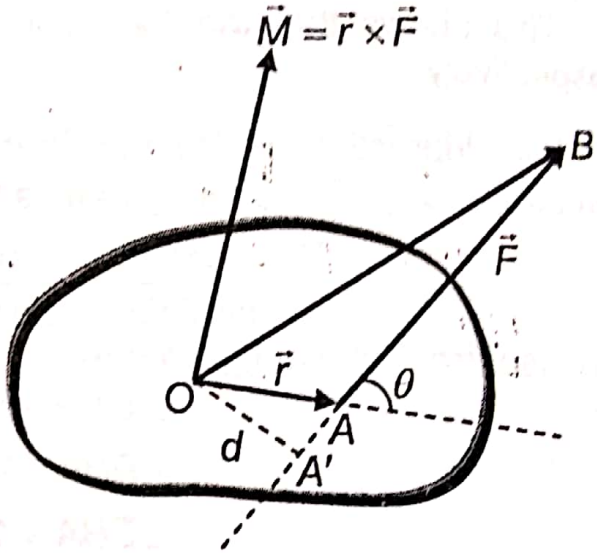


Fig.1.30

$$M = Fd \qquad ...(1.15)$$

This shows that:

> The magnitude of the moment of a force is the product of the magnitude of the force and the perpendicular distance from the point of rotation to the line of action of the force.

Arm of the Force: The perpendicular distance from the point of rotation to the line of action of the force is known as *arm of the force*.

Since d is the perpendicular distance from O (the point of rotation) to the line of action of the force in Fig.1.30, so d is arm of the force.

We conclude that:

> The magnitude of the moment of a force is the product of the magnitude of the force and arm of the force.

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Note:

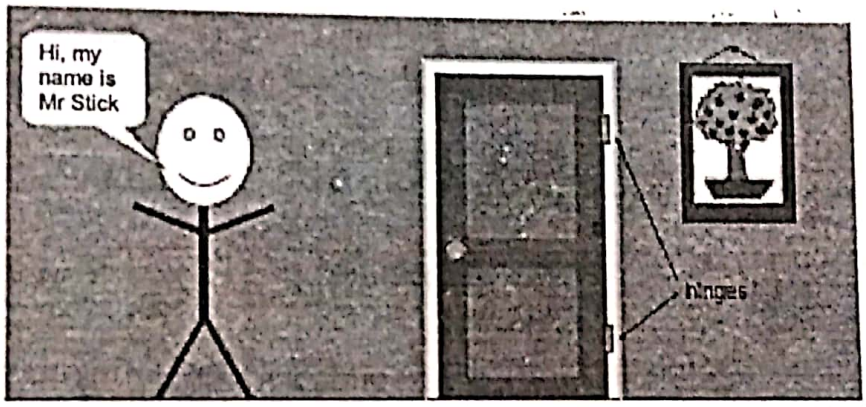
> The moment of a force about a point depends upon the magnitude of the force and the distance of the force from the point of rotation.

It is noted that if line of action of the force passes through the point O, then $\vec{r} = 0$, consequently $\vec{M} = 0$. We conclude that:

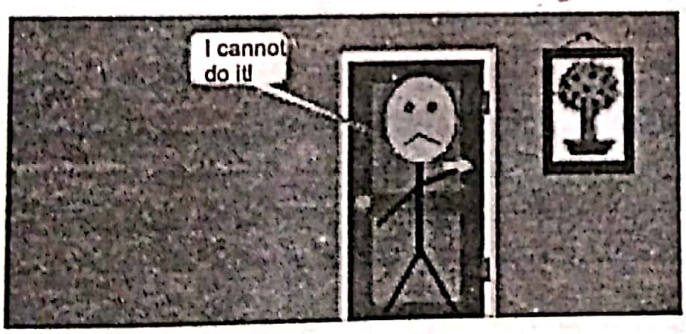
> The moment of a force about a point O is zero if the line of action of the force passes through the point O.

We explain this fact from a daily life example:

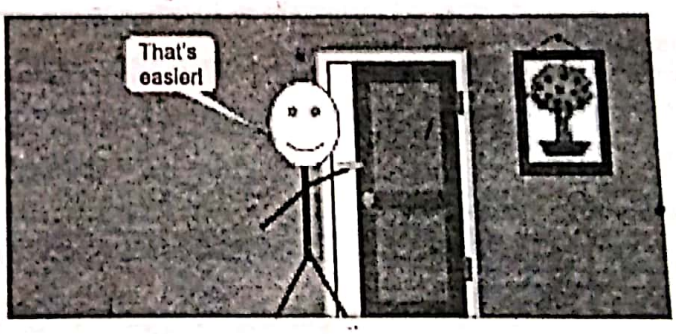
What happens when you try to open a door with one finger? Where do you push?



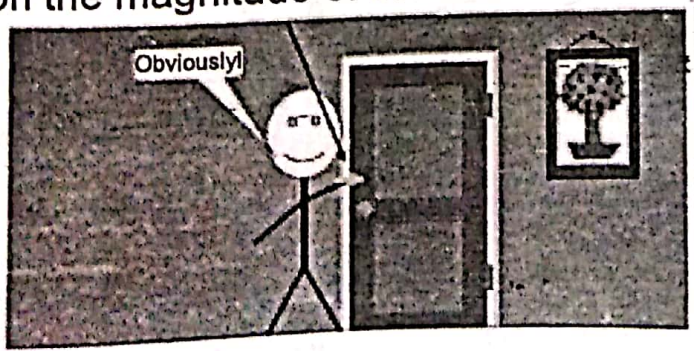
Do you push near the hinges?



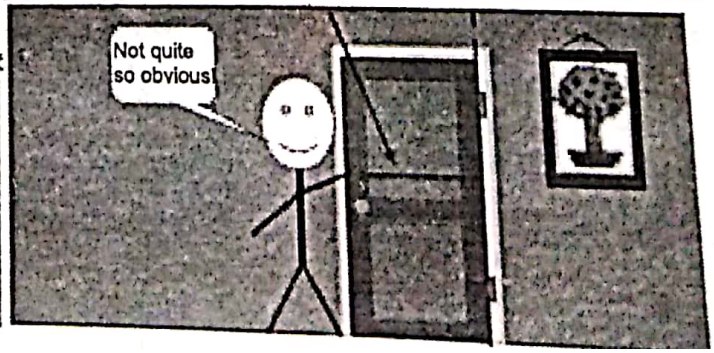
Do you push far from the hinges?



The turning effect of a force depends on the magnitude of the force.



The turning effect of a force depends on the distance from the axis of rotation.



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Example-21: *Find moments of force applied through the spanner to open the nut as shown in Fig.1.31(a) and Fig.1.31(b). What is your conclusion?*

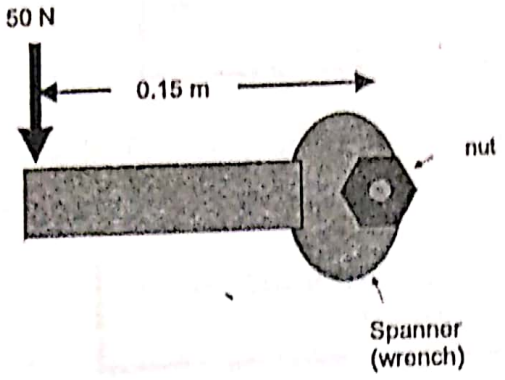


Fig.1.31(a)

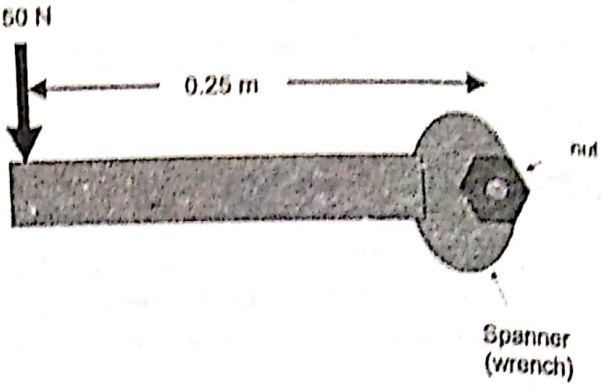


Fig.1.31(b)

Solution: It is clear from Fig.1.31(a) that

$F = 50$ N

$d = 0.15$ m

Required moment is

$M = Fd$

$M = 50 \times 0.15$

$M = 7.5$ Nm

It is clear from Fig.1.31(b) that

$F = 50$ N

$d = 0.25$ m

Required moment is

$M = Fd$

$M = 50 \times 0.25$

$M = 12.5$ Nm

We conclude that if arm of the force is greater then the moment of the force is greater.

Note:

Moment is positive if rotation is anticlockwise.

Moment is negative if rotation is clockwise.

Varignon's Theorem: *The moment about a point O of the resultant of a system of concurrent forces is equal to the sum of the moments of the various forces about the same point O.*

PU, 2011; 2010 (BS Math); AJK, 2012 (B.A/B.Sc.)

Proof: Let the system of concurrent forces $\vec{P}_1, \vec{P}_2, .., \vec{P}_n$ act at a point A as shown in Fig.1.32. Let $\vec{r}$ be the position vector of point A from O. If $\vec{R}$ is the resultant of forces $\vec{P}_1, \vec{P}_2, .., \vec{P}_n$, then

$$\vec{R} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + .. + \vec{P}_n \quad ...(1)$$

Taking cross product with $\vec{r}$, we have

$$\vec{r} \times \vec{R} = \vec{r} \times \vec{P}_1 + \vec{r} \times \vec{P}_2 + \vec{r} \times \vec{P}_3 + .. + \vec{r} \times \vec{P}_n$$

$$\vec{M} = \vec{M}_1 + \vec{M}_2 + \vec{M}_3 + .. + \vec{M}_n \quad ...(2)$$

Where $\vec{M}, \vec{M}_1, \vec{M}_2, \vec{M}_3, ..., \vec{M}_n$ are moments of $\vec{R}, \vec{P}_1, \vec{P}_2, .., \vec{P}_n$ respectively.

This completes the proof.

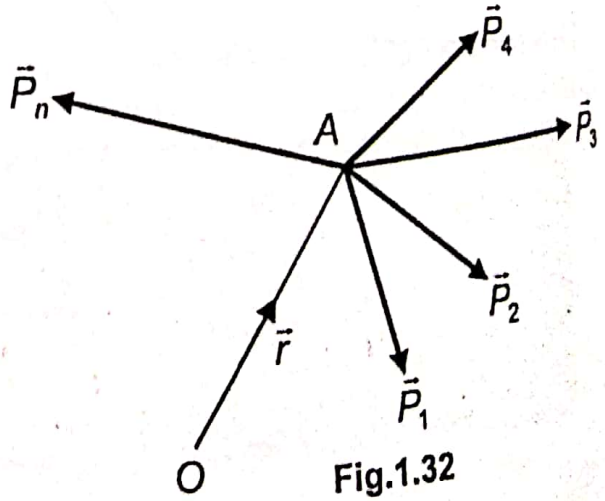


Fig.1.32

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Varignon's theorem often simplifies the calculation of the moment of a force $\vec{P}_i$ about O if we resolve the force $\vec{P}_i$ into rectangular components. Considering the two-dimensional case, let $\vec{P}_i$ have components X_i, Y_i along x-axis and y-axis as shown in Fig.1.33. Then

$$\vec{P}_i = X_i\hat{i} + Y_i\hat{j} \quad ...(1)$$

If $A_i(x_i, y_i)$ is a point on the line of action of the force $\vec{P}_i$, then the position vector $\vec{r}_i$ of A_i is

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} \quad ...(2)$$

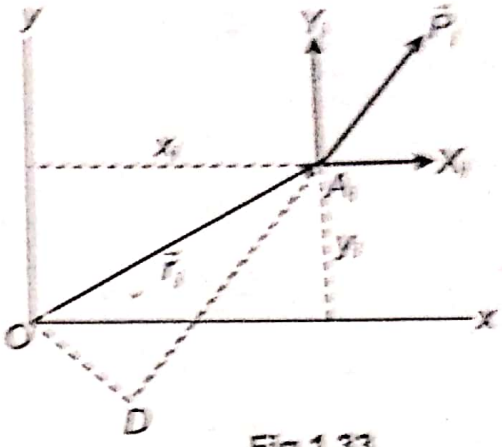


Fig.1.33

The moment of $\vec{P}_i$ about O is

$$\vec{M}_i = \vec{r}_i \times \vec{P}_i = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & 0 \\ X_i & Y_i & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(x_iY_i - y_iX_i)$$

$$= (x_iY_i - y_iX_i)\hat{k}$$

The magnitude of the moment is given by

$$M_i = x_iY_i - y_iX_i \quad ...(2)$$

Which is the sum of moments of X_i and Y_i about O, where x_iY_i is the moment of Y_i about O and $-y_iX_i$ is the moment of X_i about O.

It is clear that the moment of Y_i is positive, because it tends an anticlockwise rotation while the moment of X_i is negative, because it tends a clockwise rotation.

From above discussion, we also conclude that:

> In order to find the moment of a force about a point, we find the moments of its resolved parts along coordinate axes and then we add them.

Example-22: *Determine the moment produced by the force $\vec{F}$ about point O in Fig.1.34. Express the result as a Cartesian vector.*

Solution: It is clear from the Fig.1.34 that force acts at point A whose position vector from O is

$$\vec{r} = \overrightarrow{OA} = 12\hat{k}$$

It is also clear from figure that

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4\hat{i} + 12\hat{j}) - 12\hat{k}$$

The unit vector along $\overrightarrow{AB}$ is

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$$\hat{u}_{AB} = \frac{4\hat{i}+12\hat{j}-12\hat{k}}{\left|4\hat{i}+12\hat{j}-12\hat{k}\right|}$$

$$= \frac{4\hat{i}+12\hat{j}-12\hat{k}}{\sqrt{4^2+(12)^2+(-12)^2}}$$

$$= \frac{4\hat{i}+12\hat{j}-12\hat{k}}{\sqrt{16+144+144}}$$

$$= \frac{4\hat{i}+12\hat{j}-12\hat{k}}{\sqrt{304}}$$

$$= \frac{4\hat{i}+12\hat{j}-12\hat{k}}{17.44}$$

The given force is

$$\vec{F} = F\hat{u}_{AB} = 2\left(\frac{4\hat{i}+12\hat{j}-12\hat{k}}{17.44}\right)$$

$$= 0.4588\hat{i}+1.376\hat{j}-1.376\hat{k}$$

Moment of $\vec{F}$ about O is

$$\vec{M} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 12 \\ 0.4588 & 1.376 & -1.376 \end{vmatrix}$$

$$= \hat{i}(0-12\times1.376)-\hat{j}(0-12\times0.4588)$$

$$= -16.5\hat{i}+5.51\hat{j}\ \text{kN m}$$

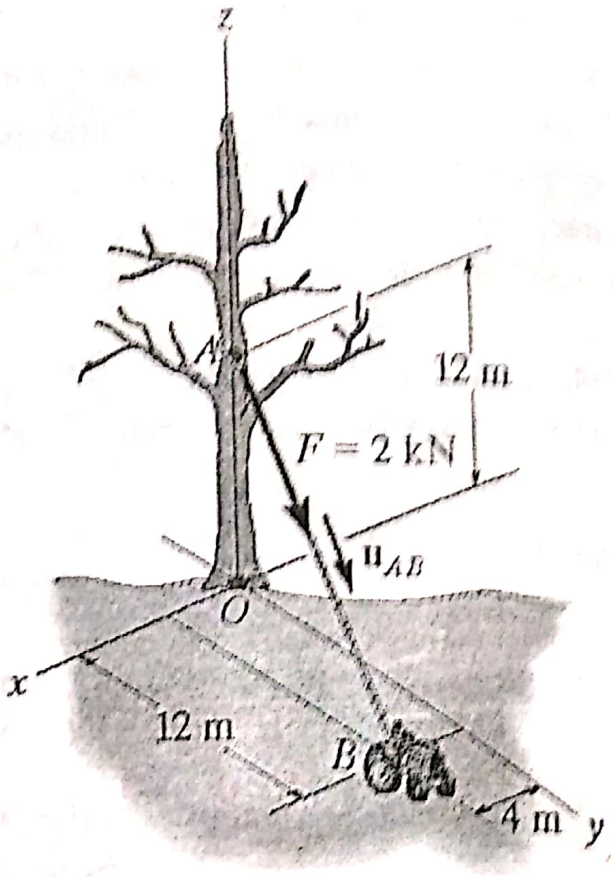


Fig.1.34

Example-23: *Two forces* $\vec{F}_1 = -60\hat{i}+40\hat{j}+20\hat{k}$, $\vec{F}_2 = 80\hat{i}+40\hat{j}-30\hat{k}$ *act on the rod shown in Fig.*1.35(a). *Determine the resultant moment they create about the flange at O. Express the result as a Cartesian vector.*

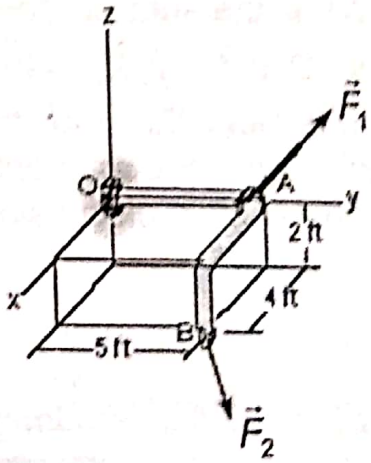


Fig.1.35(a)

Solution: Position vectors are directed from point O to each force as shown in Fig.1.35(b). These vectors are

$$\vec{r}_A = 5\hat{j}$$

$$\vec{r}_B = 4\hat{i}+5\hat{j}-2\hat{k}$$

The resultant moment about O is therefore

$$\vec{M} = \vec{M}_1 + \vec{M}_1$$

$$= \vec{r}_A\times\vec{F}_1 + \vec{r}_B\times\vec{F}_2$$

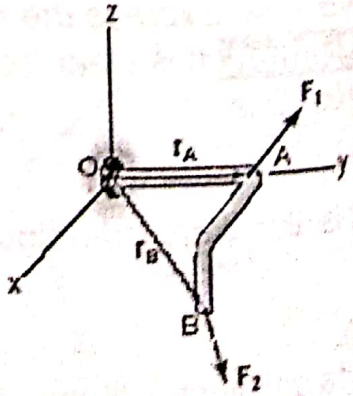


Fig.1.35(b)

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$$\vec{M} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$$= \hat{i}[5(20)-0] - \hat{j}(0-0) + \hat{k}[0-5(-60)]$$

$$+ \hat{i}[5(-30)+2(40)] - \hat{j}[4(-30)+2(80)] + \hat{k}[4(40)-5(80)]$$

$$= 100\hat{i} + 300\hat{k} - 70\hat{i} - 40\hat{j} - 240\hat{k}$$

$$= 30\hat{i} - 40\hat{j} + 60\hat{k}$$

Example-24: *Determine the moment of the force about point O in Fig.*1.36.

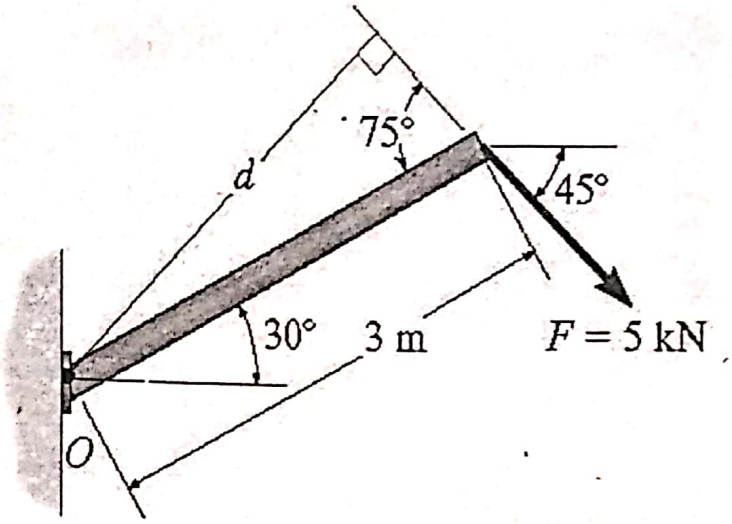


Fig.1.36

Solution: The moment arm d in Fig.1.36 can be found from trigonometry as follows:

$$d = 3\sin 75° = 3(0.97) = 2.91$$

Moment of force F about point O is given by

$$M = Fd = 5(2.91) = 14.55\ \text{kN}$$

1-7 Couples

Couple: A *couple* is defined as two parallel forces with the same magnitude but opposite in direction separated by a perpendicular distance d.

Two parallel forces $\vec{P}$ and $-\vec{P}$ with different lines of action are shown in Fig.1.37. These forces form a couple and is denoted by $(\vec{P}, -\vec{P})$.

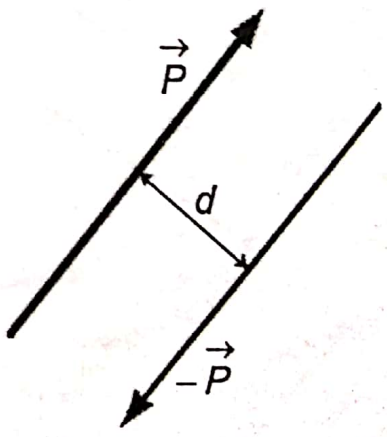
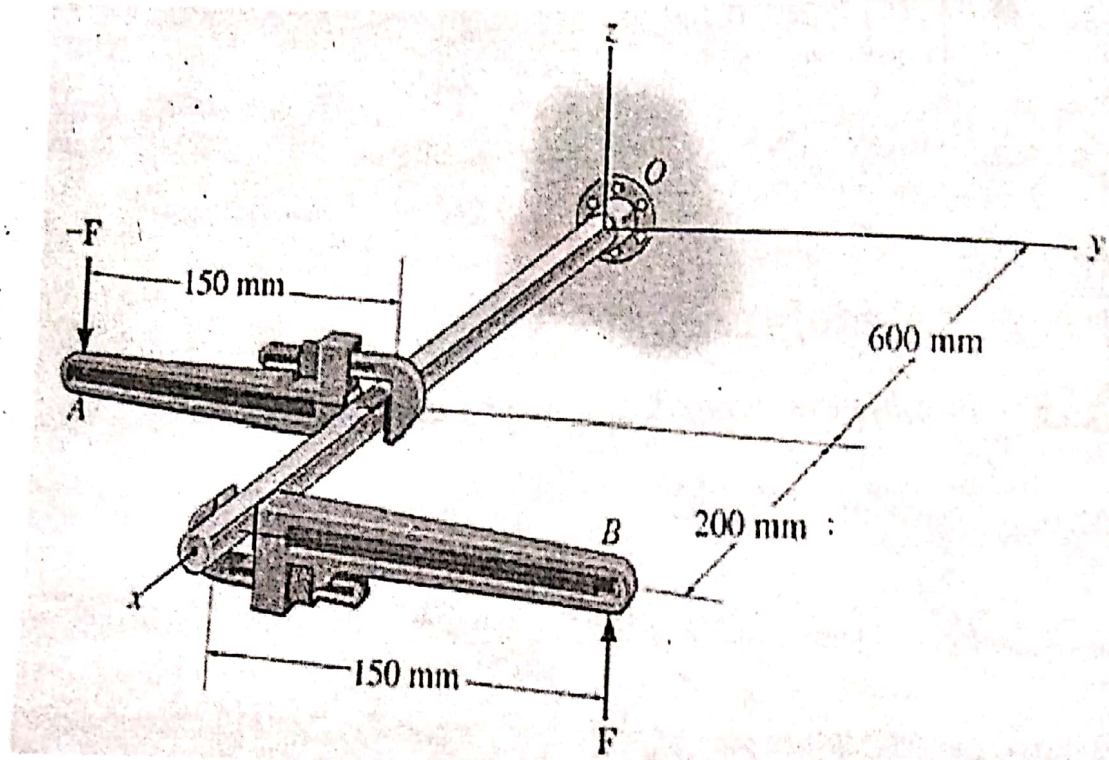


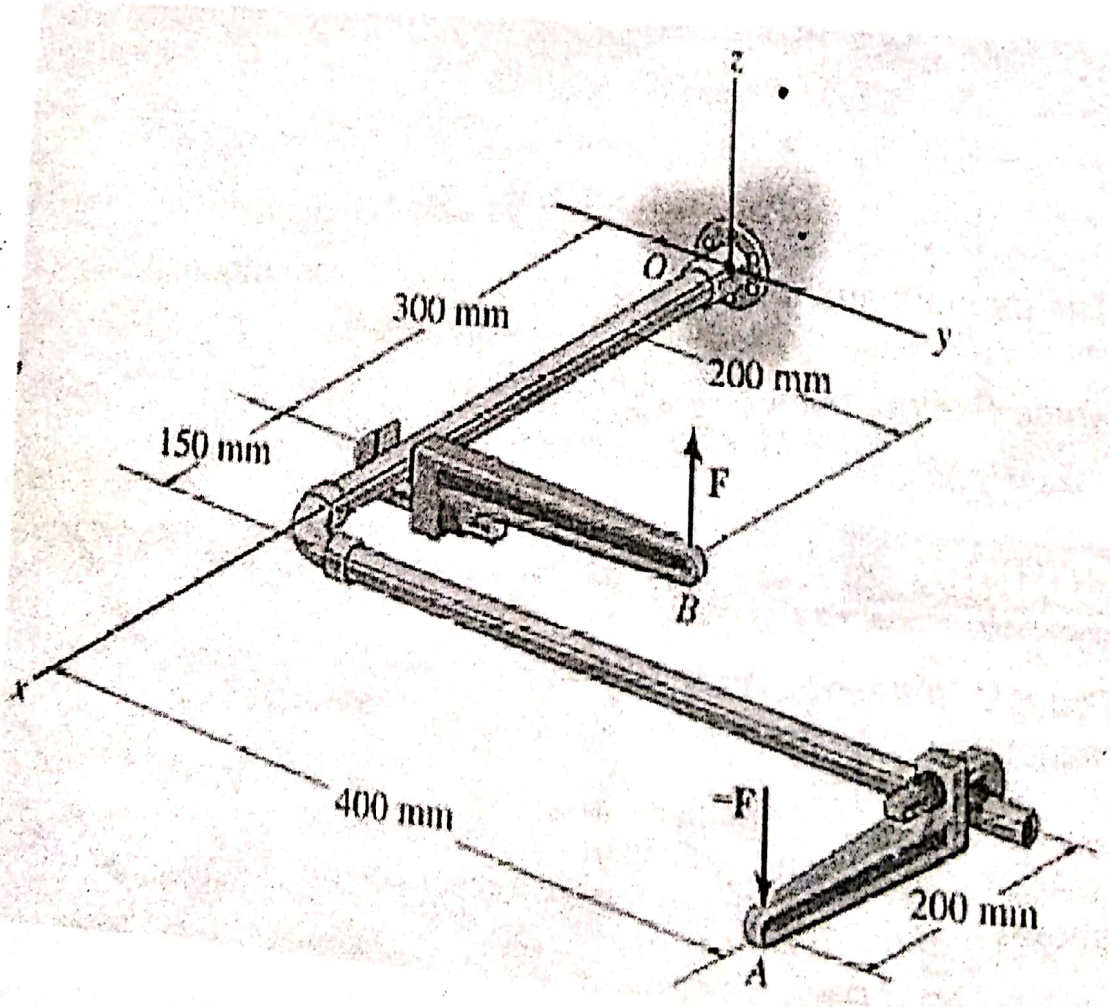
Fig.1.37

Arm of the Couple: The perpendicular distance d between the lines of actions of the forces $\vec{P}$ and $-\vec{P}$ is known as the *arm of the couple*.

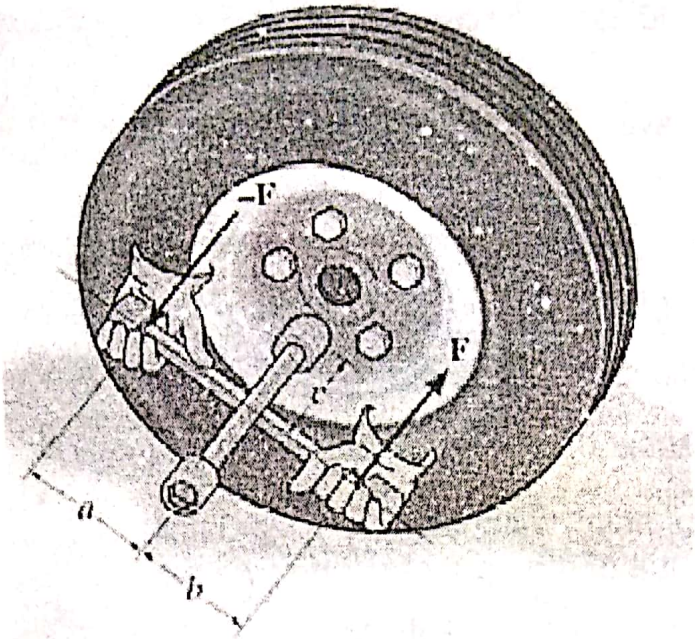
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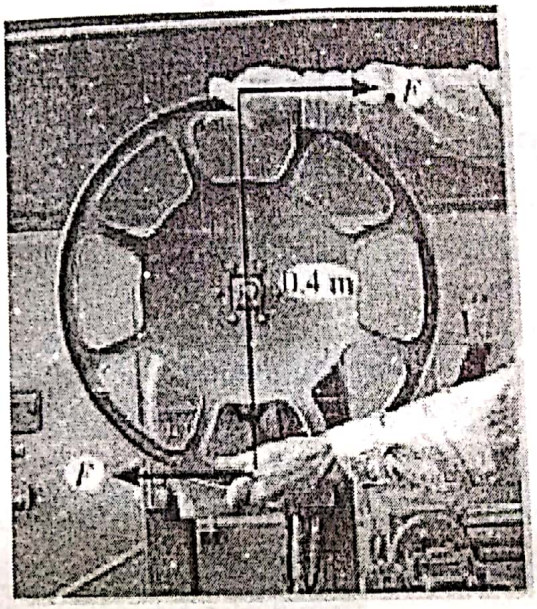
In the following figures, some couples along with their applications are shown:



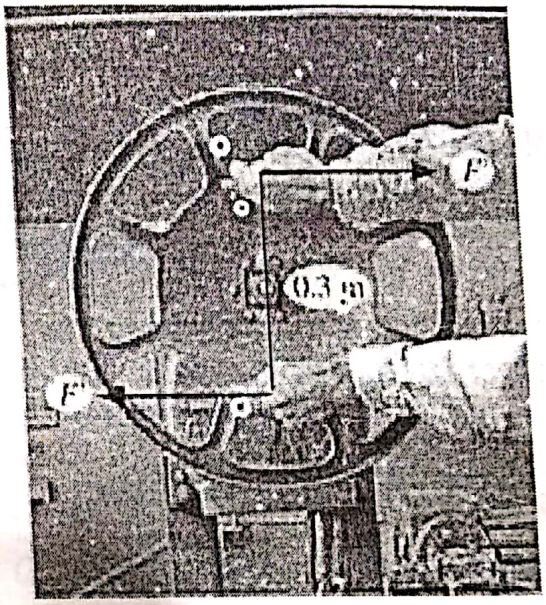


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Less force required to rotate the wheel.



Greater force required to rotate the wheel.

Theorem: *The moment of a couple about any point in the plane of forces is equal to the moment of one force of the couple about the point of application of the other force.*

Proof: Let $\vec{G}$ be the moment of the couple $(\vec{P}, -\vec{P})$ about a point O. Let $\vec{r}_1$, $\vec{r}_2$ be the position vectors of the points of applications A and B of forces $\vec{P}$ and $-\vec{P}$ respectively as shown in Fig.1.38. Then

$\vec{G}$ = Sum of moments of $\vec{P}$ and $-\vec{P}$

$$\vec{G} = \vec{r}_1 \times \vec{P} + \vec{r}_2 \times -\vec{P} = (\vec{r}_1 - \vec{r}_2) \times \vec{P} \quad ...(1)$$

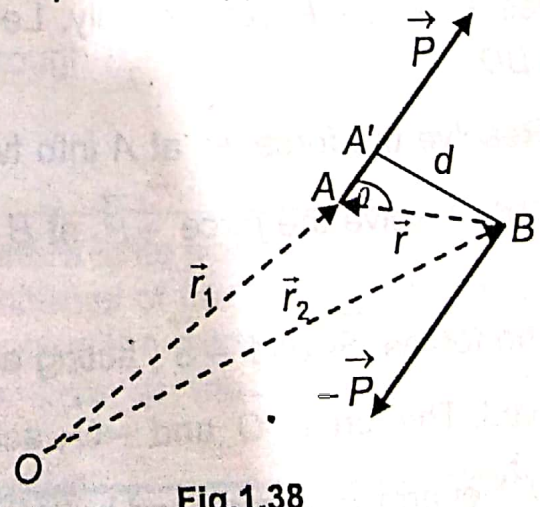


Fig.1.38

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Using law of vector addition, we have $\vec{r}_2 + \vec{r} = \vec{r}_1$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

Using this in (1), we have

$$\boxed{\vec{G} = \vec{r} \times \vec{P}} \quad \ldots(1.16)$$

This shows that the moment of the couple $(\vec{P}, -\vec{P})$ is the moment of the force $\vec{P}$ about the point of application $-\vec{P}$ (point B).

Similarly, we can show that the moment of the couple $(\vec{P}, -\vec{P})$ is the moment of the force $-\vec{P}$ about the point of application $\vec{P}$ (point A).

Next taking magnitude of moment of couple, we have

$$\left|\vec{G}\right| = \left|\vec{r} \times \vec{P}\right|$$

$$G = rP\sin\theta \quad \ldots(2)$$

In triangle ABA', $\frac{d}{r} = \sin\theta$, so $d = r\sin\theta$. Putting this value in (2), we have

$$\boxed{G = Pd} \quad \ldots(1.17)$$

This shows that the magnitude of the moment of the couple is the product of the magnitude of the force and arm of the couple.

Theorem (Equivalent Couples): *The effect of a couple upon a rigid body is unaltered if it is replaced by any other couple of the same moment lying in the same plane.*

PU, 2012; 2005; 2001; 2000 (B.A./B.Sc.)

Proof:

Let a couple $(\vec{P}, -\vec{P})$ of arm d_1 act upon a rigid body as shown in Fig.1.39. Draw two parallel lines AC and BD, where A and B are the points of applications of the forces $\vec{P}$ and $-\vec{P}$ respectively. Let d_2 be the perpendicular distance between AC and BD.

Resolve the force $\vec{P}$ at A into two components, $\vec{Q}$ along CA and $\vec{S}$ along BA. Similarly resolve the force $-\vec{P}$ at B into components $-\vec{Q}$ along BD and $-\vec{S}$ along AB.

The forces $\vec{S}$ and $-\vec{S}$, acting along the line AB, are in equilibrium and can be removed. The forces $\vec{Q}$ and $-\vec{Q}$, acting at A and B, form a couple. Thus the couple $(\vec{P}, -\vec{P})$ of arm d_1 is replaced by another couple $(\vec{Q}, -\vec{Q})$ of arm d_2.

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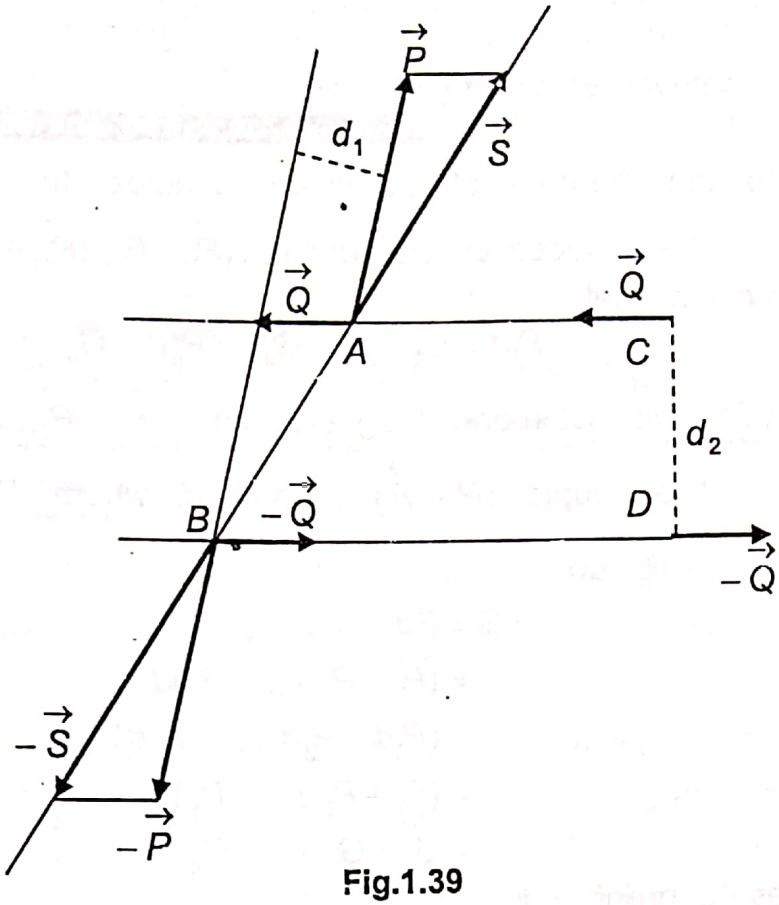


Fig.1.39

Using principle of transmissibility of forces, $\vec{Q}$ and $-\vec{Q}$ are transformed to points C and D. Since the location of the points A and B and the directions of AC and BD are arbitrary, the location of the couple $(\vec{Q}, -\vec{Q})$ in the plane is also arbitrary.

Since $\vec{P}$ is the resultant of the forces $\vec{Q}$ and $\vec{S}$, so by Varignon's theorem, the moment of force $\vec{P}$ about B is the sum of moments of forces $\vec{Q}$ and $\vec{S}$ about B, i.e.

$$Pd_1 = Qd_2$$

Where Pd_1 and Qd_2 are the moments of $\vec{P}$ and $\vec{Q}$ about B while the moment of $\vec{S}$ about B is zero as line of action of $\vec{S}$ passes through B.

This completes the proof.

From this theorem we conclude that:

1. A couple can be transferred anywhere in its plane.
2. It is possible to change the magnitudes of the forces of a couple and its arm arbitrarily without changing the moment of the couple.
3. Any two couples of equal moments in the same plane are equivalent.

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Theorem (Composition of Coplanar Couples): *Coplanar couples of moments $G_1, G_2, .., G_n$ are equivalent to a single couple lying in the same plane, whose moment G is given by $G = G_1 + G_2 + .. + G_n$.*

PU, 2013 (B.A./B.Sc.); PU, 2010 (BS Math); AJK, 2010 (B.A./B.Sc.)

Proof: Using the theorem of equivalent couples, the couples of moments $G_1, G_2, .., G_n$ can be replaced by the couples $(\vec{P}_1, -\vec{P}_1), (\vec{P}_2, -\vec{P}_2), ..., (\vec{P}_n, -\vec{P}_n)$ with a common arm d such that

$$P_1 d = G_1, P_2 d = G_2, ..., P_n d = G_n$$

The forces $\vec{P}_1, \vec{P}_2, ..., \vec{P}_n$ act in one straight line and $-\vec{P}_1, -\vec{P}_2, ..., -\vec{P}_n$ in a parallel line.

Thus we get a couple $(\vec{P}, -\vec{P})$ of arm d, where $\vec{P} = \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n$ and $-\vec{P} = -\vec{P}_1 - \vec{P}_2 - ... - \vec{P}_n$, so

$$\begin{aligned} G &= Pd \\ &= (P_1 + P_2 + ... + P_n)d \\ &= (P_1 d + P_2 d + ... + P_n d) \\ &= (P_1 + P_2 + ... + P_n)d \\ &= G_1 + G_2 + ... + G_n \end{aligned}$$

This completes the proof.

Theorem (Conditions for the Equilibrium of Couples): *The necessary and sufficient condition for the equilibrium of a system of coplanar couples is that the algebraic sum of their moments is zero.*

Proof: Let us consider a system of coplanar couples of moments $G_1, G_2 ..., G_n$. Using the theorem of equivalent couples, these couples of moments $G_1, G_2, .., G_n$ can be replaced by the couples $(\vec{P}_1, -\vec{P}_1), (\vec{P}_2, -\vec{P}_2), ..., (\vec{P}_n, -\vec{P}_n)$ with a common arm d such that

$$P_1 d = G_1, P_2 d = G_2, ..., P_n d = G_n$$

The forces $\vec{P}_1, \vec{P}_2, ..., \vec{P}_n$ act in one straight line and $-\vec{P}_1, -\vec{P}_2, ..., -\vec{P}_n$ in a parallel line.

Thus we get a couple $(\vec{P}, -\vec{P})$ of arm d, where $\vec{P} = \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n$ and $-\vec{P} = -\vec{P}_1 - \vec{P}_2 - ... - \vec{P}_n$, The moment G of couple $(\vec{P}, -\vec{P})$ is given by

$$\begin{aligned} G &= Pd \\ &= (P_1 + P_2 + ... + P_n)d \\ &= (P_1 d + P_2 d + ... + P_n d) \\ &= (P_1 + P_2 + ... + P_n)d \\ &= G_1 + G_2 + ... + G_n \end{aligned}$$

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Since moment of couple is the tendency of couple to rotate the body, so the necessary and sufficient condition for the equilibrium of a system of coplanar couples is that $G = 0$, i.e.

$$G_1 + G_2 + ... + G_n = 0$$

This shows that:

> The necessary and sufficient condition for the equilibrium of a system of coplanar couples is that the algebraic sum of their moments is zero.

Theorem (Force and a Couple):

A force $\vec{P}$ acting on a rigid body can be moved to any point O of the rigid body, provided a couple is added, whose moment is equal to the moment of $\vec{P}$ about O.

Proof:

Let the given force $\vec{P}$ act at a point A of the rigid body as shown in Fig.1.40. At the point O of the body we add two forces $\vec{P}$ and $-\vec{P}$ without altering the effect of the original force $\vec{P}$ on the body. The force $\vec{P}$ at A and $-\vec{P}$ at O form a couple $(\vec{P}, -\vec{P})$. Hence the given force $\vec{P}$ at A is equivalent to a force $\vec{P}$ at O together with a couple $(\vec{P}, -\vec{P})$ whose moment is equal to the moment about O of the force $\vec{P}$ acting at A.

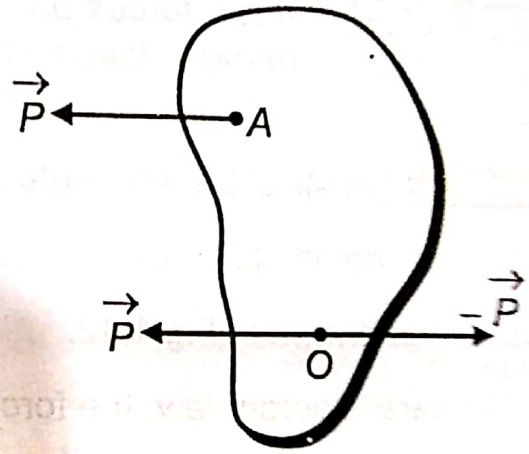


Fig.1.40

From this theorem we conclude that:

> A force can be shift to any point of the rigid body not lying on the line of action of force provided a couple is added.

Theorem (Force and a Couple): *A single force and a couple acting in the same plane upon a rigid body are equivalent to the single force acting in a direction parallel to its original direction.*

PU. 2009, 2008, 2006 (B.A./B.Sc.)

Proof:

Let a couple $(\vec{P}, -\vec{P})$, and a force $\vec{Q}$ (acting through O) as shown in Fig.1.41. Let $\vec{Q}$ meet $\vec{P}$ in O. Then $\vec{P}$, $\vec{Q}$ are equivalent to a force $\vec{P}+\vec{Q}$ acting at O along a line OC. Let OC meet the line of action of $-\vec{P}$ in O'. The point of application of $\vec{P}+\vec{Q}$

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may be shifted to O'.

The force $\vec{P}+\vec{Q}$ acting at O' can be resolved into two faces

(i) $\vec{Q}$ parallel to the original force $\vec{Q}$ and

(ii) $\vec{P}$ acting along the line of action of $-\vec{P}$.

The forces $\vec{P},-\vec{P}$ acting at O' are in equilibrium, so these cancel each other.

Therefore, the given couple $(\vec{P},-\vec{P})$, and a force $\vec{Q}$ (acting through O) is equivalent to a single force $\vec{Q}$ acting at O'.

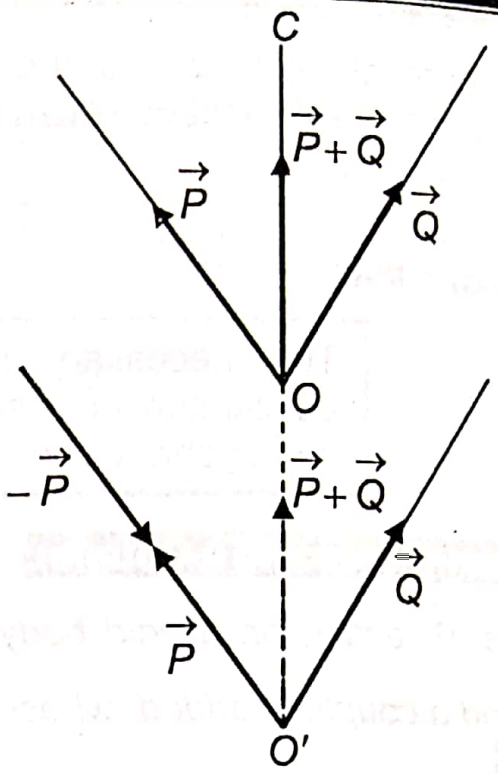


Fig.1.41

Example-25: *If three forces are represented in magnitude, direction and position by the sides of a triangle taken in order then shows that these are equivalent to a couple.*

PU, 1991 (B.A./B.Sc.)

Solution: Let ABC be a triangle and let the three forces be completely represented by $\vec{AB}$, $\vec{BC}$ and $\vec{CA}$ as shown in Fig.1.42.

By the parallelogram law, the forces $\vec{CA}$ and $\vec{AB}$ acting at A have a resultant $\vec{CB}$ which acts at A.

Thus the forces are equivalent to two forces $\vec{CB}$ acting at A and $\vec{BC}$ along the side BC of the triangle. These two forces form a couple. If d denotes the length of the perpendicular from A to BC, the magnitude of the moment of the couple is $d \times BC$ which represents twice the area of the triangle ABC.

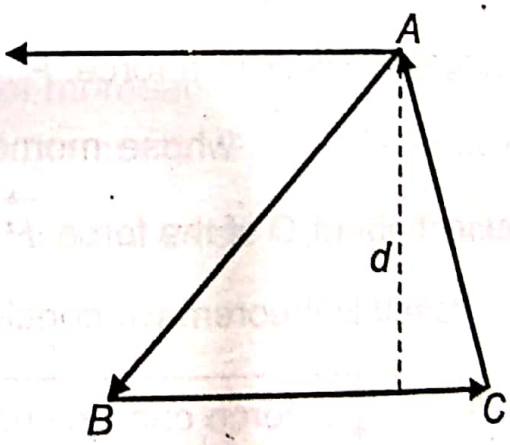


Fig.1.42

A generalisation of this result shows that

> If forces are completely represented by the sides of a plane polygon taken in order, they are equivalent to a couple whose moment is represented by twice the area of the polygon.

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Example-26: *Forces act along the sides BC, CA, AB of a triangle. Show that they are equivalent to a couple only if the forces are proportional to the sides.*

PU 2008 (B.A./B.Sc.)

Solution:

Let the forces $\lambda\vec{BC}$, $(\lambda+\mu)\vec{CA}$ and $(\lambda+\nu)\vec{AB}$ act along the sides of the triangle ABC as shown in Fig.1.43, then

$$\lambda\vec{BC}+(\lambda+\mu)\vec{CA}+(\lambda+\nu)\vec{AB}=\lambda(\vec{BC}+\vec{CA}+\vec{AB})+\mu\vec{CA}+\nu\vec{AB} \quad ...(1)$$

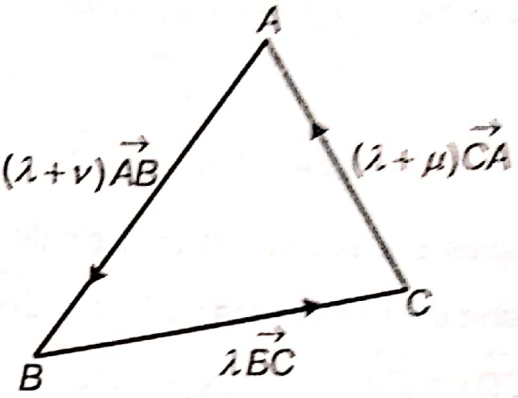


Fig.1.43

Since the forces $\vec{BC},\vec{CA},\vec{AB}$ are equivalent to a couple whose moment is twice the area of triangle ABC, so from (1), we have

$$\lambda\vec{BC}+(\lambda+\mu)\vec{CA}+(\lambda+\nu)\vec{AB}=\text{a couple}+\mu\vec{CA}+\nu\vec{AB} \quad ...(2)$$

This shows that the system is equivalent to a couple only

$$\mu\vec{CA}+\nu\vec{AB}=0$$

which holds only if $\mu=\nu=0$. Hence the forces along the sides of the triangle ABC are $\lambda\vec{BC}+\lambda\vec{CA}+\lambda\vec{AB}$. This shows that the forces are proportional to the sides of the triangle.

Example-27:

If forces $l\vec{AB}, m\vec{BC}, l\vec{CD}, m\vec{DA}$ acting along the sides of a quadrilateral are equivalent to a couple, show that either l = m or ABCD is a parallelogram.

PU, 2005; 2008 (B.A./B.Sc.)

Solution: If $ABCD$ is not a parallelogram, take a point E in its plane so that $AECD$ is a parallelogram (Fig.1.44). Now the force $\vec{AB}$ along AB is equivalent to a force $\vec{AB}$ acting at E together with a couple. But by the parallelogram

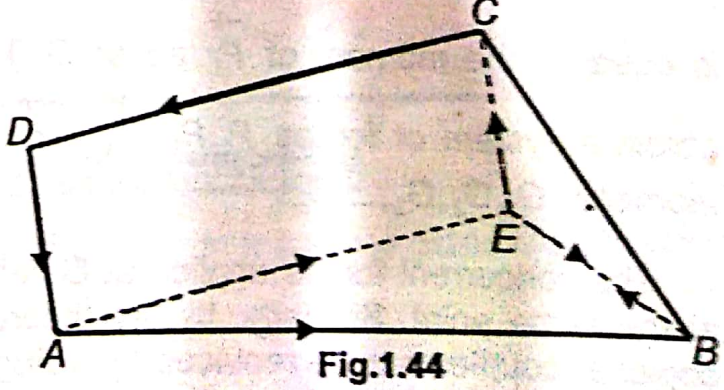


Fig.1.44

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Now, the force $\vec{AB}$ at E is the resultant of $\vec{AE}$ and $\vec{EB}$. Hence

$$\vec{AB} = \vec{AE} + \vec{EB} + \text{a couple}$$

Similarly,

$$\vec{BC} = \vec{BE} + \vec{EC} + \text{a couple}$$

Hence

$$l\vec{AB} + m\vec{BC} + l\vec{CD} + m\vec{DA} = l(\vec{AE} + \vec{EB} + \text{a couple}) + m(\vec{BE} + \vec{EC} + \text{a couple}) + l\vec{CD} + m\vec{DA}$$

$$= l(\vec{AE} + \vec{CD}) + m(\vec{EC} + \vec{DA}) + (l-m)\vec{EB} + \text{a couple}$$

Since the coplanar couples are equivalent to a single couple. Now the forces $\vec{AE}$ and $\vec{CD}$ form a couple and so do the forces $\vec{EC}$ and $\vec{DA}$. Thus

$$l\vec{AB} + m\vec{BC} + l\vec{CD} + m\vec{DA} = \text{a couple} + \text{a couple} + (l-m)\vec{EB} + \text{a couple}$$

$$= \text{a couple} + (l-m)\vec{EB}$$

$$= \text{a couple}$$

If $(l-m)\vec{EB} = 0$, i.e. either $l-m=0$ or $\vec{EB}=0$

This shows that either $l = m$ or $ABCD$ is a parallelogram.

Reduction of a System of Coplanar Forces to One Force and One Couple

Theorem: *Any system of coplanar forces acting on a rigid body can be reduced to a single force at all arbitrary point in the plane of the forces together with a couple.*

P.U. 1988 (B.A/B.Sc.)

Proof:

Consider forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, ..., \vec{P_n}$ acting in one plane upon a rigid body at the points $A_1, A_2, A_3, ..., A_n$. Let O be any point coplanar with the forces. Any force $\vec{P_i}$ acting at A_i is equivalent to a force $\vec{P_i}$ acting at O together with a couple whose moment G_i is equal to the moment of $\vec{P_i}$ about O. Transferring all the forces to act at O we obtain a system of forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, ..., \vec{P_n}$ acting at O and a system of couples of moments $G_1, G_2, G_3, ..., G_n$.

The concurrent forces applied at O can be replaced by their resultant $\vec{R}$ acting at the same point. Similarly, by the theorem of the composition of couples, all the coplanar couples can be replaced by a single couple of moment G.

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The force $\vec{R}$ and the couple G of the equivalent system are given by the equations

$$\vec{R} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \ldots + \vec{P}_n$$

$$G = G_1 + G_2 + G_3 + \ldots + G_n$$

Depending on the values of $\vec{R}$ and G, some particular cases arise:

1. If $\vec{R} = 0$ and $G = 0$, the system is in equilibrium.
2. If $\vec{R} = 0$ and $G \neq 0$, the system is equivalent to a couple whose moment does not depend on the particular single origin.
3. If $\vec{R} \neq 0$, the system can be reduced to a resultant force. Two cases are possible;

   (a) $\vec{R} \neq 0, G = 0$

   Here the system is equivalent to a single force $\vec{R}$ through the point O:

   (b) $\vec{R} \neq 0, G \neq 0$

   In this case we represent the force $\vec{R}$ by $\vec{OB}$ (Fig.1.45). We can replace the couple of moment G by another couple of forces $\vec{R}$, $-\vec{R}$ at a distance d apart such that $Rd = G$, and place them so that one of them balances the force $\vec{R}$ along OB. Then the resultant of the force $\vec{R}$ and the couple of moment G is the force $\vec{R}$ whose line of action is at a distance $d = \frac{G}{R}$ from O. These cases show that:

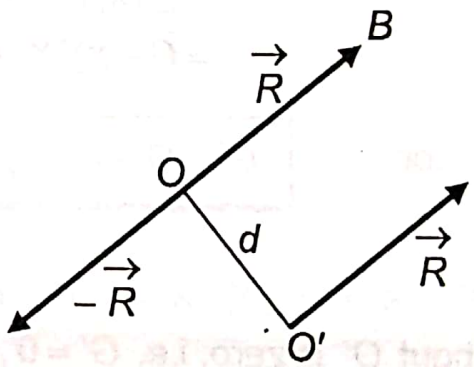


Fig.1.45

> If a coplanar system of forces is not in equilibrium, it can be reduced to either a single force (when $\vec{R} \neq 0$) or a single couple (when $\vec{R} = 0$).

Equation of Line of Action of Resultant of Coplanar Forces

Let the forces $\vec{P}_i$ act on a body at points $A_i(x_i, y_i)$ in the same plane. As shown in Fig.1.46. The moments G_i about the forces $\vec{P}_i$ about O are

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$$G_i = x_i Y_i - y_i X_i \qquad ...(1)$$

where X_i, Y_i are resolved parts of forces $\vec{P}_i$ along coordinate axes. Using Varignon's theorem, we can find the moment G of the resultant $\vec{R}$ by summing the moments in (1), i.e.

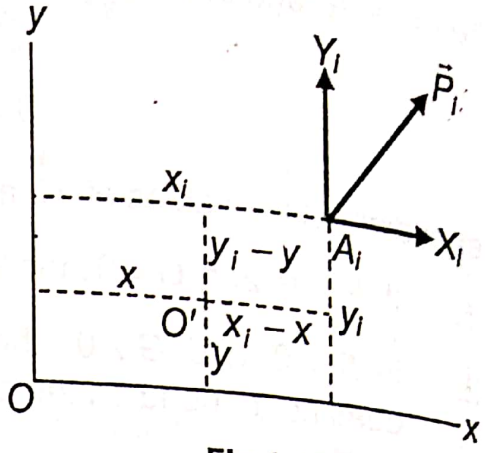


Fig.1.46

$$G = \sum G_i$$

or $$G = \sum (x_i Y_i - y_i X_i) \qquad ...(2)$$

Let $O'(x,y)$ be another point on the same plane, then the moment G' of the resultant about O' is

$$\begin{aligned} G' &= \sum [(x_i - x)Y_i - (y_i - y)X_i] \\ &= \sum (x_i Y_i - xY_i - y_i X_i + yX_i) \\ &= \sum (x_i Y_i - y_i X_i - xY_i + yX_i) \\ &= \sum (x_i Y_i - y_i X_i) + \sum (-xY_i + yX_i) \\ &= G - x\sum Y_i + y\sum X_i \end{aligned} \qquad \because G = \sum (x_i Y_i - y_i X_i)$$

or $$\boxed{G' = G - xY + yX} \qquad ...(1.18)$$

Where $Y = \sum Y_i$, $X = \sum X_i$. If the resultant passes through O', then its moment G' about O' is zero, i.e. $G' = 0$, so from (1.18), we have

or $$\boxed{G - xY + yX = 0} \qquad ...(1.19)$$

which is the equation of line of action of resultant.

We conclude that:

> If the system reduces to a single force, any point (x, y) on its line of action satisfies the equation $G - xY + yX = 0$.

Reduction of Parallel Forces:

Let $\vec{P}_1, \vec{P}_2, \vec{P}_3, ..., \vec{P}_n$ be a system of forces all parallel to a unit vector $\hat{a}$, then

$$\vec{P}_1 = k_1\hat{a},\ \vec{P}_2 = k_2\hat{a},\ \vec{P}_3 = k_3\hat{a},\ ...,\ \vec{P}_n = k_n\hat{a}$$

where k_i is a +ve or –ve constant according as $\vec{P}_i$ has the same sense as $\hat{a}$ or it has the opposite sense. The sum, $\vec{R}$, of these forces is given by

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$$\vec{R} = \vec{P_1} + \vec{P_2} + \vec{P_3} + ... + \vec{P_n}$$
$$= k_1\hat{a} + k_2\hat{a} + k_3\hat{a} + ... + k_n\hat{a}$$
$$= (k_1 + k_2 + k_3 + ... + k_n)\hat{a}$$
$$= k\hat{a}$$

Where $k = k_1 + k_2 + k_3 + ... + k_n$

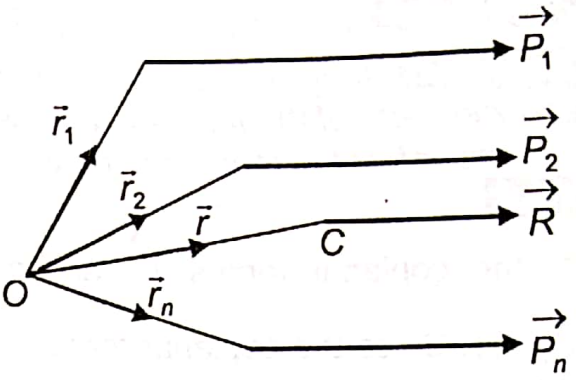


Fig.1.47

Let the position vectors of the points of application of the forces $\vec{P_i}$, relative to an origin O be $\vec{r_i}$ as shown in Fig.1.47. Then the total moment $\vec{G}$ of these forces about O is

$$\vec{G} = \vec{r_1} \times \vec{P_1} + \vec{r_2} \times \vec{P_2} + \vec{r_3} \times \vec{P_3} + ... + \vec{r_n} \times \vec{P_n}$$
$$= \vec{r_1} \times k_1\hat{a} + \vec{r_2} \times k_2\hat{a} + \vec{r_3} \times k_3\hat{a} + ... + \vec{r_n} \times k_n\hat{a}$$
$$= (k_1\vec{r_1} + k_2\vec{r_2} + k_3\vec{r_3} + ... + k_n\vec{r_n}) \times \hat{a}$$
$$= \frac{k_1\vec{r_1} + k_2\vec{r_2} + k_3\vec{r_3} + ... + k_n\vec{r_n}}{k} \times k\hat{a}$$

or $$\boxed{\vec{G} = \vec{r} \times \vec{R}} \quad ...(1.20)$$

Where

$$\boxed{\vec{r} = \frac{k_1\vec{r_1} + k_2\vec{r_2} + k_3\vec{r_3} + ... + k_n\vec{r_n}}{k}} \quad ...(1.21)$$

Thus, in general, (unless $k \neq 0$) the system of parallel forces $\vec{P_1}, \vec{P_2}, \vec{P_3}, ..., \vec{P_n}$ is equivalent to a force $\vec{R}$ parallel to the forces. It is clear from equation (1.20) that the force $\vec{R}$ acts at a point C whose position vector $\vec{r}$ is given by equation (1.21). It may be noticed that if $k = 0$, the force $\vec{R}$ vanishes and the point $\vec{r}$ recedes to infinity. In such a case, the expression (1.20) becomes indeterminate and $\vec{G}$ may have a finite value. If $\vec{G} \neq 0$, the forces reduce to a couple of moment $\vec{G}$. It is interesting to notice that the position vector $\vec{r}$ depends only on k_i and $\vec{r_i}$.

Thus if the forces are turned about their points of application through the same angle so that they remain parallel, the vector $\vec{r}$ remains the same and thus the position of C is not altered. The point C is known as the *centre of the parallel forces.*

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Conditions of Equilibrium of a Coplanar Force System:

Theorem-1: *A system of coplanar forces is in equilibrium if the algebraic sum of the resolved parts of the forces in any two directions is zero and the algebraic sum of the moments of all the forces about an arbitrary point is zero.*

Proof:

Let the coplanar forces $\vec{P}_i$ acting at the points $A_i(x_i, y_i)$ have resolved parts (X_i, Y_i). Since the coplanar forces can be reduced to a single force $\vec{R}$ acting at an arbitrary point of the plane of the forces together with a couple G, so coplanar forces are in equilibrium if

$$\vec{R} = 0 \quad \text{and} \quad G = 0 \qquad \ldots(1)$$

Since $\vec{R} = X\hat{i} + Y\hat{j} = \sum X_i \hat{i} + \sum Y_i \hat{j}$

and $G = \sum (x_i Y_i - y_i X_i)$

Using these values in (1), we have

$$\boxed{\begin{aligned} X &= \sum X_i = 0 \\ Y &= \sum Y_i = 0 \\ G &= \sum (x_i Y_i - y_i X_i) = 0 \end{aligned}} \qquad (1.22)$$

This completes the proof that a system of coplanar forces is in equilibrium if the algebraic sum of the resolved parts of the forces in any two directions is zero and the algebraic sum of the moments of all the forces about an arbitrary point is zero.

Note: Equations (1.22) are known as the *basic equations of equilibrium.*

Theorem-2: *A system of coplanar forces is in equilibrium if the algebraic sum of moments of all the forces about two different points O and A is zero and the algebraic sum of the resolved parts in some one direction, not perpendicular to OA, is zero.*

Proof: Let us take O as the origin and the direction as the x-axis. Let A have the rectangular coordinates (x, y). Since OA is not perpendicular to the x-axis, $x \neq 0$. If X, Y are the algebraic sums of the resolved parts of the forces and G and G' be the algebraic sum or their moments about O and A respectively, then

$$G' = G - xY + yX \qquad \ldots(1)$$

Now we are given that

$$G = 0$$

$$G' = G - xY + yX = 0$$

$$X = 0$$

These equations give

$$X = 0,\ Y = 0,\ G = 0$$

This shows that that the forces are in equilibrium.

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Theorem-3: *A system of coplanar forces is in equilibrium if the algebraic sum of the moments of all the forces about three different points O, A, B, not in the same straight line, are separately zero.*

Proof: Let O be the origin and OA the x-axis. Let A and B have the coordinates $(x,0)$ and (x',y') respectively. Since B does not lie on the x-axis, $y' \neq 0$. Let X, Y be the sums of the resolved parts and G, G' and G'' be the moments of the forces about O, A and B respectively. Then

$$G' = G - xY \quad ...(1)$$

$$G'' = G - x'Y + y'X \quad ...(2)$$

Now we are given that

$$G = 0,\ G' = 0,\ G'' = 0$$

Putting $G = 0,\ G' = 0$ in (1), we have

$$-xY = 0$$

$$\Rightarrow Y = 0 \qquad \because x \neq 0$$

Putting $G = 0,\ G'' = 0,\ Y = 0$ in (2), we have

$$y'X = 0$$

$$\Rightarrow X = 0 \qquad \because y' \neq 0$$

Thus, we have

$$X = 0,\ Y = 0,\ G = 0$$

This shows that that the forces are in equilibrium.

Example-28: *Forces of magnitude P, $2P$, $3P$, $4P$ act respectively along the sides AB, BC, CD, DA of a square $ABCD$, of side a, and forces each of magnitude $(8\sqrt{2})P$ act along the diagonals BD, AC. Find the magnitude of the resultant force and the distance of its line of action from A.*

PU, 1999 (B.A./B.Sc.)

Solution: Take AB, AD as x-axis and y-axis respectively as shown in Fig.1.48. Resolving all the forces at A, we note that the forces P, $(8\sqrt{2})P$, $2P$, $3P$, $(8\sqrt{2})P$, and $4P$ make angles $0°$, $45°$, $90°$, $135°$, $180°$, $270°$ with positive x-axis. If X, Y are the resolved parts of the single force and G is the moment of the couple, then we have

$$X = P\cos 0° + (8\sqrt{2})P\cos 45° + 2P\cos 90° + (8\sqrt{2})P\cos 135° + 3P\cos 180° + 4P\cos 270°$$

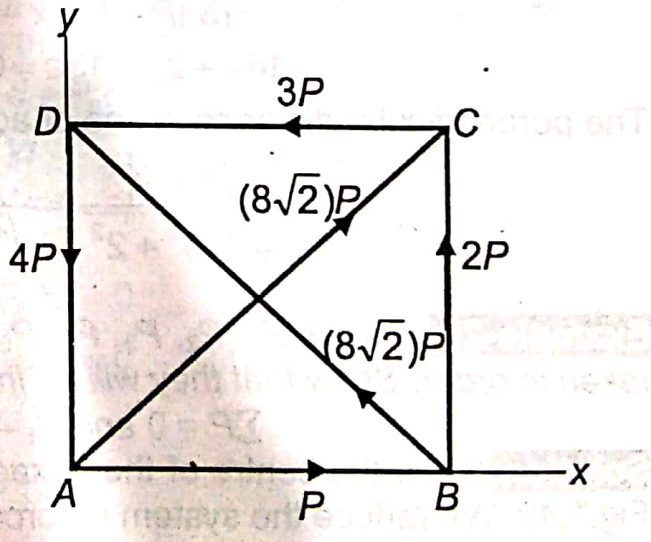


Fig.1.48

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$$X = P + (8\sqrt{2})P\left(\frac{1}{\sqrt{2}}\right) + 0 + (8\sqrt{2})P\left(-\frac{1}{\sqrt{2}}\right) + 3P(-1) + 0$$

$$X = P + 8P - 8P - 3P$$

$$X = -2P$$

Similarly,

$$Y = P\sin 0° + (8\sqrt{2})P\sin 45° + 2P\sin 90° + (8\sqrt{2})P\sin 135° + 3P\sin 180° + 4P\sin 270°$$

$$Y = 0 + (8\sqrt{2})P\left(\frac{1}{\sqrt{2}}\right) + 2P(1) + (8\sqrt{2})P\left(\frac{1}{\sqrt{2}}\right) + 0 + 4P(-1)$$

$$Y = 8P + 2P + 8P - 4P$$

$$Y = 14P$$

Hence magnitude of the resultant is

$$R = \sqrt{X^2 + Y^2}$$

$$= \sqrt{(-2P)^2 + (14P)^2} = \sqrt{4P^2 + 196P^2}$$

$$= \sqrt{200P^2} = 10\sqrt{2}P$$

Now

G = Sum of the moments of the forces about A

$$= (8\sqrt{2})P.\frac{a}{\sqrt{2}} + 2P.a + 3P.a$$

$$= 8Pa + 2Pa + 3Pa$$

$$= 13Pa$$

The equation of the line of action of the resultant is

$$G - xY + yX = 0$$

$$13aP - 14Px - 2Py = 0 \qquad \because G = 13aP,\ X = -2P,\ Y = 14P$$

$$14x + 2y - 13a = 0 \qquad \ldots(1)$$

The perpendicular distance of line of action of resultant from A is

$$\frac{|-13a|}{\sqrt{(14)^2 + 2^2}} = \frac{13a}{\sqrt{196 + 4}} = \frac{13a}{\sqrt{200}} = \frac{13a}{10\sqrt{2}}$$

Example-29: *Forces P_1, P_2, P_3, P_4, P_5, P_6 act along the sides of a regular hexagon taken in order. Show that they will be in equilibrium if*

$$\Sigma P = 0 \text{ and } P_1 - P_4 = P_3 - P_6 = P_5 - P_2$$

Solution: Take the centre of the hexagon as origin O and axes Ox, Oy as shown in Fig.1.49. We reduce the system of forces at O. The forces P_1, P_2, P_3, P_4, P_5, P_6 make angles 0°, 60°, 120°, 180°, 240°, 300° with positive x-axis. Sum of resolved parts of all forces along x-axis is

$$X = P_1\cos 0° + P_2\cos 60° + P_3\cos 120° + P_4\cos 180° + P_5\cos 240° + P_6\cos 300°$$

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$$X = P_1 + \frac{1}{2}P_2 - \frac{1}{2}P_3 - P_4 - \frac{1}{2}P_5 + \frac{1}{2}P_6$$

$$X = P_1 - P_4 + \frac{1}{2}(P_2 - P_3 - P_5 + P_6) \quad ...(1)$$

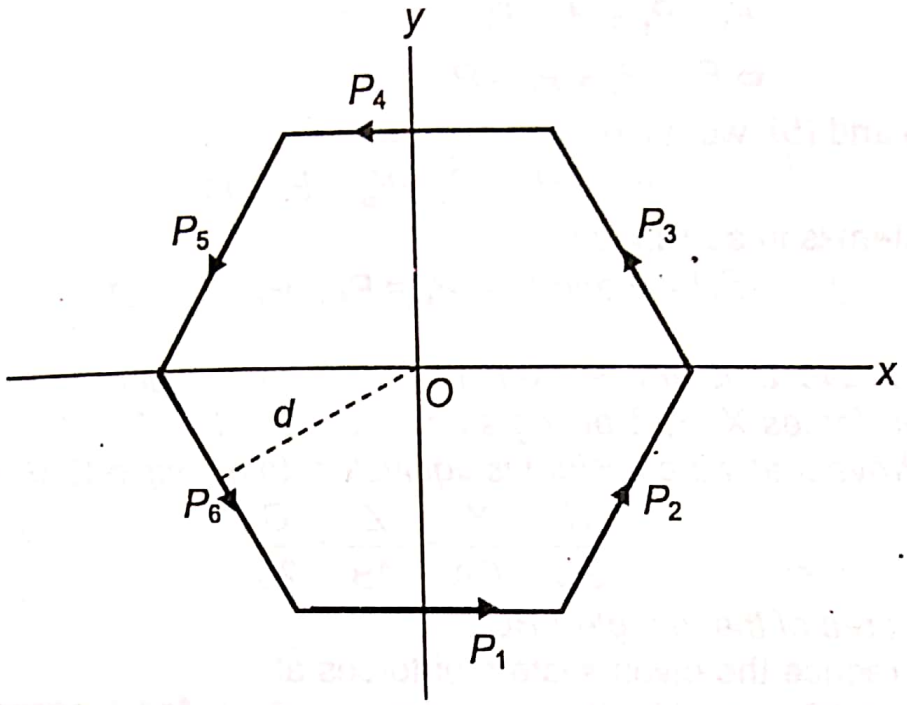


Fig.1.49

Similarly, the sum of resolved parts of all forces along *y*-axis is

$$Y = P_1 \sin 0° + P_2 \sin 60° + P_3 \sin 120° + P_4 \sin 180° + P_5 \sin 240° + P_6 \sin 300°$$

$$Y = 0 + \frac{\sqrt{3}}{2}P_2 + \frac{\sqrt{3}}{2}P_3 + 0 - \frac{\sqrt{3}}{2}P_5 - \frac{\sqrt{3}}{2}P_6$$

$$Y = \frac{\sqrt{3}}{2}(P_2 + P_3 - P_5 - P_6) \quad ...(2)$$

If *d* is perpendicular distance of all the forces from *O* as shown in Fig.1.46, then

G = Sum of the moments of all the forces about O

$$G = P_1 d + P_2 d + P_3 d + P_4 d + P_5 d + P_6 d$$

$$G = (P_1 + P_2 + P_3 + P_4 + P_5 + P_6)d \quad ...(3)$$

The forces are in equilibrium if $G = 0$, $X = 0$, $Y = 0$

$G = 0$ gives

$$(P_1 + P_2 + P_3 + P_4 + P_5 + P_6)d = 0$$

$$\Rightarrow P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 0 \quad \because d \neq 0$$

$$\Rightarrow \Sigma P = 0$$

Now $Y = 0$ gives

$$\frac{\sqrt{3}}{2}(P_2 + P_3 - P_5 - P_6) = 0$$

$$\Rightarrow P_2 + P_3 - P_5 - P_6 = 0$$

$$\Rightarrow P_3 - P_6 = P_5 - P_2 \quad ...(4)$$

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Similarly, $X = 0$ gives

$$P_1 - P_4 + \frac{1}{2}(P_2 - P_3 - P_5 + P_6) = 0$$

$$P_1 - P_4 + \tfrac{1}{2}(2P_2 - 2P_5) = 0 \qquad \text{(using (4))}$$

$$P_1 - P_4 + P_2 - P_5 = 0$$

$$\Rightarrow P_1 - P_4 = P_5 - P_2$$

Combining (4) and (5), we have

$$P_1 - P_4 = P_3 - P_6 = P_5 - P_2 \qquad \ldots(5)$$

Hence the system is in equilibrium if

$$\Sigma P = 0 \text{ and } P_1 - P_4 = P_3 - P_6 = P_5 - P_2$$

Example-30: *Prove that any system of forces in a plane is equivalent to three suitably chosen forces X, Y, Z acting along the sides BC, CA, AB of a given triangle in the plane. Prove that if the system is equivalent to a couple G, then*

$$\frac{X}{BC} = \frac{Y}{CA} = \frac{Z}{AB} = \frac{G}{2\Delta},$$

where Δ is the area of the triangle ABC.

Solution: We reduce the given system of forces at the point A (Fig.1.50). Let R be the single force and G_A the moment of the couple. We replace the couple by a force X along BC and an equal and opposite force at A. If h_A is the perpendicular from A to BC, the force X is determined from

$$Xh_A = G_A$$

The two forces R and X acting at A are equivalent to a single force P at A. Let the line of action of P cut BC in D. Then we can choose λ and μ so that

$$\vec{P} = (\lambda + \mu)\vec{AD} \qquad \ldots(1)$$

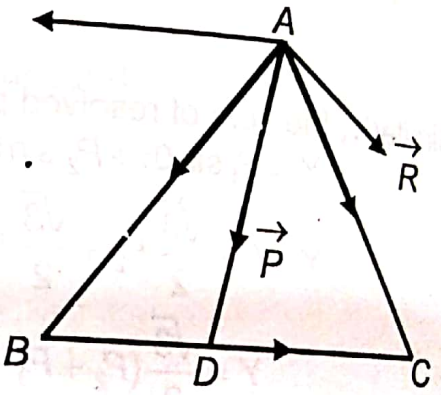


Fig.1.50

But by the (λ, μ) theorem we can replace $(\lambda + \mu)\vec{AD}$ by forces $\lambda\vec{AB}$ and $\mu\vec{AC}$ provided we choose the ratio $\lambda : \mu$ such that

$$\lambda BD = \mu DC \qquad \ldots(2)$$

Equations (1) and (2) determine the scalars λ, μ. Thus the given force system is replaced by three forces X along BC, $Y(= \mu CA)$ along CA and $Z(= \lambda AB)$ along AB.

If the given system is equivalent to a couple, the sum of the moments of the forces about any point is G. By taking moments about A, B, C respectively we get

$$Xh_A = G,\ Yh_B = G,\ Zh_C = G \qquad \ldots(3)$$

where h_A, h_B, h_C are the altitudes of the triangle through A, B, C respectively. But

$$BC.h_A = 2\Delta,\ CA.h_B = 2\Delta,\ AB.h_C = 2\Delta \qquad \ldots(4)$$

Dividing equations (3) by (4), we have

$$\frac{X}{BC} = \frac{Y}{CA} = \frac{Z}{AB} = \frac{G}{2\Delta}$$

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Example-31: *Coplanar forces (X_r, Y_r) act at points (x_r, y_r), $r = 1, 2, \ldots, n$. If each force is rotated about its point of application through an angle θ, show that the resultant passes through a fixed point, for all values of θ.*

Solution: Let the system reduce to a single force (X, Y) at the origin O together with a couple of moment G. Then

$$X = \sum X_r,\ Y = \sum Y_r,\ G = \sum (x_r Y_r - y_r X_r)$$

When each force is turned through an angle θ as shown in Fig.1.51, the resolved parts of the force (X_r, Y_r) along Ox, Oy are

$$X_r \cos\theta - Y_r \sin\theta \text{ and } X_r \sin\theta + Y_r \cos\theta$$

After rotation, let the system reduce to a force (X', Y') at O and a couple of moment G'. Then

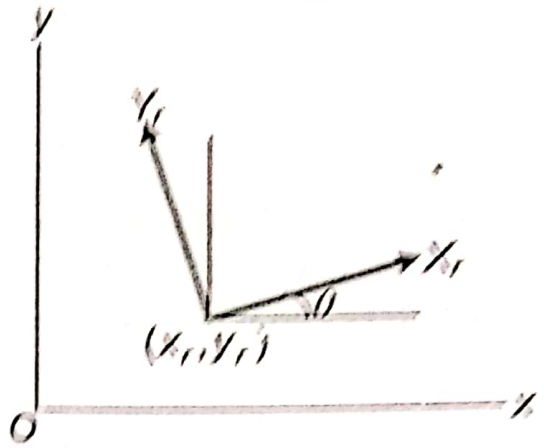


Fig.1.51

$$X' = \sum (X_r \cos\theta - Y_r \sin\theta)$$
$$= \cos\theta \sum X_r - \sin\theta \sum Y_r$$
$$= X\cos\theta - Y\sin\theta \qquad \because X = \sum X_r,\ Y = \sum Y_r$$
$$Y' = \sum (X_r \sin\theta + Y_r \cos\theta)$$
$$= \sin\theta \sum X_r + \cos\theta \sum Y_r$$
$$= X\sin\theta + Y\cos\theta \qquad \because X = \sum X_r,\ Y = \sum Y_r$$
$$G' = \sum [x_r (X_r \sin\theta + Y_r \cos\theta) - y_r (X_r \cos\theta - Y_r \sin\theta)]$$
$$G' = \sum (x_r X_r \sin\theta + x_r Y_r \cos\theta - y_r X_r \cos\theta + y_r Y_r \sin\theta)$$
$$G' = \sum (x_r Y_r \cos\theta - y_r X_r \cos\theta + x_r X_r \sin\theta + y_r Y_r \sin\theta)$$
$$G' = \sum [(x_r Y_r - y_r X_r)\cos\theta + (x_r X_r + y_r Y_r)\sin\theta]$$
$$G' = \cos\theta \sum (x_r Y_r - y_r X_r) + \sin\theta \sum (x_r X_r + y_r Y_r)$$
$$G' = \cos\theta G + \sin\theta \sum (x_r X_r + y_r Y_r) \qquad \because G = \sum (x_r Y_r - y_r X_r)$$
$$G' = G\cos\theta + V\sin\theta$$

Where $V = \sum (x_r X_r + y_r Y_r)$ is called the *Virial* of the force system.

The equation of line of action of the new resultant of the system is

$$G' + yX' - xY' = 0$$
$$\Rightarrow G\cos\theta + V\sin\theta + y(X\cos\theta - Y\sin\theta) - x(X\sin\theta + Y\cos\theta) = 0$$
$$\Rightarrow G\cos\theta + yX\cos\theta - xY\cos\theta + V\sin\theta - yY\sin\theta - xX\sin\theta = 0$$
$$\Rightarrow (G + yX - xY)\cos\theta + (V - yY - xX)\sin\theta = 0 \qquad \ldots(1)$$

This shows that whatever be the value of θ, the resultant passes through the point whose coordinates satisfy the equations

$$G + yX - xY = 0 \quad \ldots(2), \qquad V - yY - xX = 0 \quad \ldots(3)$$

Multiplying (2) by Y and (3) by X, we have

$$GY + yXY - xY^2 = 0 \quad \ldots(4), \qquad VX - yXY - xX^2 = 0 \quad \ldots(5)$$

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Adding (4) and (5), we have

$$GY + yXY - xY^2 = 0$$
$$VX - yXY - xX^2 = 0$$
$$VX + GY - x(X^2 + Y^2) = 0$$
$$x(X^2 + Y^2) = VX + GY$$
$$x = \frac{VX + GY}{X^2 + Y^2}$$

Multiplying (2) by X and (3) by Y, we have

$$GX + yX^2 - xXY = 0 \quad ...(6), \qquad VY - yY^2 - xXY = 0 \quad ...(7)$$

Subtracting (7) from (6), we have

$$GX + yX^2 - xXY = 0$$
$$VY - yY^2 - xXY = 0$$
$$GX - VY + y(X^2 + Y^2) = 0$$
$$y(X^2 + Y^2) = VY - GX$$
$$y = \frac{VY - GX}{X^2 + Y^2}$$

This shows that the resultant passes through a fixed point $\left(\frac{VX + GY}{X^2 + Y^2}, \frac{VY - GX}{X^2 + Y^2}\right)$, for all values of θ. This point is called the *astatic centre.*

Example-32: *Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC. Find the condition that their resultant is parallel to BC and determine its magnitude.*

PU, 2011 (B.A./B.Sc.)

Solution: Forces P, Q, R acting along the sides BC, CA, AB of a triangle ABC are shown in Fig.1.52(a). Taking BC along x-axis and resolving forces P, Q, R at B as shown in Fig.1.52(b), we note that the forces P, Q, R make angles $0°$, $180°–C$, $180°+B$ respectively with positive x-axis.

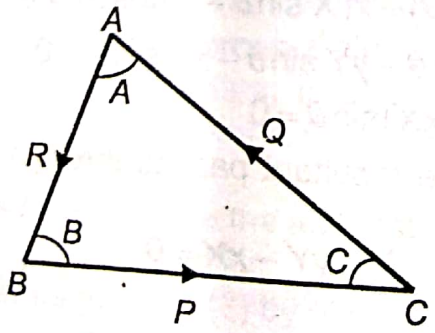


Fig.1.52(a)

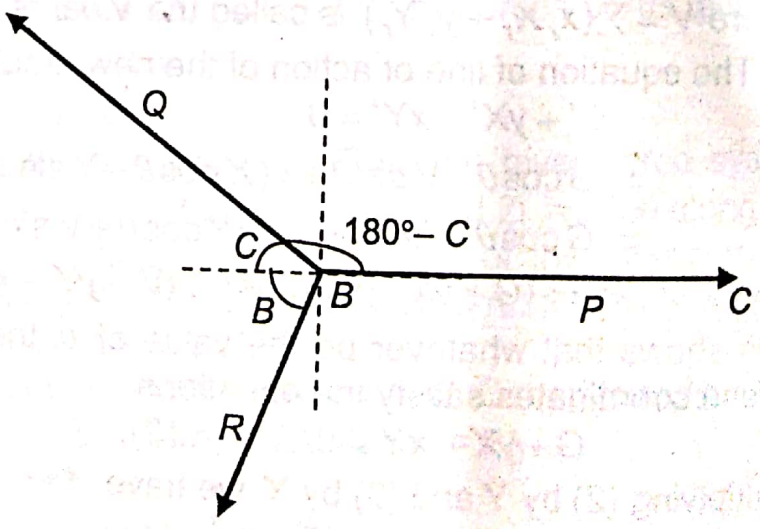


Fig.1.52(b)

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Sum of resolved parts of the forces P, Q, R along x-axis is

$$X = P\cos 0° + Q\cos(180° - C) + R\cos(180° + B)$$

$$X = P - Q\cos C - R\cos B \quad ...(1)$$

Sum of resolved parts of the forces P, Q, R along y-axis is

$$Y = P\sin 0° + Q\sin(180° - C) + R\sin(180° + B)$$

$$Y = Q\sin C - R\sin B \quad ...(2)$$

The condition that the resultant is parallel to BC is that

$$Y = 0$$

$$\Rightarrow Q\sin C - R\sin B = 0$$

$$\Rightarrow Q\sin C = R\sin B$$

$$\Rightarrow \frac{Q}{\sin B} = \frac{R}{\sin C}$$

$$\Rightarrow \frac{b}{Q} = \frac{c}{R} \quad \because \frac{b}{\sin B} = \frac{c}{\sin C} \text{ (law of sine)}$$

$$\Rightarrow \boxed{R = \frac{Qc}{b}} \quad ...(3)$$

Magnitude of the resultant is given by

$$R' = \sqrt{X^2 + Y^2} = \sqrt{X^2 + 0} = X = P - Q\cos C - R\cos B$$

$$R' = P - Q\cos C - \frac{Qc}{b}\cos B \quad \because R = \frac{Qc}{b}$$

$$R' = P - \frac{Q}{b}(b\cos C + c\cos B) \quad ...(4)$$

Using law of cosines, we have

$$b\cos C = \frac{a^2 + b^2 - c^2}{2a}, \qquad c\cos B = \frac{c^2 + a^2 - b^2}{2a}$$

Putting these in (4), we have

$$R' = P - \frac{Q}{b}\left(\frac{a^2 + b^2 - c^2}{2a} + \frac{c^2 + a^2 - b^2}{2a}\right)$$

$$= P - \frac{Q}{b}\left(\frac{a^2 + b^2 - c^2 + c^2 + a^2 - b^2}{2a}\right) = P - \frac{Q}{b}\left(\frac{2a^2}{2a}\right) = \boxed{P - \frac{Qa}{b}}$$

Example-33:

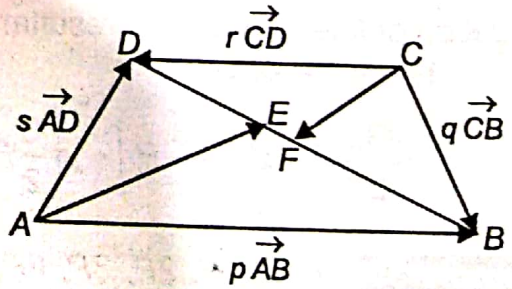
If forces $p\overrightarrow{AB}, q\overrightarrow{CB}, r\overrightarrow{CD}, s\overrightarrow{AD}$ acting along the sides of a plane quadrilateral are in equilibrium, show that $pr = qs$.

PU, 2014; 2010 (B.A./B.Sc.); PU, 2011 (BS Math)

Solution: By (λ, μ) theorem, the resultant of forces $p\overrightarrow{AB}$ and $s\overrightarrow{AD}$ is $(p+s)\overrightarrow{AE}$, where E divides DB in ratio

$$\frac{DE}{EB} = \frac{p}{s} \quad ...(1)$$

Once again by (λ, μ) theorem, the resultant of forces $q\overrightarrow{CB}$ and $r\overrightarrow{CD}$ is $(q+r)\overrightarrow{AF}$,



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where F divides DB in ratio $\frac{DF}{FB} = \frac{q}{r}$ ...(2)

Since the given forces are in equilibrium, which is only possible if F coincides E, so from (2), we have

$$\frac{DE}{EB} = \frac{q}{r} \quad \text{...(3)}$$

Equating (1) and (3), we have $\frac{p}{s} = \frac{q}{r}$

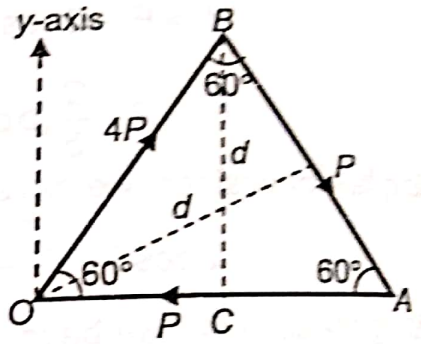
$$\Rightarrow pr = qs$$

Example-34: OAB is an equilateral triangle of side a; C is the midpoint of OA. Forces $4P$, P and P act along the sides OB, BA and AO respectively. If OA and OY (parallel to BC) are taken as x- and y-axes, prove that the resultant of the forces is $3P$ and the equation of its line of action is

$$3y = \sqrt{3}(3x + a)$$

PU, 2014 (B.A./B.Sc.)

Solution: Since OAB is an equilateral triangle of side a, so $|OA| = |OB| = |AB| = a$.



In ΔOCB, $d = |BC| = |OB| \sin 60° = \frac{\sqrt{3}}{2} a$.

Resolving forces along x-axis and y-axis, we have

$$X = P\cos 180° + 4P\cos 60° + P\cos 300°$$

$$X = P(-1) + 4P(\tfrac{1}{2}) + P(\tfrac{1}{2}) = -P + 2P + \tfrac{1}{2}P = \tfrac{3}{2}P$$

$$Y = P\sin 180° + 4P\sin 60° + P\sin 300°$$

$$Y = P(0) + 4P(\tfrac{\sqrt{3}}{2}) + P(-\tfrac{\sqrt{3}}{2}) = 2\sqrt{3}P - \tfrac{\sqrt{3}}{2}P = \tfrac{3\sqrt{3}}{2}P$$

Magnitude of the resultant is

$$R = \sqrt{X^2 + Y^2} = \sqrt{(\tfrac{3\sqrt{3}}{2}P)^2 + (\tfrac{3}{2}P)^2} = \sqrt{\tfrac{27}{4}P^2 + \tfrac{9}{4}P^2} = \sqrt{\tfrac{36}{4}P^2} = \sqrt{9P^2} = 3P$$

Taking moments about O, we not that the P and $4P$ pass through O, so their moments about O are zero, the other force P tends to rotate the body clockwise, so its moment is negative, i.e.

$$G = 0 + 0 - Pd = -\frac{\sqrt{3}a}{2}P \qquad \because d = \frac{\sqrt{3}a}{2}$$

Equation of line of action of resultant is

$$G - xY + yX = 0$$

$$-\frac{\sqrt{3}a}{2}P - x\left(\frac{3\sqrt{3}}{2}P\right) + y\left(\frac{3}{2}P\right) = 0$$

$$-\sqrt{3}a - 3\sqrt{3}x + 3y = 0$$

$$3y = 3\sqrt{3}x + \sqrt{3}a$$

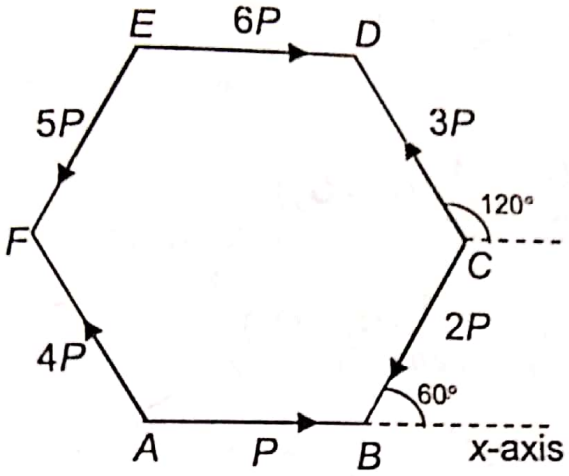
$$3y = \sqrt{3}(3x + a)$$

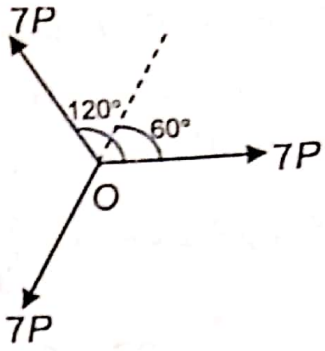
Example-35: Forces P, $2P$, $3P$, $6P$, $5P$ and $4P$ act respectively along the sides AB, ... forces being

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indicated by the order of the letters. Prove that the six forces are equivalent to a couple.

Solution: Resolving six forces at point O, we note that these are equivalent to three forces $7P$, $7P$ and $7P$ making angles 0°, 120°, 240° respectively with positive x-axis.





Sum of resolved parts along x-axis and y-axis are given below:

$$X = 7P\cos 0° + 7P\cos 120° + 7P\cos 240°$$

$$X = 7P + 7P(-\tfrac{1}{2}) + P(-\tfrac{1}{2}) = 7P - \tfrac{7}{2}P - \tfrac{7}{2}P = 0$$

$$Y = 7P\sin 0° + 7P\sin 120° + 7P\sin 240°$$

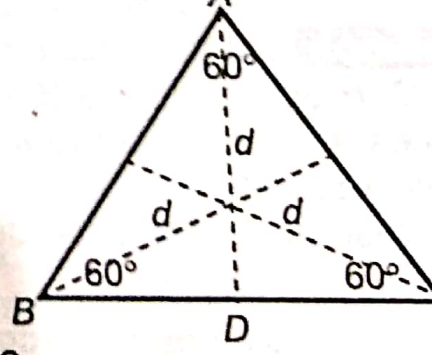
$$Y = 7P(\tfrac{\sqrt{3}}{2}) + 7P(-\tfrac{\sqrt{3}}{2}) = 0$$

$$\Rightarrow R = \sqrt{X^2 + Y^2} = 0$$

Since the given forces are equivalent to a singe force R at O along with a couple, $R = 0$ shows that the given forces are equivalent to a couple.

Example-36: *A system of forces acts on a plate in the form of an equilateral trian of side 2a. The moments of the forces about the three vertices are G_1, G_2, respectively. Find the magnitude of the resultant.*

Solution: Let forces be of magnitudes P_1, P_2, P_3 acting along the sides BC, CA, AB respectively. Since ABC is an equilateral triangle of side 2a, so $|OA| = |OB| = |AB| = 2a$. In ΔABD,



$$d = |AD| = |AB|\sin 60° = (2a)\frac{\sqrt{3}}{2} = \sqrt{3}a$$

Since G_1, G_2, G_3 are moments of forces about vertices, so

$$G_1 = P_1 d = P_1(\sqrt{3}a), \quad G_2 = P_2 d = P_2(\sqrt{3}a), \quad G_3 = P_3 d = P_3(\sqrt{3}a)$$

$$\Rightarrow P_1 = \frac{G_1}{\sqrt{3}a}, \quad P_2 = \frac{G_2}{\sqrt{3}a}, \quad P_3 = \frac{G_3}{\sqrt{3}a}$$

Considering BC as positive x-axis, the resolved parts of the forces along x-axi y-axis are:

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$$X = P_1\cos 0° + P_2\cos 120° + P_3\cos 240°$$

$$X = \frac{G_1}{\sqrt{3}a} + \frac{G_2}{\sqrt{3}a}\left(-\frac{1}{2}\right) + \frac{G_3}{\sqrt{3}a}\left(-\frac{1}{2}\right)$$

$$X = \frac{G_1}{\sqrt{3}a} - \frac{G_2}{2\sqrt{3}a} - \frac{G_3}{2\sqrt{3}a} = \frac{1}{2\sqrt{3}a}(2G_1 - G_2 - G_3)$$

$$Y = P_1\sin 0° + P_2\sin 120° + P_3\sin 240°$$

$$Y = \frac{G_2}{\sqrt{3}a}\left(\frac{\sqrt{3}}{2}\right) + \frac{G_3}{\sqrt{3}a}\left(-\frac{\sqrt{3}}{2}\right) = \frac{G_2}{2a} - \frac{G_3}{2a} = \frac{1}{2a}(G_2 - G_3)$$

$$R^2 = X^2 + Y^2 = \left(\frac{1}{2\sqrt{3}a}(2G_1 - G_2 - G_3)\right)^2 + \left(\frac{1}{2a}(G_2 - G_3)\right)^2$$

$$= \frac{1}{12a^2}(2G_1 - G_2 - G_3)^2 + \frac{1}{4a^2}(G_2 - G_3)^2$$

$$= \frac{1}{12a^2}[(2G_1 - G_2 - G_3)^2 + 3(G_2 - G_3)^2]$$

$$= \frac{1}{12a^2}(4G_1^2 + G_2^2 + G_3^2 - 4G_1G_2 + 2G_2G_3 - 4G_3G_1 + 3G_2^2 + 3G_3^2 - 6G_2G_3)$$

$$= \frac{1}{12a^2}(4G_1^2 + 4G_2^2 + 4G_3^2 - 4G_1G_2 - 4G_2G_3 - 4G_3G_1)$$

$$= \frac{4}{12a^2}(G_1^2 + G_2^2 + G_3^2 - G_1G_2 - G_2G_3 - G_3G_1)$$

$$R = \sqrt{\frac{G_1^2 + G_2^2 + G_3^2 - G_1G_2 - G_2G_3 - G_3G_1}{3a^2}}$$

Example 37 A couple of

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Putting this value of R in (1), we have

$$P-\frac{1}{\sqrt{2}}Q-\frac{1}{\sqrt{2}}Q=0 \Rightarrow P-\sqrt{2}Q=0$$

$$\Rightarrow P-\frac{2}{\sqrt{2}}Q=0 \Rightarrow P=\sqrt{2}Q \quad ...(3)$$

Taking moments about A, we have

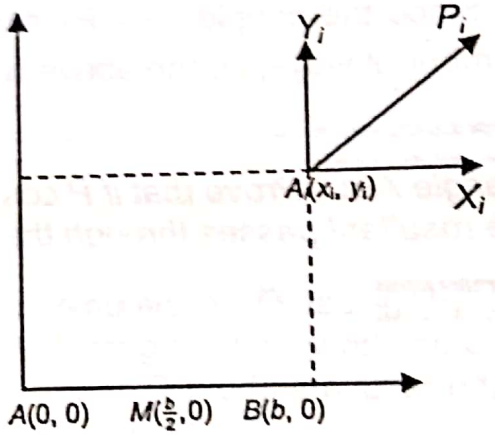
$$G=P(0)+Q(AE)+R(0)=Q(AE) \quad ...(4)$$

ΔAEB is a right triangle in which

$$AE=AB\sin 45^\circ=a\left(\frac{1}{\sqrt{2}}\right)$$

Putting this value of AE in (4), we have $G=Q\frac{a}{\sqrt{2}} \Rightarrow Q=\frac{\sqrt{2}G}{a}$. Putting this value of Q, in (2) and (3), have the required forces $\frac{2G}{a}, \frac{\sqrt{2}G}{a}, \frac{\sqrt{2}G}{a}$.

Example-38: *A, B are any two points in a lamina, on which a system of forces coplanar with it are acting, and when the forces are reduced to a single force at each of these points and a couple, the moments of the couples are G_a and G_b respectively. Prove that when the reduction is made to a force at the midpoint of AB and a couple, the moment of the couple is $\frac{1}{2}(G_a+G_b)$.*



Solution: Let P_i be the system of forces acting at points A_i (x_i, y_i) as shown in the adjoining figure. If points A and B have coordinate $A(0, 0)$ and $B(b, 0)$, then the coordinates of the midpoint M are $M\left(\frac{b}{2}, 0\right)$.

Now G_a is the sum of moments of the resolved parts of forces P_i about A, so

$$G_a=\sum(x_iY_i-y_iX_i) \quad ...(1)$$

Similarly, G_b is the sum of moments of the resolved parts of forces P_i about B, so

$$G_b=\sum[(x_i-b)Y_i-y_iX_i]=\sum(x_iY_i-bY_i-y_iX_i)=\sum(x_iY_i-y_iX_i)-b\sum Y_i$$

$$G_b=G_a-bY \quad \because G_a=\sum(x_iY_i-y_iX_i),\ \sum Y_i=Y$$

$$bY=G_a-G_b \quad ...(2)$$

The sum of moments of the resolved parts of forces P_i about M is

$$G=\sum\left[\left(x_i-\frac{b}{2}\right)Y_i-y_iX_i\right]=\sum\left[(x_iY_i-\frac{b}{2}Y_i-y_iX_i\right]=\sum(x_iY_i-y_iX_i)-\frac{b}{2}\sum Y_i$$

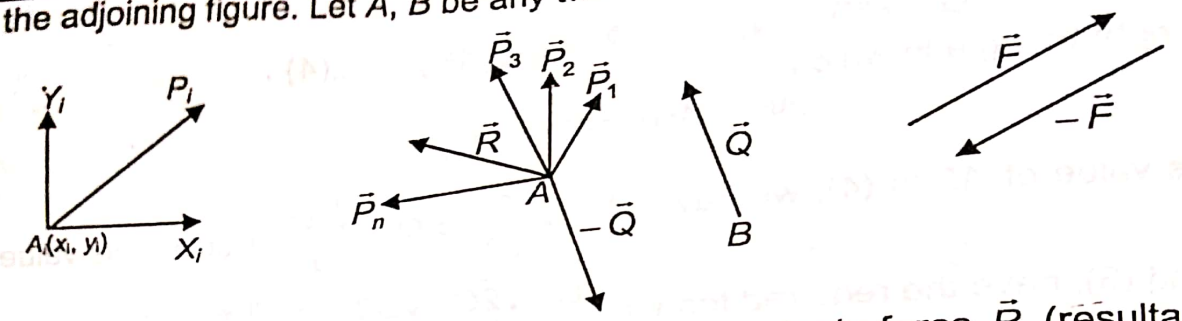
$$G=G_a-\frac{1}{2}bY=G_a-\frac{1}{2}(G_a-G_b) \quad \because G_a=\sum(x_iY_i-y_iX_i),\ \sum Y_i=Y,\ bY=G_a-G_b$$

$$G=\frac{2G_a-G_a+G_b}{2}=\frac{1}{2}(G_a+G_b)$$

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Example-39: *Show that a system of coplanar forces may be represented in an infinite number of ways by two forces, one through each of two fixed points in their plane.*

Solution: Let P_i be the system of coplanar forces acting at points $A_i(x_i, y_i)$ as shown in the adjoining figure. Let A, B be any two fixed points in the plane of forces.



When the reduction is made at point A, we have a single force $\vec{R}$ (resultant of forces $\vec{P}_i$) at A and a couple $(\vec{F}, -\vec{F})$. Next we replace this couple by another equivalent couple $(\vec{Q}, -\vec{Q})$ such that the forces $-\vec{Q}$ and $\vec{Q}$ pass through the points A and B respectively. Consequently we have two forces $\vec{P} = R + (-\vec{Q})$ and $\vec{Q}$ through A and B. Since the couple $(\vec{F}, -\vec{F})$ can be replaced by an equivalent couple in an infinite number of ways, so the above reduction can be done by an infinite number of ways.

Example-40: *Three forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC. Prove that if $P \cos A + Q \cos B + R \cos C = 0$, then the line of action of the resultant passes through the circumcentre of the triangle.*

PU, 2011; 2000 (B.A./B.Sc.)

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It is given that $P\cos A + Q\cos B + R\cos C = 0$, so (3) reduces to

$$G = |OA|(0) = 0$$

This shows that the moment of the resultant about O is zero, so through O, the circumcentre of the triangle ABC.

Example-11: *Three forces P, Q, R act along the sides BC, CA, triangle ABC. Prove that, if P sec A + Q sec B + R sec C = 0, t of the resultant passes through, the orthocentre of the triangle.*

PU, 2012; 2010; 2002 (EXAMS of Punjab University of

Solution: Let O be the orthocentre of the ΔABC, i.e. the point v the ΔABC meet as shown in the figure. AL, BM, CN are altitude clear from figure that ΔBOL is a right triangle in which

$$\frac{|OL|}{|BL|} = \tan(90^\circ - C)$$

$$\Rightarrow |OL| = |BL|\cot C \qquad \ldots(1)$$

ΔABL is also a right triangle in which

$$\frac{|BL|}{|AB|} = \cos B$$

$$\Rightarrow |BL| = c\cos B \qquad \because |AB| = c$$

Putting this value of $|BL|$ in (1), we have

$$|OL| = c\cos B\cot C \qquad \ldots(2)$$

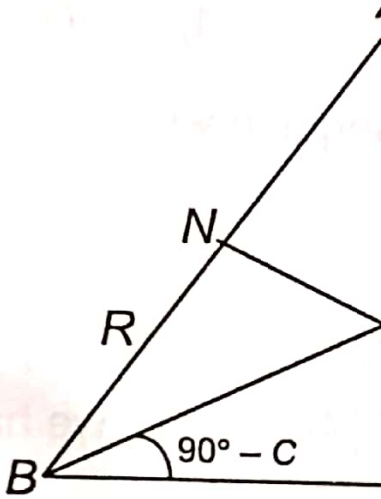
Similarly, using cyclic order, we have

$$|OM| = a\cos C\cot A \qquad \ldots(3) \quad \text{and} \quad |ON| = b\cos$$

If G is the moment of the resultant about O, then

$$G = P|OL| + Q|OM| + R|ON|$$

$$= P(c\cos B\cot C) + Q(a\cos C\cot A) + R(b\cos A\cot B)$$

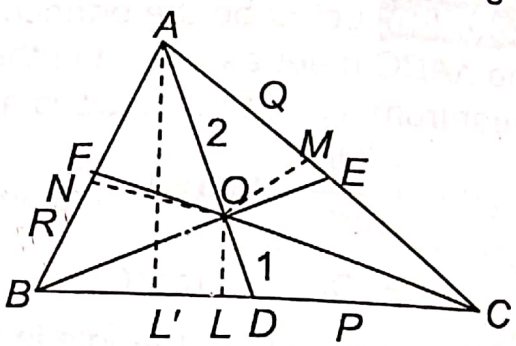


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This shows that the moment of the resultant about O is zero, so the resultant passes through O, the orthocentre of the triangle ABC.

Example-42: *Three forces, P, Q, R, act along sides BC, CA, AB of triangle ABC. Prove that if P cosec A + Q cosec B + cosec C = 0, then line of action passes through centroid of triangle ABC.*

Solution: Let O be the centroid of the ΔABC, i.e. the point where the medians of the ΔABC intersect. Let AD, BE, CF be the medians of the triangle ABC as shown in the figure. We draw perpendiculars from O and A on BC. Since AD is median, so O divides AD in the ratio $AO : OD = 2 : 1$. Since $\Delta AL'D$ and ΔGLD are similar triangles, so



$$\frac{|OL|}{|OD|} = \frac{|AL'|}{|AD|}$$

$$\Rightarrow |OL| = \frac{|AL'|}{3} \qquad \ldots(1) \because |OD| = 1, |AD| = 3$$

$\Delta AL'B$ is a right triangle in which

$$\frac{|AL'|}{|AB|} = \sin B$$

$$|AL'| = c\sin B \qquad \because |AB| = c$$

Putting this value of $|AL'|$ in (1), we have

$$|OL| = \frac{c\sin B}{3}$$

Similarly, using cyclic order, we have

$$|OM| = \frac{a\sin C}{3} \text{ and } |ON| = \frac{b\sin A}{3}$$

If G is the moment of the resultant about O, then

$$G = P\left(\frac{c\sin B}{3}\right) + Q\left(\frac{a\sin C}{3}\right) + R\left(\frac{b\sin A}{3}\right)$$

$$= \frac{1}{3}(P\sin C\sin B + Q\sin A\sin C + R\sin B\sin A) \qquad \because \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$= \frac{1}{3}\sin A\sin C\sin B\left(\frac{P}{\sin A} + \frac{Q}{\sin B} + \frac{R}{\sin C}\right)$$

$$= \frac{1}{3}\sin A\sin C\sin B(P\,\mathrm{cosec}A + Q\,\mathrm{cosec}B + R\,\mathrm{cosec}C)$$

$$= \cos A\cos B\cos C(0)$$

$$= 0 \qquad \because P\,\mathrm{cosec}A + Q\,\mathrm{cosec}B + R\,\mathrm{cosec}C = 0$$

This shows that the moment of the resultant about O is zero, so the resultant passes through O, the centroid of the tri...

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Example-43: *Three forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC. If their resultant touches the inscribed circle, prove that*

$$\frac{\cos^2 A\backslash 2}{P}+\frac{\cos^2 B\backslash 2}{Q}+\frac{\cos^2 C\backslash 2}{R}=0$$

PU, 2001 (B.A./B.Sc.)

Solution: Let S be the resultant of the forces P, Q, R touching the inscribed circle as shown in the Fig.1.53(a). Taking BC along positive x-axis and resolving forces P, Q, R at B we note that these forces make angles $0°$, $180° - C$, $180° + B$ with positive x-axis as shown in Fig.1.53(b).

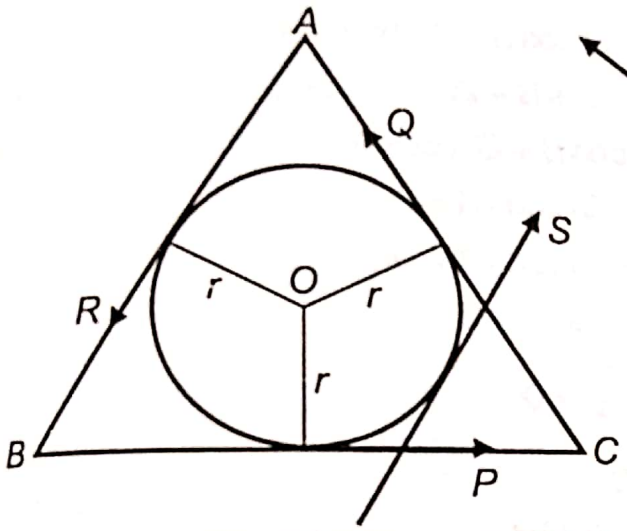


Fig.1.53(a)

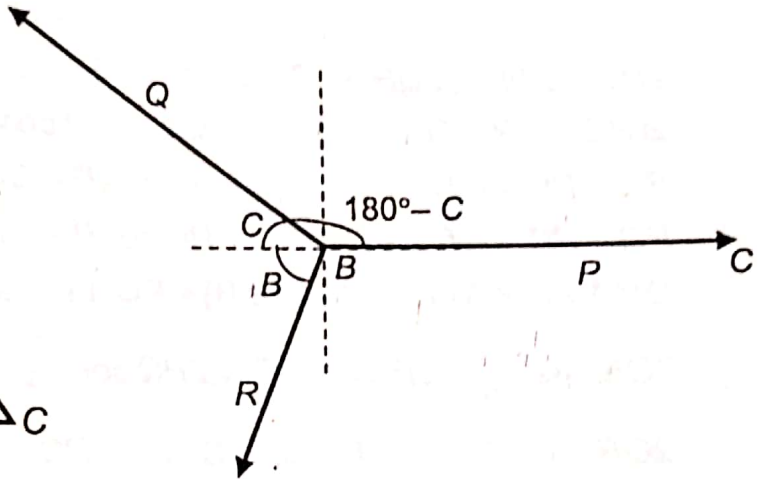


Fig.1.53(b)

Sum of resolved parts of forces P, Q, R along and perpendicular to x-axis are:

$$X = P\cos 0° + Q\cos(180° - C) + R\cos(180° + B)$$

$$X = P - Q\cos C - R\cos B$$

$$Y = P\sin 0° + Q\sin(180° - C) + R\sin(180° + B)$$

$$Y = Q\sin C - R\sin B$$

$$S^2 = X^2 + Y^2 = (P - Q\cos C - R\cos B)^2 + (Q\sin C - R\sin B)^2$$

$$= P^2 + (Q\cos C)^2 + (R\cos B)^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos B\cos C$$

$$+ (Q\sin C)^2 + (R\sin B)^2 - 2QR\sin B\sin C$$

$$= P^2 + Q^2\cos^2 C + R^2\cos^2 B - 2PQ\cos C - 2PR\cos B + 2QR\cos B\cos C$$

$$+ Q^2\sin^2 C^2 + R^2\sin^2 B - 2QR\sin B\sin C$$

$$= P^2 + Q^2\cos^2 C + Q^2\sin^2 C + R^2\cos^2 B + R^2\sin^2 B - 2PQ\cos C$$

$$- 2PR\cos B + 2QR\cos B\cos C - 2QR\sin B\sin C$$

$$= P^2 + Q^2(\cos^2 C + \sin^2 C) + R^2(\cos^2 B + \sin^2 B) - 2PQ\cos C$$

$$- 2PR\cos B + 2QR(\cos B\cos C - \sin B\sin C)$$

$$= P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos(B + C)$$

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$= P^2 + Q^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos(180° - A) \because A + B + C = 180°$

$= P^2 + Q^2 - 2PQ\cos C - 2PR\cos B - 2QR\cos A$ ...(1)

Since resultant S touches the inscribed circle, so its moment about O, the centre of the circle is Sr, so taking moments about O, we have

$Sr = Pr + Qr + Rr$

$S = P + Q + R$

$S^2 = (P + Q + R)^2 = P^2 + Q^2 + R^2 + 2PQ + 2PR + 2QR$

Putting this value in (1), we have

$P^2 + Q^2 + R^2 + 2PQ + 2PR + 2QR = P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B - 2QR\cos A$

$2PQ + 2PR + 2QR = -2PQ\cos C - 2PR\cos B - 2QR\cos A$

$2(PQ + PR + QR) = -2(PQ\cos C + PR\cos B + QR\cos A)$

$PQ + PR + QR + PQ\cos C + PR\cos B + QR\cos A = 0$

$PQ + PR + QR + PQ\cos C + PR\cos B + QR\cos A = 0$

$QR(1 + \cos A) + PR(1 + \cos B) + PQ(1 + \cos C) = 0$

$2QR\cos^2\frac{A}{2} + 2PR\cos^2\frac{B}{2} + 2PQ\cos^2\frac{C}{2} = 0$

$\frac{2QR\cos^2 A\backslash 2}{2PQR} + \frac{2PR\cos^2 B\backslash 2}{2PQR} + \frac{2PQ\cos^2 C\backslash 2}{2PQR} = 0$ (dividing by $2PQR$)

$\frac{\cos^2 A\backslash 2}{P} + \frac{\cos^2 B\backslash 2}{Q} + \frac{\cos^2 C\backslash 2}{R} = 0$

EXERCISE 1

Multiple Choice Questions (MCQs)

Four options are given in each of the following questions, fill the circle in front of that choice which you think is correct. Cutting or filling two or more circles is not allowed:

Q.1

(i) The quantity of matter in a body is called its
(a) acceleration (b) velocity (c) mass (d) tension
ⓐ ⓑ ⓒ ⓓ

(ii) A body of negligible small dimensions is termed as
(a) heavy body (b) small body (c) particle (d) none of these
ⓐ ⓑ ⓒ ⓓ

(iii) The concept of a particle is an ideal one and represents a geometrical _____ which posses mass.
(a) ellipse (b) triangle (c) circle (d) point
ⓐ ⓑ ⓒ ⓓ

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(iv) A body may be regarded as an aggregate of
(a) ellipses (b) triangles (c) circles (d) particles
ⓐ ⓑ ⓒ ⓓ

(v) A body is said to be a rigid body if the distance between any two of its particles _____ by any means.
(a) changes (b) remains unchanged
(c) remains variable (d) none of these
ⓐ ⓑ ⓒ ⓓ

(vi) A body is said to the rigid body if it _____ its shape in any way.
(a) changes (b) does not change
(c) changes rapidly (d) none of these
ⓐ ⓑ ⓒ ⓓ

(vii) A body is said to be in motion if it _____ its position with respect to its surroundings in the course of time.
(a) does not change (b) changes
(c) never changes (d) none of these
ⓐ ⓑ ⓒ ⓓ

(viii) A body is said to at rest if it _____ its position with respect to its surroundings in the course of time.
(a) does not change (b) changes
(c) rapidly changes (d) none of these
ⓐ ⓑ ⓒ ⓓ

(ix) Rest and motion of a body are _____ concepts.
(a) absolute (b) not relative (c) relative (d) none of these
ⓐ ⓑ ⓒ ⓓ

(x) The physical quantity which changes or tends to change the state of rest or motion of a body is called
(a) force (b) acceleration (c) velocity (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xi) The line passing through the point of application in the direction of the force is called the _____ of action of force

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(xiv) If a force system is equivalent to a single force $\vec{R}$, then $\vec{R}$ is called the _____ of the system.
(a) resultant (b) component (c) resolved part (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xv) The branch of mechanics which deals with bodies at rest is
(a) dynamics (b) statics (c) statistics (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xvi) The branch of mechanics which deals with bodies in motion is
(a) dynamics (b) statics (c) statistics (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xvii) _____ treats of the geometry of motion of bodies without taking into account their masses or the forces acting on them.
(a) Kinematics (b) Kinetics (c) Statistics (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xviii) _____ is concerned with the relations between the forces acting on bodies and the resulting motion.
(a) Kinematics (b) Kinetics (c) Statistics (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xix) Classical mechanics is also known as _____ mechanics.
(a) Fluid (b) Quantum (c) Newtonian (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xx) _____ mechanics is based on Newton's laws of motion.
(a) Fluid (b) Quantum (c) Classical (d) none of these
ⓐ ⓑ ⓒ ⓓ

Q.2

(i) In Newtonian mechanics, space time and mass are _____ concepts independent of each other.
(a) relative (b) absolute (c) not absolute (d) none of these
ⓐ ⓑ ⓒ ⓓ

(ii) Newton's _____ law of motion states that a body at rest will remain at rest, and a body moving with uniform velocity will continue to do so, unless acted on by some unbalanced external force.
(a) first (b) second (c) third (d) none of these
ⓐ ⓑ ⓒ ⓓ

(iii) Newton's _____ law of motion states that a force applied on a body produces acceleration in its own direction which is proportional to the applied force and inversely proportional to the mass of the body.
(a) first (b) second (c) third (d) none of these
ⓐ ⓑ ⓒ ⓓ

(iv) Newton's _____ law of motion states that action and reaction are equal and opposite.
(a) first (b) second (c) third (d) none of these
ⓐ ⓑ ⓒ ⓓ

(v) The product of the mass of the object and velocity is called its
(a) velocity (b) momentum (c) acceleration (d) none of these
ⓐ ⓑ ⓒ ⓓ

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(vi) If a particle of mass m is moving with a velocity $\vec{v}$, then its linear momentum is

(a) $m\vec{a}$ (b) $m^2\vec{a}$ (c) $m^2\vec{v}$ (d) $m\vec{v}$

ⓐ ⓑ ⓒ ⓓ

(vii) Mathematical form of Newton's second law of motion is $\vec{F} =$

(a) $m\vec{a}$ (b) $m^2\vec{a}$ (c) $m^2\vec{v}$ (d) $m\vec{v}$

ⓐ ⓑ ⓒ ⓓ

(viii) The forces passing through a single point are called _____ forces.

(a) parallel (b) concurrent (c) non-concurrent (d) none of these

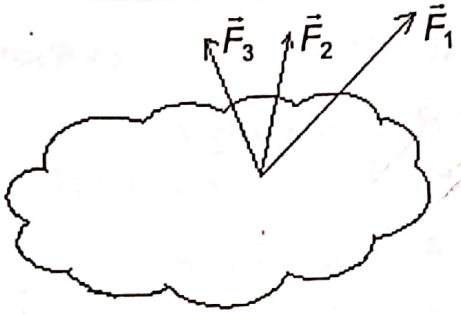
ⓐ ⓑ ⓒ ⓓ

(ix) The forces lying in the same plane are called _____ forces.

(a) non-coplanar (b) coplanar (c) skew (d) none of these

ⓐ ⓑ ⓒ ⓓ

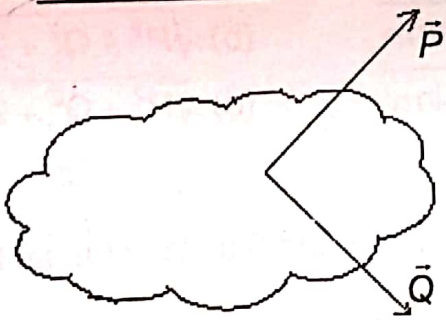
(x) Forces $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ are _____ forces.



(a) parallel (b) concurrent (c) non-concurrent (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xi) Forces $\vec{P}$ and $\vec{Q}$ have _____ of action.



(a) same direction (b) different lines (c) same line (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xii) The action of a given force system on a body _____ if we add to or subtract from them another force system in equilibrium.

(a) is changed (b) remains unchanged

(c) is changed rapidly (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xiii) The point of application of a force acting on a rigid body _____ be transferred to any other point of the body lying on the line action of the force without changing its effect.

(a) can (b) cannot (c) can never (d) none of these

ⓐ ⓑ ⓒ ⓓ

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(xiv) The magnitude w of the weight of the particle can be written as $w =$ _____, where m is the mass of the particle and g is the gravitational acceleration.

(a) $m\sqrt{g}$ (b) mg^2 (c) m^2g (d) mg

ⓐ ⓑ ⓒ ⓓ

(xv) When two forces $\vec{P}$ and $\vec{Q}$ act on a particle their resultant $\vec{R}$ is found by the

(a) Newton's first law of motion (b) Newton's second law of motion
(c) Newton's third law of motion (d) parallelogram law of forces

ⓐ ⓑ ⓒ ⓓ

(xvi) If a force $\vec{R}$ makes an angle θ with x-axis, then the resolved part of $\vec{R}$ along x-axis is

(a) $R\cot\theta$ (b) $R\tan\theta$ (c) $R\sin\theta$ (d) $R\cos\theta$

ⓐ ⓑ ⓒ ⓓ

(xvii) If a force $\vec{R}$ makes an angle θ with x-axis, then the resolved part of $\vec{R}$ along y-axis is

(a) $R\cot\theta$ (b) $R\tan\theta$ (c) $R\sin\theta$ (d) $R\cos\theta$

ⓐ ⓑ ⓒ ⓓ

(xviii) The resolved part of $\vec{R}$ along x-axis is

(a) $\vec{R}\cdot\hat{i}$ (b) $\vec{R}\times\hat{i}$ (c) $\vec{R}\cdot\hat{j}$ (d) $\vec{R}\times\hat{j}$

ⓐ ⓑ ⓒ ⓓ

(xix) The resolved part of $\vec{R}$ along y-axis is

(a) $\vec{R}\cdot\hat{i}$ (b) $\vec{R}\times\hat{i}$ (c) $\vec{R}\cdot\hat{j}$ (d) $\vec{R}\times\hat{j}$

ⓐ ⓑ ⓒ ⓓ

(xx) The magnitude R of the resultant of forces $\vec{P}$ and $\vec{Q}$ acting at an angle α is

(a) $\sqrt{P^2+Q^2+2PQ\tan\alpha}$ (b) $\sqrt{P^2+Q^2+2PQ\cot\alpha}$

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(a) $\tan^{-1}\left(\frac{P}{Q}\right)$ (b) $\tan^{-1}\left(\frac{Q\sin\alpha}{P+Q\cos\alpha}\right)$

(c) $\tan^{-1}\left(\frac{P\sin\alpha}{Q+P\cos\alpha}\right)$ (d) $\tan^{-1}\left(\frac{Q}{P}\right)$

(v) If two forces $\vec{P}$ and $\vec{Q}$ act at an angle α, then the angle of their resultant with $\vec{Q}$ is ⓐ ⓑ ⓒ ⓓ

(a) $\tan^{-1}\left(\frac{P}{Q}\right)$ (b) $\tan^{-1}\left(\frac{Q\sin\alpha}{P+Q\cos\alpha}\right)$

(c) $\tan^{-1}\left(\frac{P\sin\alpha}{Q+P\cos\alpha}\right)$ (d) $\tan^{-1}\left(\frac{Q}{P}\right)$

(vi) The resultant of two forces of equal magnitude will _____ the angle between them. ⓐ ⓑ ⓒ ⓓ

(a) not bisect (b) trisect (c) bisect (d) none of these

(vii) Two forces of magnitudes 7 and 4 units act at an angle $60°$, the angle of their resultant with the force of magnitude 7 is ⓐ ⓑ ⓒ ⓓ

(a) $\tan^{-1}\left(\frac{2\sqrt{3}}{9}\right)$ (b) $\tan^{-1}\left(\frac{7\sqrt{3}}{15}\right)$ (c) $\tan^{-1}\left(\frac{7}{4}\right)$ (d) none of these

(viii) Two forces of magnitudes 7 and 4 units act at an angle $60°$, the angle of their resultant with the force of magnitude 4 is ⓐ ⓑ ⓒ ⓓ

(a) $\tan^{-1}\left(\frac{2\sqrt{3}}{9}\right)$ (b) $\tan^{-1}\left(\frac{7\sqrt{3}}{15}\right)$ (c) $\tan^{-1}\left(\frac{7}{4}\right)$ (d) none of these

(ix) The magnitude of the resultant of forces $\vec{P}$ and $\vec{Q}$ acting perpendicularly is ⓐ ⓑ ⓒ ⓓ

(a) $\sqrt{P^2-Q^2}$ (b) P^2-Q^2 (c) P^2+Q^2 (d) $\sqrt{P^2+Q^2}$

(x) The resultant of two concurrent forces $\lambda\vec{OA}$ and $\mu\vec{OB}$ is _____, where C divides AB such that $AC:CB=\mu:\lambda$. ⓐ ⓑ ⓒ ⓓ

(a) $(\lambda+\mu)\vec{OC}$ (b) $(\lambda-\mu)\vec{OC}$ (c) $\lambda\vec{OC}$ (d) $\mu\vec{OC}$

(xi) Forces P, Q act at a point O and their resultant is R. If any transversal cuts the lines of action of the forces in the points A, B, C respectively, then $\frac{P}{OA}+\frac{Q}{OB}=$ ⓐ ⓑ ⓒ ⓓ

(a) $\frac{R^2}{OC}$ (b) $\frac{R}{OC}$ (c) $\frac{OC}{R}$ (d) none of these

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(xii) The necessary and sufficient condition for a particle to be in equilibrium under the action of the forces $\vec{P}_1, \vec{P}_2, \ldots, \vec{P}_n$ is that $\vec{P}_1 + \vec{P}_2 + \ldots + \vec{P}_n =$

(a) $\hat{i}$ (b) $\hat{j}$ (c) $\hat{k}$ (d) 0

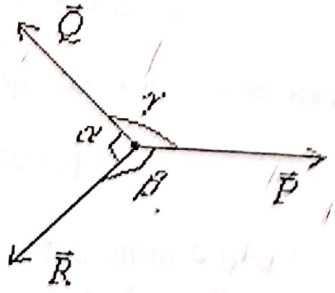
ⓐ ⓑ ⓒ ⓓ

(xiii) If a particle is in equilibrium under the action of three forces, then the magnitude of each force is proportional to the _____ of the angle between the other two.

(a) cosine (b) sine (c) cosecant (d) secant

ⓐ ⓑ ⓒ ⓓ

(xiv) If the particle is in equilibrium under the action of forces in the adjoining figure,



then

(a) $\dfrac{P}{\cot\alpha} = \dfrac{Q}{\cot\beta} = \dfrac{R}{\cot\gamma}$ (b) $\dfrac{P}{\tan\alpha} = \dfrac{Q}{\tan\beta} = \dfrac{R}{\tan\gamma}$

(c) $\dfrac{P}{\cos\alpha} = \dfrac{Q}{\cos\beta} = \dfrac{R}{\cos\gamma}$ (d) $\dfrac{P}{\sin\alpha} = \dfrac{Q}{\sin\beta} = \dfrac{R}{\sin\gamma}$

ⓐ ⓑ ⓒ ⓓ

(xv) The tendency of a force to turn a body about a point is called the _____ of that force.

(a) magnitude (b) direction (c) moment (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xvi) If $\vec{r}$ is the position vector of the point of application A of the force $\vec{F}$ from a fixed point O, then the moment of the force $\vec{F}$ about O is $\vec{M} =$

(a) $\vec{r} \cdot \vec{F}$ (b) $\vec{r} \times \vec{F}$ (c) $\vec{r} + \vec{F}$ (d) $\vec{r} - \vec{F}$

ⓐ ⓑ ⓒ ⓓ

(xvii) The moment of a force $\vec{F}$ about a point O is _____ to the plane containing O and $\vec{F}$.

(a) not perpendicular (b) parallel (c) perpendicular (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xviii) The moment of the resultant of concurrent forces about a point is equal to the _____ of moments of the forces about that point.

(a) difference (b) sum (c) product (d) none of these

ⓐ ⓑ ⓒ ⓓ

(xix) The moment of a force is _____ if the rotation is anticlockwise.

(a) zero (b) positive (c) negative (d) none of these

ⓐ ⓑ ⓒ ⓓ

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(xx) The moment of a force is _____ if the rotation is clockwise.
(a) zero (b) positive (c) negative (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xxi) The necessary and sufficient condition for the equilibrium of a system of coplanar couples is that the algebraic sum of their moments is
(a) zero (b) positive (c) negative (d) none of these
ⓐ ⓑ ⓒ ⓓ

(xxii) If forces are completely represented by the sides of a plane polygon taken in order, then they are equivalent to a couple whose moment is represented by _____ the area of the polygon.
(a) half of (b) twice (c) negative (d) none of these
ⓐ ⓑ ⓒ ⓓ

Short Questions

Q.4 Solve / write answers of the following short questions:

(i) Define the following:
(a) Body **(b)** Mass **(c)** Particle

(ii) Define the concept of rigid body.

(iii) Define the concept of force.

(iv) State Newton's First Law of Motion.

(v) State Newton's Second Law of Motion.

(vi) When forces are in equilibrium?

(vii) State Newton's Third Law of Motion.

(viii) State parallelogram of forces.

(ix) State equilibrium law.

(x) State law of superposition.

(xi) State principle of transmissibility of forces.

(xii) A force of 10 N is acting at an angle 30° with x-axis. Find its resolved parts along x-axis and y-axis.

(xiii) A force of 10 N is acting at an angle 30° with y-axis. Find its resolved parts along x-axis and y-axis.

(xiv) Forces of 8 N and 7 N are acting at an angle 60°. Find the magnitude of the resultant.

(xv) Forces of 12 N and 10 N act at an angle 30°. Find the magnitude of the resultant.

(xvi) Forces of 8 N and 7 N act at an angle 60°. Find the angle which the resultant makes with force 8 N.

(xvii) Forces of 8 N and 7 N act at an angle 60°. Find the angle which the resultant makes with force 7 N.

(xviii) Find the greatest and the least resultants of the forces of 8 N and 7 N.

(xix) The greatest and least resultants of two forces acting at an angle 60° are of magnitudes P and Q respectively. Find the magnitude of the resultant of the forces.

(xx) What are concurrent forces?

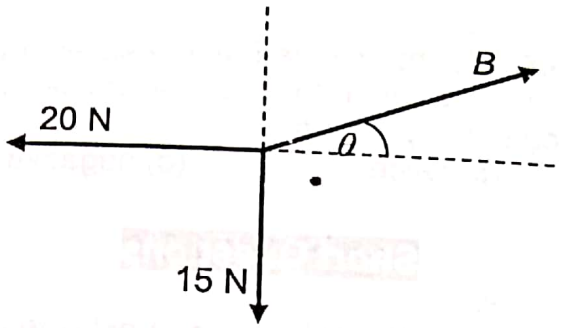
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Q.5 Solve / write answers of the following short questions:

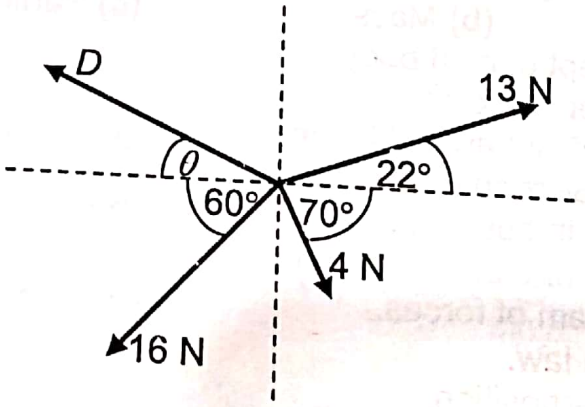
(i) Forces P, Q, R act at a point parallel to the sides of an equilateral triangle ABC taken in the same order. Find the magnitude of the resultant.

(ii) State of (λ, μ) theorem.

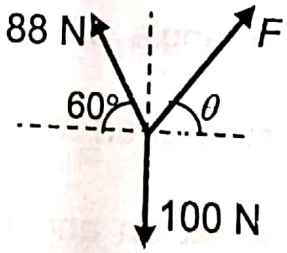
(iii) The three forces in the diagram are in equilibrium. What are the values of B and θ?



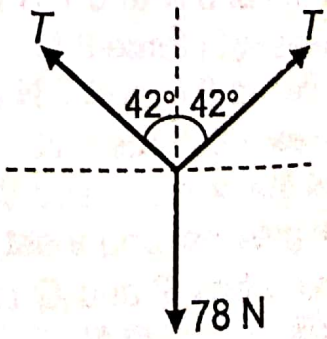
(iv) The four forces in the diagram are in equilibrium. What are the values of D and θ?



(v) The three forces in the diagram are in equilibrium. What are the values of F and θ?

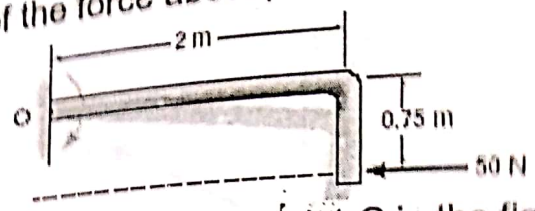


(vi) Two light inextensible strings suspend a particle of weight 78 N. The angle between each string and the vertical is 42°, as shown in diagram. What is the tension in each string?

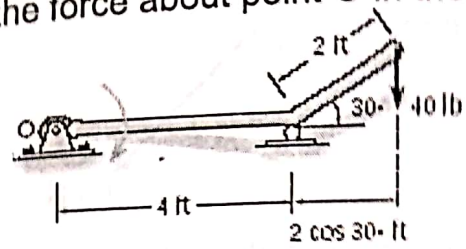


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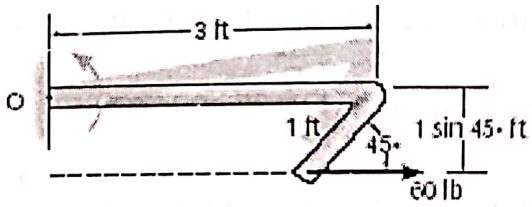
(vii) Find the moment of the force about point O in the figure.



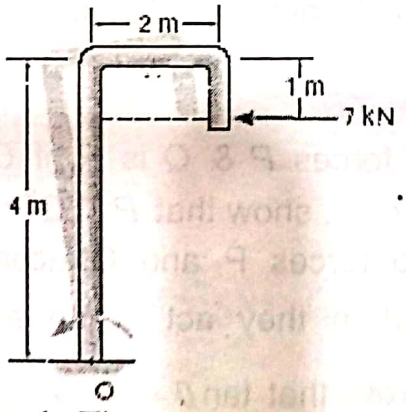
(viii) Find the moment of the force about point O in the figure.



(ix) Find the moment of the force about point O in the figure.



(x) Find the moment of the force about point O in the figure.

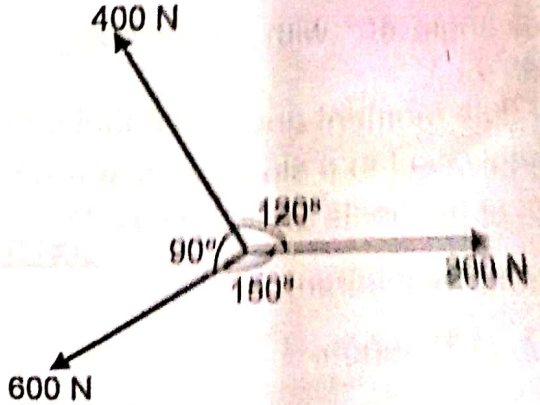


(xi) State and prove Varignon's Theorem.

PU, 2011; 2010 (BS Maths); A.K. 2012 (B.A/B.Sc.)

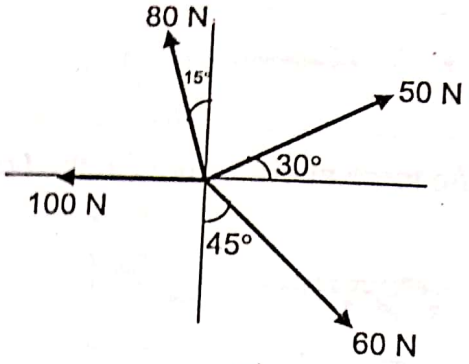
Long Questions

Q.6 Find the resultant of all the forces shown in figure below:



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Q.7 Find the resultant of all the forces shown in figure below:



Q.8 The resolved part of the resultant $\vec{R}$ of two coplanar forces $\vec{P}$ and $\vec{Q}$ acting at a point O in the direction of $\vec{Q}$ is of magnitude P. Show that the angle between the forces is $2\sin^{-1}\sqrt{\frac{Q}{2P}}$.

Q.9 Let $\vec{R}$ be the resultant of two coplanar forces $\vec{P}$ and $\vec{Q}$ acting at a point O. Find $\vec{R}$ if α is the angle between $\vec{P}$ and $\vec{Q}$, what will be the value of R if $P = Q$?

Q.10 The resultant of two forces P & Q is R, if Q is double R is treble, if Q is reversed R is again treble, show that $P^2 : Q^2 : R^2 :: 1 : 4 : 1$. PU, 2003 (B.A./B.Sc.)

Q.11 The resultant of two forces P and Q acting at an angle θ is equal to $(3m+1)\sqrt{P^2+Q^2}$; when they act at an angle $90°-\theta$, the resultant is $(3m-1)\sqrt{P^2+Q^2}$, prove that $\tan\theta = \frac{3m-2}{3m+2}$.

Q.12 Two forces $\vec{P}$ and $\vec{Q}$ act at a point and α is the angle between them. Find the magnitude and the direction of the resultant.

Q.13 Two forces $\vec{P}+\vec{Q}$ and $\vec{P}-\vec{Q}$ make an angle 2α with one another and their resultant makes an angle $45°$ with the bisector of the angle between them. Show that $P = Q\tan\alpha$. PU, 2014 (B.A./B.Sc.)

Q.14 Define a couple and its moment and prove that coplanar couples of moments $\vec{G}_1, \vec{G}_2, \ldots, \vec{G}_n$ are equivalent to a single couple lying in the same plane, whose moment is the sum of moments of all the couples.

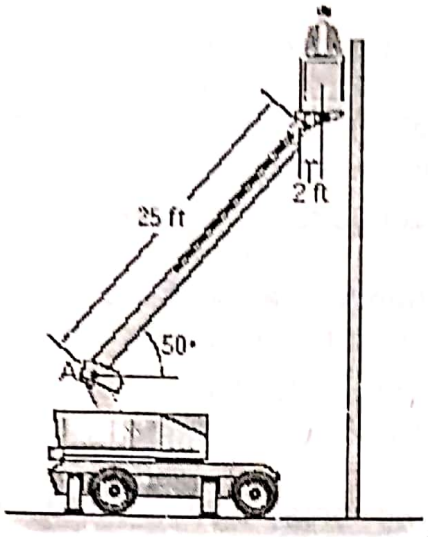
Q.15 State and prove Lammi's theorem. PU, 2013 (B.A./B.Sc.); PU, 2010 (SS Math)

Q.16 State and prove (λ, μ) Theorem. PU, 2009 (B.A./B.Sc.)

PU, 2011 (SS Math)

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Q.17 The snorkel boom lift is extended into the position shown. If the worker weighs 160 lb, determine the moment of this force about the connection at *A*.



SUMMARY

- Space, time and matter are the fundamental concepts.
- Matter which occupies a limited portion of space is called a body.
- The quantity of matter in a body is called its mass.
- A body of negligibly small dimensions is termed a particle.
- A body is said to be a rigid body if the distance between any two particles of the body remains unchanged so that that the body does not expand or contract or change its shape in any way.
- Force is defined as a measure of push or pull on an object.
- The point of body where the force acts is called its point of application.
- The line passing through the point of application in the direction of the force is called its line of action.
- A set of forces acting on a particle or rigid body is called a force system.
- Two force systems are said to be equivalent if one system can be replaced by the other system without disturbing the instantaneous state of rest or motion of a body.
- Mechanics deals with the action of forces on bodies which we encounter in nature and technology.
- Branch of mechanics which deals with bodies at rest is known as statics.
- The branch of mechanics which deals with bodies in motion is known as dynamics.
- The branch of dynamics which treats of the geometry of motion of bodies, without taking into account their masses or the forces acting on them is known as kinematics.
- The branch of dynamics which is concerned with the relations between the forces acting on bodies and the resulting motion is known as kinetics.

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- Inertia is a property of matter that resists changes in motion.
- The linear momentum of a particle is the vector $\vec{p} = m\vec{v}$, where m is the mass of the particle and v its velocity relative to a reference system.
- The forces acting on a particle are said to be in equilibrium if these do not change the state of rest or character of motion of the particle. In this case the state of the particle is referred to as an equilibrium state.
- A given force $\vec{R}$ can be replaced by an infinite number of pairs of possible components.
- If the two components of a force are mutually perpendicular, then such components are called the resolved parts of the given force.
- Scalar component of $\vec{R}$ along vector $\vec{u}$ is the dot product $\vec{R} \cdot \hat{u}$.
- Forces whose lines of action intersect at one point are called concurrent forces.
- The resultant of two forces bisects the angle between them if the forces have equal magnitudes.
- The magnitude R of the resultant of two forces $\vec{P}$ and $\vec{Q}$ acting at an angle α is given by $R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$.
- Magnitude of the resultant of the forces P, Q, R acting at a point parallel to the sides of a triangle ABC taken in the same order is $(P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C)^{\frac{1}{2}}$.
- A particle is in equilibrium if the resultant of all forces acting on the particle is equal to zero.
- If a particle is in equilibrium under the action of three coplanar forces, each force has a magnitude proportional to the sine of the angle between the other two.
- The tendency of a force to turn a body about a point is called the moment of that force.
- The perpendicular distance from the point of rotation to the line of action of the force is known as arm of the force.
- The moment about a point O of the resultant of a system of concurrent forces is equal to the sum of the moments of the various forces about the same point O.
- A couple is defined as two parallel forces with the same magnitude but opposite in direction separated by a perpendicular distance d.
- Any system of coplanar forces acting on a rigid body can be reduced to a single force at all arbitrary point in the plane of the forces together with a couple.
- A system of coplanar forces is in equilibrium if the algebraic sum of the resolved parts of the forces in any two directions is zero and the algebraic sum of the moments of all the forces about an arbitrary point is zero.

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