MTH304 Statics (Final)

VU

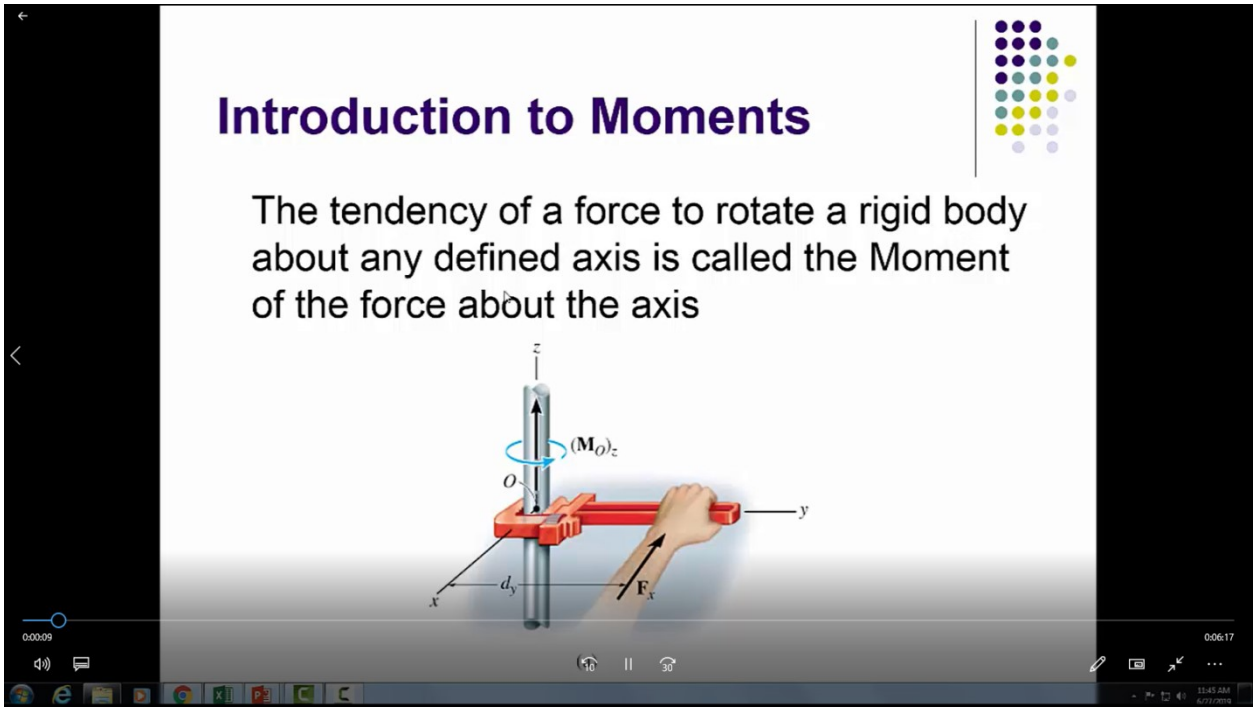


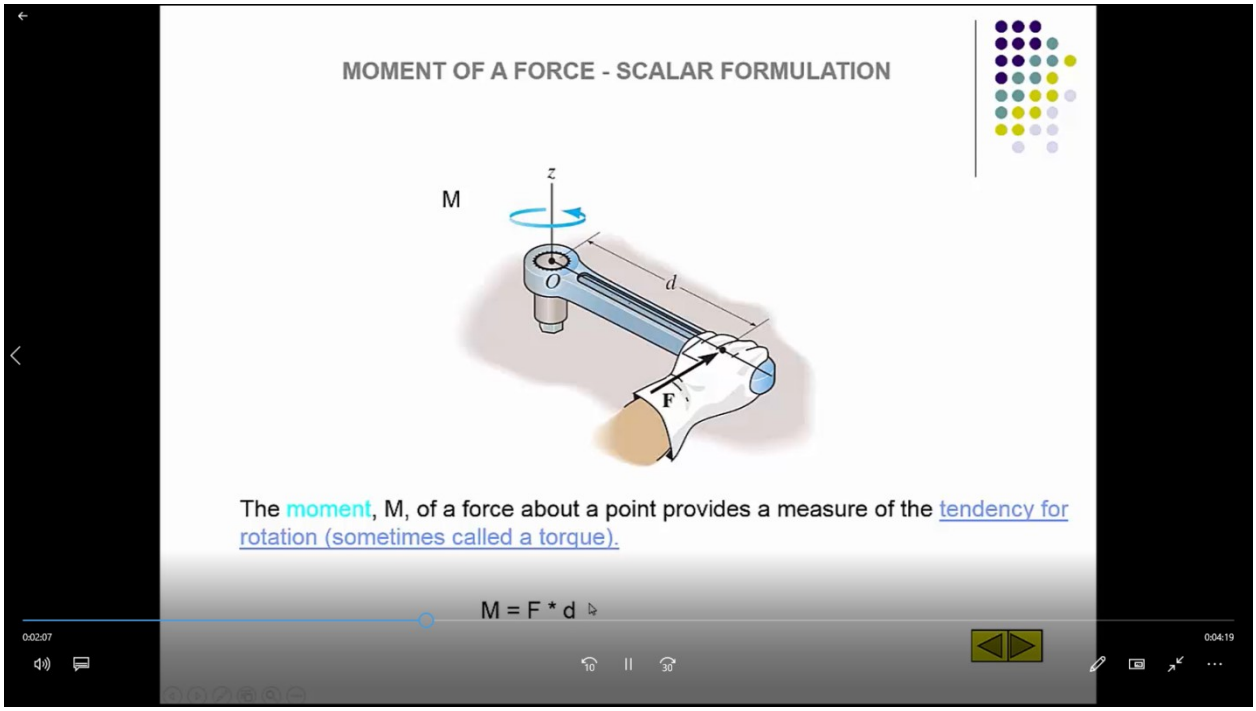
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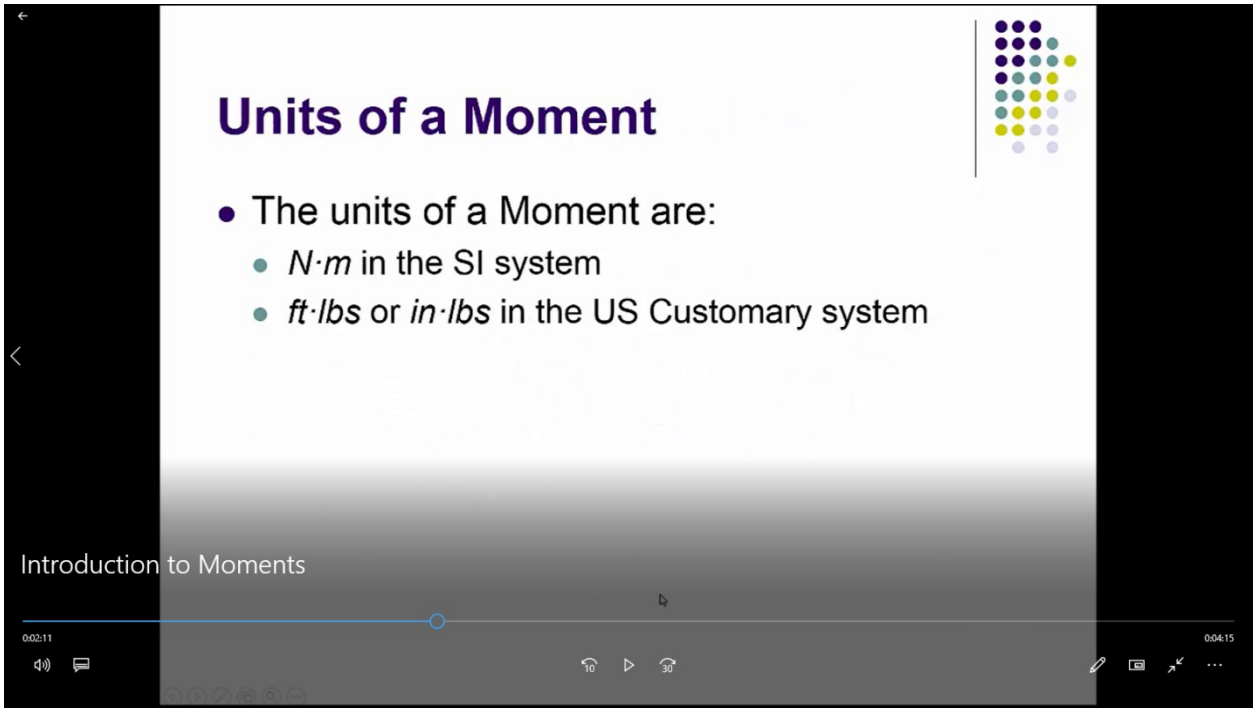
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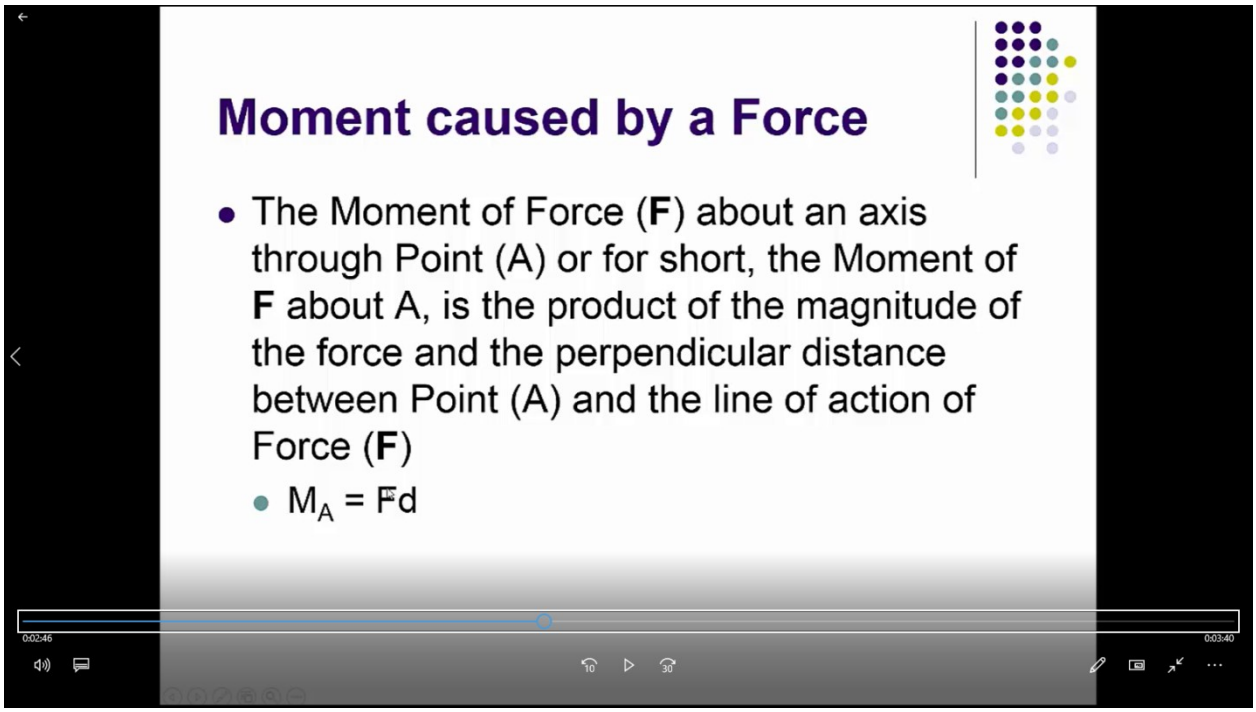
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Lecture 23

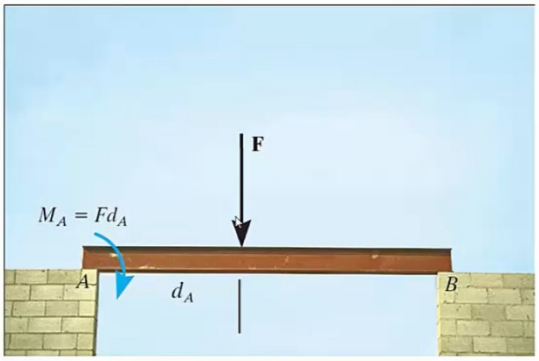

Introduction to Moments
The tendency of a force to rotate a rigid body about any defined axis is called the Moment of the force about the axis
z
$(\mathbf{M}_O)_z$
O
y
x
d_y
$\mathbf{F}_x$
0:00:09
0:06:17


MOMENT OF A FORCE - SCALAR FORMULATION
M
z
O
d
F
The moment, M, of a force about a point provides a measure of the tendency for rotation (sometimes called a torque).
M = F * d
0:02:07
0:04:19


Units of a Moment
The units of a Moment are:
N·m in the SI system
ft·lbs or in·lbs in the US Customary system
Introduction to Moments
0:02:11
0:04:15


Moment caused by a Force
The Moment of Force (F) about an axis through Point (A) or for short, the Moment of F about A, is the product of the magnitude of the force and the perpendicular distance between Point (A) and the line of action of Force (F)
$M_A = Fd$
0:02:46
0:03:40

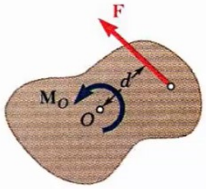
APPLICATIONS



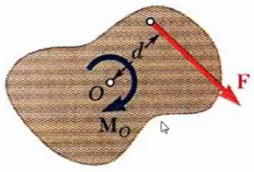
Beams are often used to bridge gaps in walls. We have to know what the effect of the force on the beam will have on the beam supports.

Properties of a Moment

- Moments not only have a magnitude, they also have a sense to them.
- The sense of a moment is clockwise or counter-clockwise depending on which way it will tend to make the object rotate



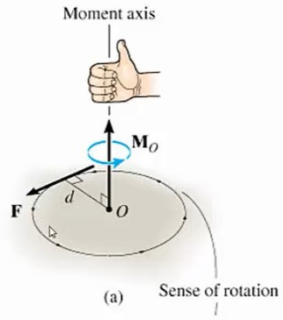
(a) $M_O = +Fd$



(b) $M_O = -Fd$

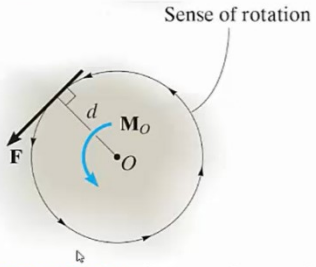
Properties of a Moment

- The sense of a Moment is defined by the direction it is acting on the Axis and can be found using Right Hand Rule.



MOMENT OF A FORCE - SCALAR FORMULATION (continued)

In the 2-D case, the magnitude of the moment is $M_o = F\,d$



As shown, d is the perpendicular distance from point O to the line of action of the force.

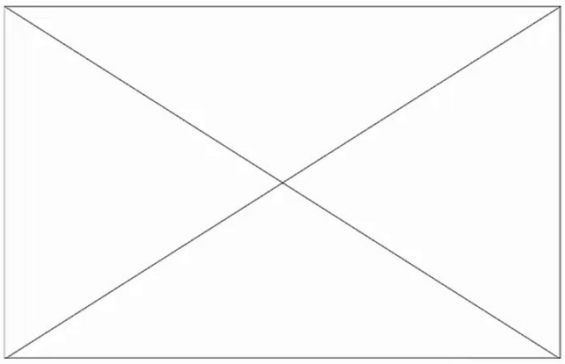
In 2-D, the direction of M_O is either clockwise or counter-clockwise, depending on the tendency for rotation.

Moments

The moment of a force is a measure of the force's capacity to rotate an object around a fixed point (pivot) or axis.

Moment = Force × Perpendicular distance

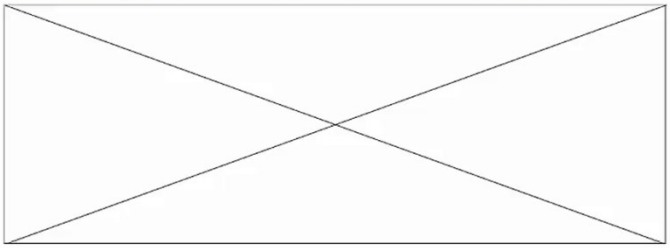
'Perpendicular distance' refers to the perpendicular distance between the line of action of the force and the centre of the rotation.



The moment of a force is measured in **Newton-metres (Nm)**.

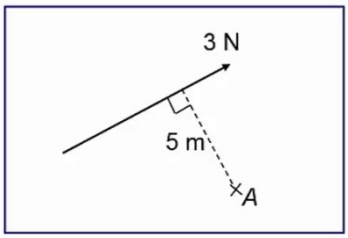
Adding and subtracting moments

Moments can act in a clockwise or anti-clockwise direction.
↻ is used to represent **clockwise** and ↺, **anti-clockwise**.
If two or more moments are acting on an object, the **overall moment** is the **difference** between the total clockwise moment and the total anti-clockwise moment.



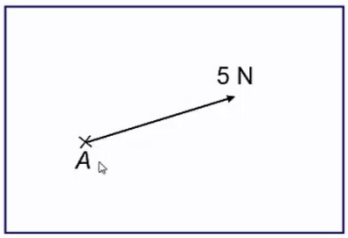
Calculating moments

Calculate the moment about *A* for each of the following forces:



Moment = 3 N × 5 m

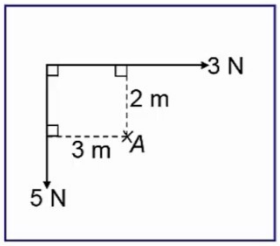
= **15 Nm** ↺



The moment is **zero** since the perpendicular distance is zero.

Calculating moments

Calculate the sum of the moments about *A* for each diagram.



↻ moment = 3 N × 2 m

= 6 Nm

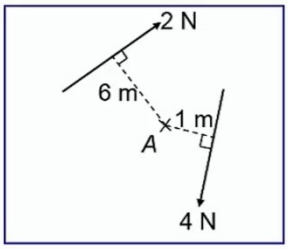
↺ moment = 5 N × 3 m

= 15 Nm

Resultant moment

= 15 Nm – 6 Nm

= **9 Nm** ↺



Both forces are acting in a clockwise direction.

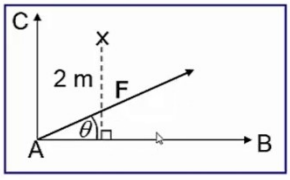
Resultant moment

= (2 × 6) + (4 × 1)

= **16 Nm** ↻

Non-perpendicular forces

If the distance given is not perpendicular to the force, you can use **trigonometry** to work out the perpendicular distance.



Force F is not perpendicular to its given distance to X. The line AB is. You can work out the component of F that is exerted in the direction AB.

The component of force F in the direction AB is equal to the magnitude of F, multiplied by the **cosine** of the angle between the force and the direction AB.

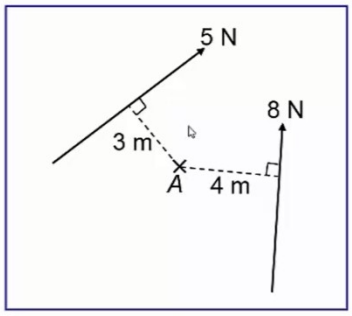
The component of the force in the direction AB = $F\cos\theta$.

The component of the force in the direction AC = $F\cos(90-\theta)$
= $F\sin\theta$.

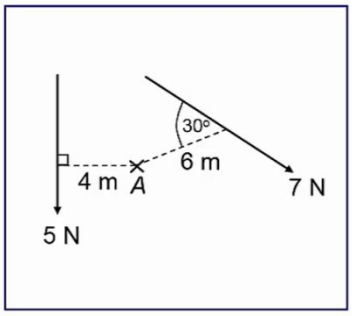
Calculating moments questions

For each of the following diagrams, calculate the sum of the moments about *A*. Remember to state whether the resulting moments are acting in a clockwise or anti-clockwise direction.

1.



2.



Calculating moments solutions

1. The 5 N force is acting in a clockwise direction and the 8 N force is acting in an anti-clockwise direction.

   ↻ moment = 5 N × 3 m = 15 Nm

   ↺ moment = 8 N × 4 m = 32 Nm

   Resultant moment = 32 Nm – 15 Nm = 17 Nm ↺

2. The 5 N force is acting in an anti-clockwise direction whilst the 7 N force is acting in a clockwise direction.

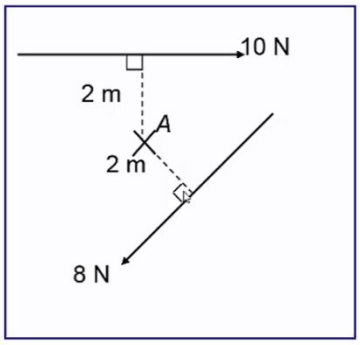
   ↺ moment = 5 N × 4 m = 20 Nm

   ↻ moment = 7sin30° N × 6 m = 21 Nm

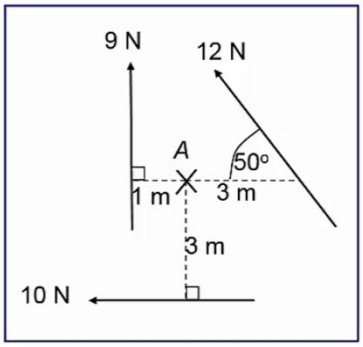
   Resultant moment = 21 Nm – 20 Nm = 1 Nm ↻

Calculating moments questions

3.



4.



Calculating moments solutions

3. Both forces are acting in a clockwise direction.

↻ moments = (10 N × 2 m) + (8 N × 2 m) = 20 Nm + 16 Nm

Resultant moment = **36 Nm ↻**

4. The 9 N and 10 N forces are acting in a clockwise direction. The 12 N force is acting in an anti-clockwise direction.

↻ moments = (9 N × 1 m) + (10 N × 3 m) = 9 Nm + 30 Nm

= 39 Nm

↺ moment = 12sin50° N × 3 m = 27.6 Nm (to 3 s.f.)

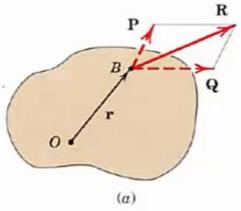
Resultant moment = 39 Nm – 27.6 Nm

= **11.4 Nm ↻** (to 3 s.f.)

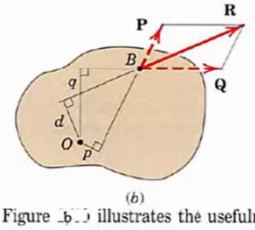
Lecture 24

Varignon's Theorem

- The moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.



(a)



(b)

Figure (b) illustrates the usefulness of Varignon's theorem. The moment of **R** about point O is Rd. However, if d is more difficult to determine than p and q, we can resolve **R** into the components **P** and **Q**, and compute the moment as

$$M_O = Rd = -pP + qQ$$

where we take the clockwise moment sense to be positive.

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{R}$$

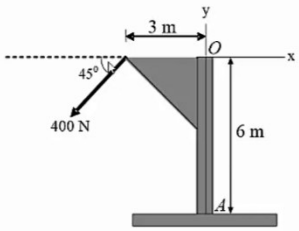
$$\mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{P} + \mathbf{Q})$$

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q}$$

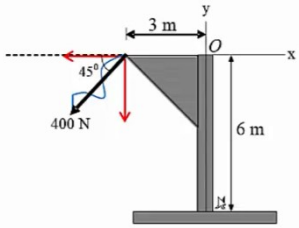
Problem

For the figure shown below, find moment of 400 N force about point *A* using:

1. Scalar approach
2. vector approach.



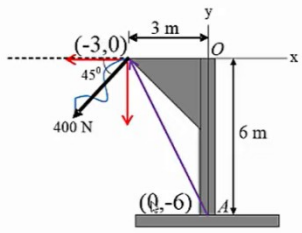
Scalar Approach



$$M_A = 400\cos 45^0 \times 6 + 400\sin 45^0 \times 3$$

$$\Rightarrow M_A = 2545.2 \text{ N.m (Counterclock wise)} \quad Ans.$$

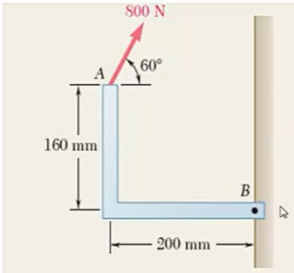
Vector Approach



$$\vec{F} = 400\cos 45^0(-\hat{i}) + 400\sin 45^0(-\hat{j}) = -282.8\hat{i} - 282.8\hat{j}$$

$$\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} = (-3-0)\hat{i} + (0-(-6))\hat{j} = -3\hat{i} + 6\hat{j}$$

$$\vec{M}_A = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 6 & 0 \\ -282.8 & -282.8 & 0 \end{vmatrix} = 2545.2\hat{k} \quad \text{N.m} \qquad \textit{Ans.}$$



PROBLEM

A force of 800 N acts on a bracket as shown. Determine the moment of the force about *B*.

SOLUTION

The moment $\mathbf{M}_B$ of the force $\mathbf{F}$ about *B* is obtained by forming the vector product

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F}$$

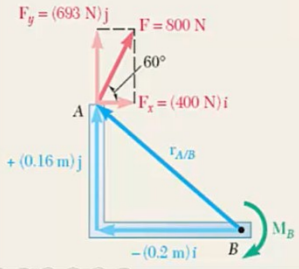
where $\mathbf{r}_{A/B}$ is the vector drawn from *B* to *A*. Resolving $\mathbf{r}_{A/B}$ and $\mathbf{F}$ into rectangular components, we have

$$\mathbf{r}_{A/B} = -(0.2\text{ m})\mathbf{i} + (0.16\text{ m})\mathbf{j}$$
$$\mathbf{F} = (800\text{ N})\cos 60°\mathbf{i} + (800\text{ N})\sin 60°\mathbf{j}$$
$$= (400\text{ N})\mathbf{i} + (693\text{ N})\mathbf{j}$$

Recalling the relations (3.7) for the cross products of unit vectors (Sec. 3.5), we obtain

$$\mathbf{M}_B = \mathbf{r}_{A/B} \times \mathbf{F} = [-(0.2\text{ m})\mathbf{i} + (0.16\text{ m})\mathbf{j}] \times [(400\text{ N})\mathbf{i} + (693\text{ N})\mathbf{j}]$$
$$= -(138.6\text{ N}\cdot\text{m})\mathbf{k} - (64.0\text{ N}\cdot\text{m})\mathbf{k}$$
$$= -(202.6\text{ N}\cdot\text{m})\mathbf{k} \qquad \mathbf{M}_B = 203\text{ N}\cdot\text{m}\ \curvearrowright$$

The moment $\mathbf{M}_B$ is a vector perpendicular to the plane of the figure and pointing *into* the paper.



Alternative Solution (Using Varignon's Theorem):

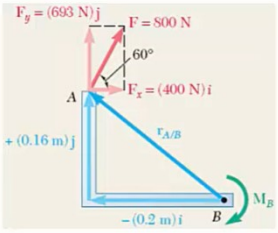
Assuming that the counterclockwise moment is positive;

$M_B = - F_x.(0.16m) - F_y.(0.2m)$

where $F_x = 400$ N and $F_y = 693$ N

$M_B = -400N(0.16m)-693N(0.2m) = -202.6$ N.m

We can say $M_B = -203$ N.m



Since we assumed that the counterclockwise moment is positive and we obtained a negative moment value, this implies that the direction of M_B is actually clockwise.

So; the result is exactly the same as before, i.e.

The Cross Product

Cross product of two vectors

0:00:02

0:08:30

The Cross Product

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, it is very useful to be able to find a nonzero vector **c** that is perpendicular to both **a** and **b**.

If $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is such a vector, then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$ and so

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \quad \boxed{1}$$

$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \quad \boxed{2}$$

The Cross Product

To eliminate c_3 we multiply 1 by b_3 and 2 by a_3 and subtract:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0 \quad \boxed{3}$$

Equation 3 has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So a solution of 3 is

$$c_1 = a_2b_3 - a_3b_2 \qquad c_2 = a_3b_1 - a_1b_3$$

The Cross Product

Substituting these values into [1] and [2], we then get

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both **a** and **b** is

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the *cross product* of **a** and **b** and is denoted by $\mathbf{a} \times \mathbf{b}$.

The Cross Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Notice that the **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors **a** and **b**, unlike the dot product, is a vector. For this reason it is also called the **vector product**.

Note that $\mathbf{a} \times \mathbf{b}$ is defined only when **a** and **b** are *three-dimensional* vectors.

The Cross Product

In order to make Definition easier to remember, we use the notation of determinants.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

The Cross Product

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 5 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears.

The Cross Product

Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0)$$

$$= -38$$

The Cross Product

If we now rewrite Definition using second-order determinants and the standard basis vectors **i**, **j**, and **k**, we see that the cross product of the vectors

$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

The Cross Product

In view of the similarity between Equations 5 and 6, we often write

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 7 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 5, we obtain Equation 6.

The symbolic formula in Equation 7 is probably the easiest way of remembering and computing cross products.

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Example 1

If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k}$$

$$= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

11

Net Moment

- The net moment is the sum of all the moments produced by all the forces
  - Remember to account for the direction of the tendency for rotation
    - Counterclockwise moments are positive
    - Clockwise moments are negative

Moment and Equilibrium

- First Condition of Equilibrium
  - The net external force must be zero

$$\Sigma \vec{\mathbf{F}} = 0 \text{ or}$$

$$\Sigma \vec{\mathbf{F}}_x = 0 \quad \text{and} \quad \Sigma \vec{\mathbf{F}}_y = 0$$

  - This is a necessary, but not sufficient, condition to ensure that an object is in complete mechanical equilibrium
  - This is a statement of translational equilibrium

Moment and Equilibrium, cont.

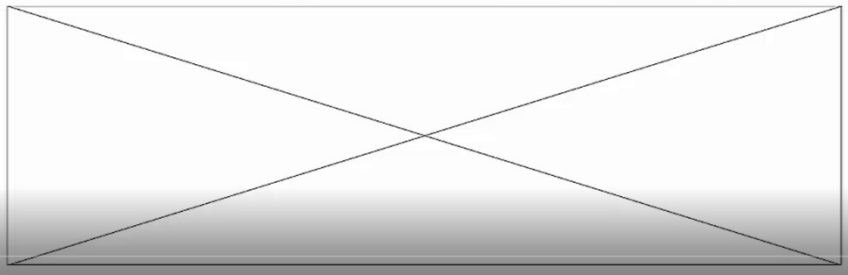
- To ensure mechanical equilibrium, you need to ensure rotational equilibrium as well as translational
- The Second Condition of Equilibrium states
  - The net Moment must be zero

$$\Sigma M_i = 0$$

Total Clockwise Moments=total Anti-clockwise moments

Rods

- If a rod is **uniform**, then its weight can be taken to act at the mid-point of the rod.
- If a rod is **pivoted** at a point, then there will be a reaction force at this point.
- If a rod is resting on **two or more supports**, then there will be reaction forces (not necessarily equal) at these points.



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Light uniform rods

A **light rod** may have a mass so small compared to the masses acting on it, that its own mass can be **ignored**.

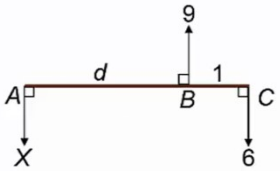
Moments of uniform rod about a point

0:00:58 0:11:25

Light uniform rods

A **light rod** may have a mass so small compared to the masses acting on it, that its own mass can be **ignored**.

1. The light rod shown below is in equilibrium.
   Calculate the size of the force X and the value of d.



Resolving forces vertically:

$X + 6 = 9$

so, $X = 3$

Taking moments about C

$9 \times 1 = 3(d + 1)$

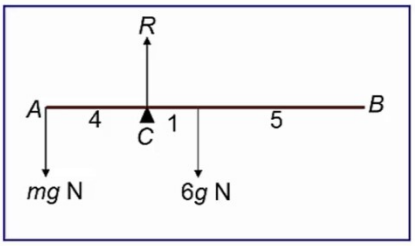
$9 - 3 = 3d$

$2 = d$

Uniform rod questions

2. A uniform rod AB has mass of 6 kg and is pivoted at C. The length of the rod is 10 m and distance AC = 4 m.

   Calculate the mass of the weight that must be attached at A to keep the rod in equilibrium.



Taking moments about C:

$m \times 4 = 6 \times 1$

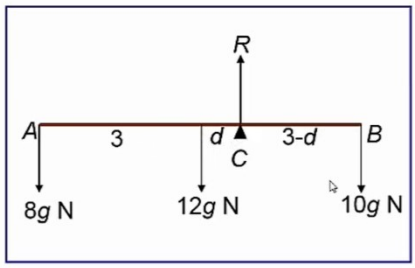
$4m = 6$

$m = 1.5$

Therefore, the mass of the weight that must be attached at A is **1.5 kg**.

Uniform rod questions

3. A uniform rod AB has length of 6 m and mass of 12 kg. A weight of mass 8 kg is attached at A and a weight of mass 10 kg is attached at B. The beam is pivoted at C and is in equilibrium. Find the distance AC.



Taking moments about C:

↻ moment

$= 10 \times (3 - d)$

$= 30 - 10d$

↺ moment

$= 8 \times (3 + d) + 12d$

$= 24 + 8d + 12d$

$= 24 + 20d$

So, $30 - 10d = 24 + 20d$

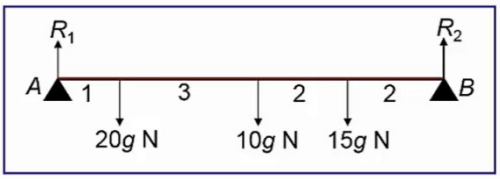
$6 = 30d$

$d = 0.2$ m

$\therefore$ The distance AC = **3.2 m**

Uniform rod questions

4. A uniform beam AB of mass 10 kg is 8 m long and is resting on two supports at A and B. A mass of 20 kg is placed on the beam 1 m from A. A mass of 15 kg is placed on the beam 2 m from B. Calculate the reaction forces at A and B.



Resolving forces vertically:

$R_1 + R_2 = 45g$ N

Taking moments about *A*:

$20 \times 1 + 10 \times 4 + 15 \times 6 = R_2 \times 8$

$150 = 8R_2$

$R_2 = 18.75g$ N (183.75 N)

So, $R_1 = 45 - R_2 = 45 - 18.75$

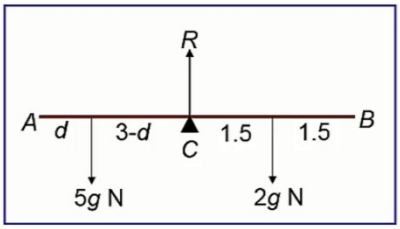
$= 26.25g$ N (257.25 N)

Lecture 26

Non-uniform rod questions

The centre of mass of a **non-uniform** rod is not necessarily at the rod's mid-point.

1. A non-uniform beam AB of length 6 m and mass 5 kg rests on a pivot at the mid-point C. When a mass of 2 kg rests on the beam 1.5 m from B, the system is in equilibrium. Find the distance between A and the centre of mass of the beam.



Taking moments about *C*:

$5g \times (3 - d) = 2g \times 1.5$

$15 - 5d = 3$

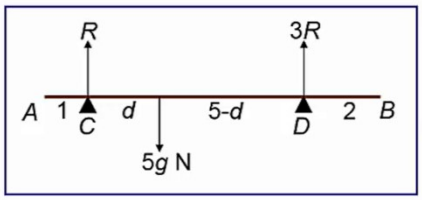
So, $5d = 12$

$d = 2.4$ m

Therefore, the centre of mass of the beam is **2.4 m** from A.

Non-uniform rod questions

2. A non-uniform rod AB has length 8 m and mass 5 kg. It is resting on two supports at C and D, where C is 1 m from A and D is 2 m from B. The reaction force at C is a third of that at D. Calculate the distance between A and the centre of mass of the rod.



Resolving vertically:

$4R = 5g$ N

$R = 1.25g$ N

Taking moments about *D*:

$5R = 5g(5 - d)$

$6.25g = g(25 - 5d)$

$6.25 = 25 - 5d$

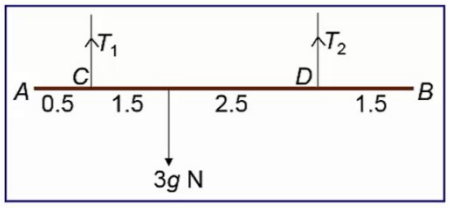
$5d = 18.75$

$d = 3.75$ m

So, the centre of mass is **4.75 m** from A.

Non-uniform rod questions

3. A non-uniform rod AB of length 6 m and mass 3 kg, has its centre of mass 2 m from A. It is suspended from the ceiling by two vertical strings attached at C and D, where C is 0.5 m from A and D is 1.5 m from B. Calculate the tension in the strings at C and D.



Resolving forces vertically:

$T_1 + T_2 = 3g$ N

Taking moments about *C*:

$3g \times 1.5 = T_2 \times 4$

$4T_2 = 4.5g$

$T_2 = 1.125g$ N

So, $T_1 = 1.875g$ N

So, the tension in the string at **C** is **1.875*g* N** (18.375 N) and the tension in the string at **D** is **1.125*g* N** (11.025 N)

Question

1. A footbridge across a stream consists of a tree trunk of length 6 m and mass 100 kg, supported at the ends A and B.

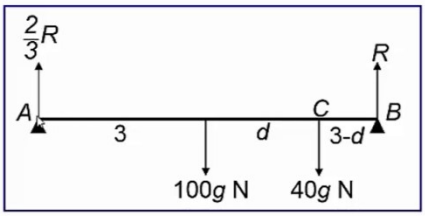
   A child of mass 40 kg is standing at a point C on the footbridge.

   Given that the magnitude of the force exerted by the support at A is two thirds of the magnitude of the force exerted by the support at B, calculate;

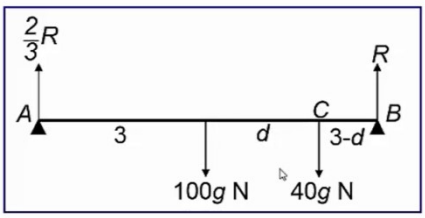
   a) the magnitude, in Newtons of the force exerted by the support at A,

   b) the distance AC.

Solution



Solution



a) **Resolving forces vertically**:

$$\frac{5}{3}R = 140g \text{ N}$$

$$R = 84g \text{ N}$$

$$\text{Force at } A = \frac{2 \times 84g \text{ N}}{3}$$

So, the force exerted by the support at $A = 56g$ N

$56 \times 9.8 =$ **548.8 N**

b) **Taking moments about *A***: $(100 \times 3) + 40(3 + d) = 84 \times 6$

$300 + 120 + 40d = 504$

$40d = 84$

$d = 2.1$ m

Therefore the distance AC is **5.1m**.

Example

A non-uniform rod AB has length 4 m and weight 100 N. The rod rests horizontally on two supports C and D, where C is 1 m from A and D is 0.5 m from B. The centre of mass of the rod is d m from A.

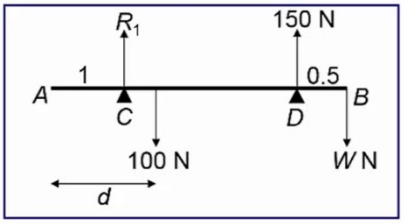
An object of weight W Newtons is placed at B. The magnitude of the reaction force on the rod at D is now 150 N.

a) Show that $100d + 3W = 475$

The object is now moved from B and placed at A. The rod remains in equilibrium and the magnitude of the reaction force at D is now 60 N.

b) Write another equation connecting W and d, and hence calculate W and d.

Solution

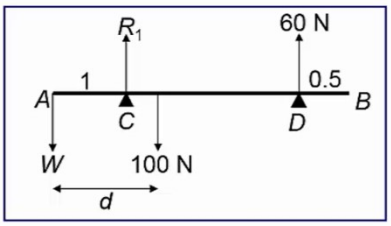


a) **Taking moments about *C*:**

$100 \times (d - 1) + 3W = 150 \times 2.5$

$100d - 100 + 3W = 375$

So, $100d + 3W = 475$



b) **Taking moments about *C*:**

$100 \times (d - 1) = 1W + 60 \times 2.5$

$100d - 100 = W + 150$

So, $100d - W = 250$

Solving simultaneously:

$$\begin{array}{r} 100d + 3W = 475 \\ -100d - W = 250 \\ \hline 4W = 225 \end{array}$$

$W = 56.25$ N and $d = 3.06$ m (3 s.f.).

Lecture 27

Example

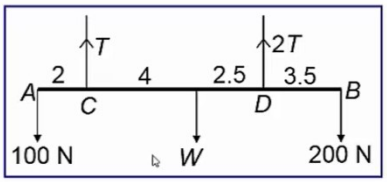
A heavy uniform steel rod AB has length 12 m. A particle of weight 100 N is placed at A and a particle of weight 200 N is placed at B.

The rod is attached to the ceiling by two steel cables at C and D where C is 2 m from A and D is 3.5 m from B. The tension in the cable at D is twice the tension in the cable at C.

a) Calculate the tension in the cable at C and state any assumptions that you have made.

b) Calculate the weight of the steel rod.

Solution



a) **Taking moments about the centre of the rod**:

$2T \times 2.5 + 100 \times 6 = T \times 4 + 200 \times 6$

$5T + 600 = 4T + 1200$

So, $T = 600$ N

Therefore, the tension in the cable at C is **600 N**.
It was assumed that the steel cables were light and inextensible.

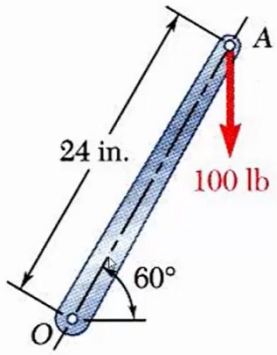
b) Resolving forces vertically: $600 + 1200 = 100 + W + 200$

$1800 = 300 + W$

So, $W = 1500$ N

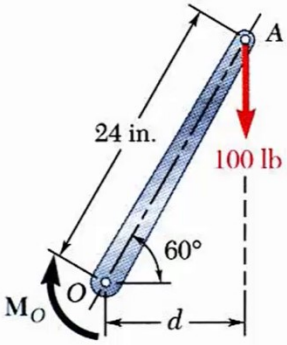
Therefore the steel rod has a weight of **1500 N**.

Example



- A 100-lb vertical force is applied to the end of a lever which is attached to a shaft at *O*.
- Determine:
  a) Moment about *O*,
  b) Horizontal force at *A* which creates the same moment,
  c) Smallest force at A which produces the same moment,
  d) Location for a 240-lb vertical force to produce the same moment,
  e) Whether any of the forces from b, c, and d is equivalent to the original force.

Example

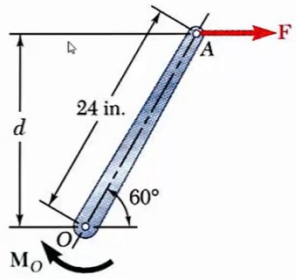


a) Moment about *O* is equal to the product of the force and the perpendicular distance between the line of action of the force and *O*. Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper.

$$M_O = Fd$$
$$d = (24\text{ in.})\cos 60° = 12\text{ in.}$$
$$M_O = (100\text{ lb})(12\text{ in.})$$
$$\boxed{M_O = 1200\text{ lb}\cdot\text{in}}$$

Example

b) Horizontal force at A that produces the same moment,



$$d = (24\text{ in.})\sin 60° = 20.8\text{ in.}$$

$$M_O = Fd$$

$$1200\text{ lb}\cdot\text{in.} = F(20.8\text{ in.})$$

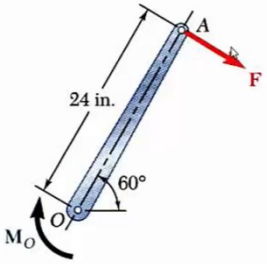
$$F = \frac{1200\text{ lb}\cdot\text{in.}}{20.8\text{ in.}}$$

$$\boxed{F = 57.7\text{ lb}}$$

0:03:14 0:04:28

Example

c) The smallest force at A to produce the same moment occurs when the perpendicular distance is a maximum or when F is perpendicular to OA.



$$M_O = Fd$$

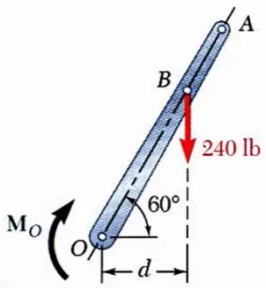
$$1200\text{ lb}\cdot\text{in.} = F(24\text{ in.})$$

$$F = \frac{1200\text{ lb}\cdot\text{in.}}{24\text{ in.}}$$

$$\boxed{F = 50\text{ lb}}$$

0:05:11 0:02:31

Example



d) To determine the point of application of a 240 lb force to produce the same moment,

$$M_O = Fd$$

$$1200\ \text{lb}\cdot\text{in.} = (240\ \text{lb})d$$

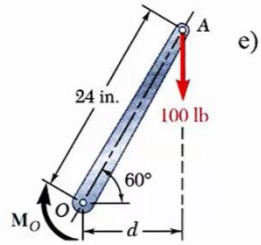
$$d = \frac{1200\ \text{lb}\cdot\text{in.}}{240\ \text{lb}} = 5\ \text{in.}$$

$$OB\cos 60° = 5\ \text{in.}$$

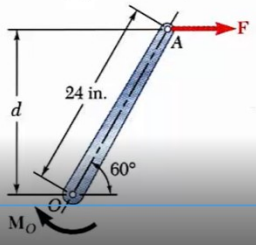
$$\boxed{OB = 10\ \text{in.}}$$

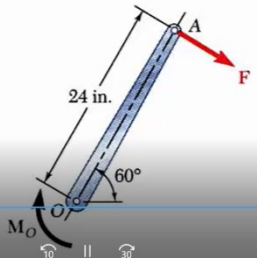
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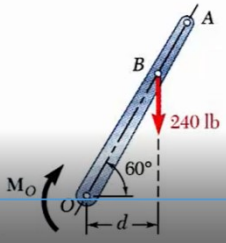
Example



e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 lb force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 100 lb force.

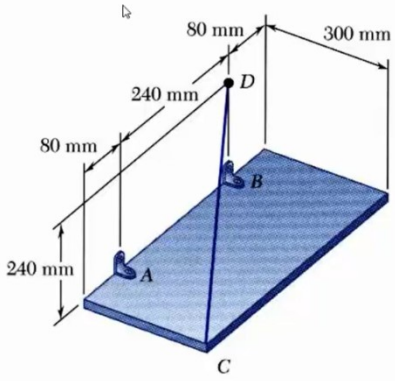






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Example



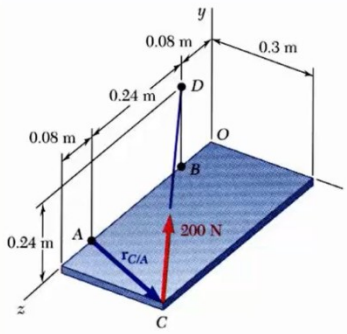
The rectangular plate is supported by the brackets at A and B and by a wire CD. Knowing that the tension in the wire is 200 N, determine the moment about A of the force exerted by the wire at C.

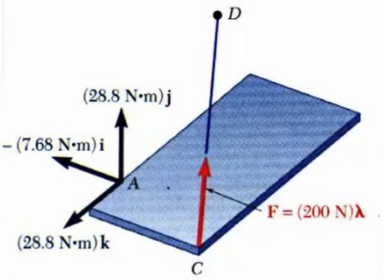
SOLUTION:

The moment M_A of the force F exerted by the wire is obtained by evaluating the vector product,

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

Example





SOLUTION:

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\text{ m})\vec{i} + (0.08\text{ m})\vec{j}$$

$$\vec{F} = F\vec{\lambda} = (200\text{ N})\frac{\vec{r}_{C/D}}{r_{C/D}}$$

$$= (200\text{ N})\frac{-(0.3\text{ m})\vec{i} + (0.24\text{ m})\vec{j} - (0.32\text{ m})\vec{k}}{0.5\text{ m}}$$

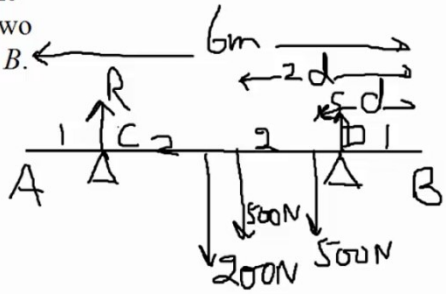
$$= -(120\text{ N})\vec{i} + (96\text{ N})\vec{j} - (128\text{ N})\vec{k}$$

$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

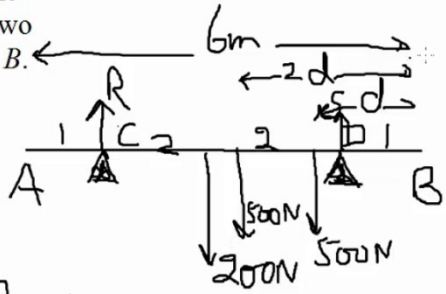
$$\boxed{\vec{M}_A = -(7.68\text{ N}\cdot\text{m})\vec{i} + (28.8\text{ N}\cdot\text{m})\vec{j} + (28.8\text{ N}\cdot\text{m})\vec{k}}$$

Example

A beam AB has length 6 m and weight 200 N. The beam rests in a horizontal position on two supports at the points C and D, where AC = 1 m and DB = 1 m. Two children, Sophie and Tom, each of weight 500 N, stand on the beam with Sophie standing twice as far from the end B as Tom. The beam remains horizontal and in equilibrium and the magnitude of the reaction at D is three times the magnitude of the reaction at C. By modelling the beam as a uniform rod and the two children as particles, find how far Tom is standing from the end B.



N, stand on the beam with Sophie standing twice as far from the end B as Tom. The beam remains horizontal and in equilibrium and the magnitude of the reaction at D is three times the magnitude of the reaction at C. By modelling the beam as a uniform rod and the two children as particles, find how far Tom is standing from the end B.



Sol Let Tom's dist. = d

Soph = 2d

$S = 3R$ — ①

$S = 3(300)$

$= 900\,N$

Moment about B

$R + S = 200 + 500 + 500$

$R + 3R = 1200$ — ②

$4R = 1200$

$R = 300$, $S = 900\,N$

Moment about B

$R = 300,\ S = 900\ N$

$$R \times 5 + 5 \times 1 = 200 \times 3 + 500 \times d + 500 \times 2d$$

$$1500d + 600 = 5R + 5$$

$$1500d + 600 = 1500 + 900$$

$$1500d = 1800$$

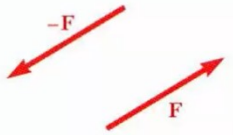
$$d = \frac{18}{15}$$

$$d = 1.2\ m$$

Lecture 28

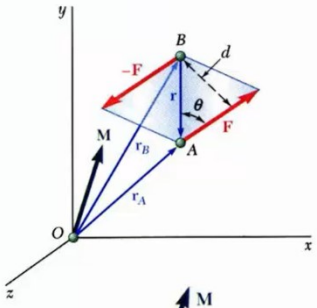
Moment of a Couple

- Two forces ***F*** and ***-F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.

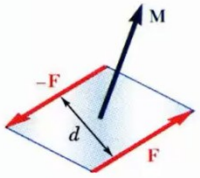


- Moment of the couple,

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times (-\vec{F})$$
$$= (\vec{r}_A - \vec{r}_B) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF \sin\theta = Fd$$



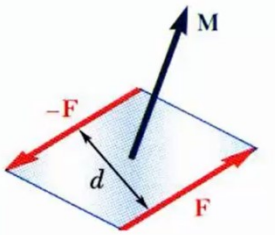
- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.

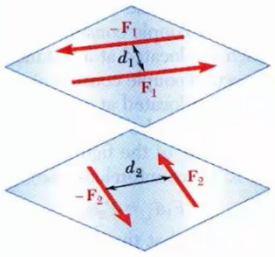


Moment of a Couple

Two couples will have equal moments if

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.



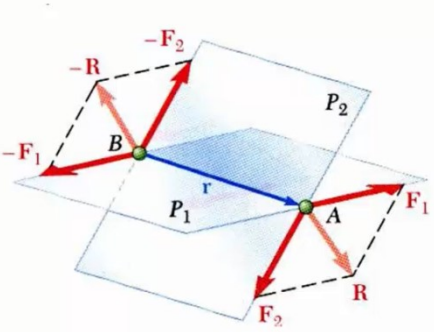


Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

  $\vec{M}_1 = \vec{r} \times \vec{F}_1$ in plane P_1

  $\vec{M}_2 = \vec{r} \times \vec{F}_2$ in plane P_2



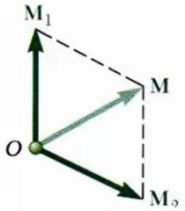
- Resultants of the vectors also form a couple

  $\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$

- By Varigon's theorem

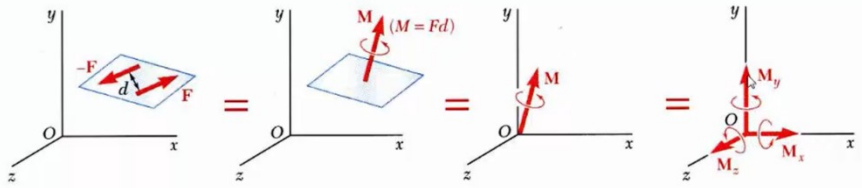
  $\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$

  $= \vec{M}_1 + \vec{M}_2$



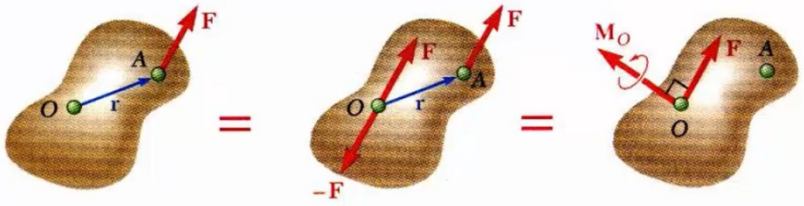
- Sum of two couples is also a couple that is equal to the vector sum of the two couples

Couples Can Be Represented by Vectors



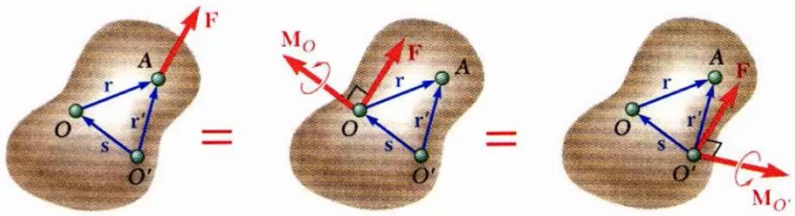
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.
- *Couple vectors* obey the law of addition of vectors.
- Couple vectors are free vectors, i.e., the point of application is not significant.
- Couple vectors may be resolved into component vectors.

Resolution of a Force Into a Force at O and a Couple



- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body.
- Attaching equal and opposite force vectors at *O* produces no net effect on the body.
- The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

Resolution of a Force Into a Force at O and a Couple



- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector $\boldsymbol{M_{O'}}$

$$\vec{M}_{O'} = \vec{r}' \times \vec{F}$$

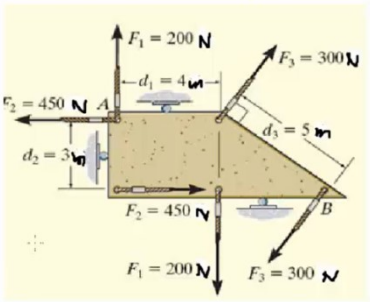
- The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}' \times \vec{F} = (\vec{r} + \vec{s}) \times \vec{F} = \vec{r} \times \vec{F} + \vec{s} \times \vec{F}$$
$$= \vec{M}_O + \vec{s} \times \vec{F}$$

- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

Example

Determine the Resultant couple moment of three couples acting on the plate as shown in fig.



Sol.

C-1

$F_1 = 200\ N$

$d_1 = 4\ m$

$M_1 = -F_1 d_1$

$= -200 \times 4$

$= -800\ Nm$

C-2 $F_2 = 450N$ $d_2 = 3\ m$

$M_2 =$

$= -800\ Nm$

C-2 $F_2 = 450N$ $d_2 = 3\ m$

$M_2 = +F_2 d_2 = 450 \times 3m$

$= 1350\ Nm$

C-3 $F_3 = 300\ N$ $d_3 = 5m$

$M_3 = -F_3 d_3$

$= -300 \times 5$

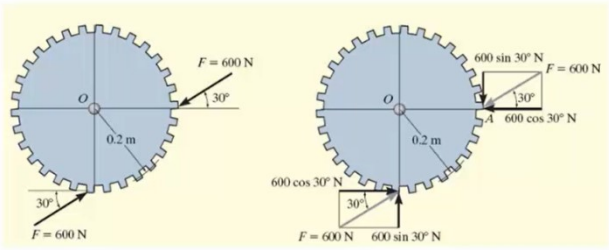
$= -1500\ Nm$

$M_R = -800 + 1350 - 1500$

$= -950\ Nm$

Example

Determine the magnitude and the direction of Couple moment acting on the gear as shown in the fig.



Sol

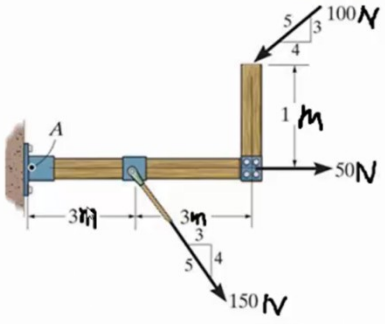
$$F_y = F\sin\theta = 600\sin 30^\circ$$

$$F_x = F\cos\theta = 600\cos 30^\circ$$

$$M_O = M_1 + M_2 = 600\cos 30^\circ \times 0.2 - 600\sin 30^\circ \times 0.2$$

$$= 103.9 - 60 = 43.9 \text{ N m}$$

EXAMPLE



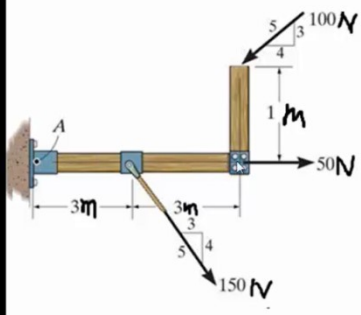
Given: A 2-D force system with geometry as shown.

Find: The equivalent resultant force and couple moment acting at A and then the equivalent single force location measured from A.

Plan:

1) Sum all the x and y components of the forces to find F_{RA}.
2) Find and sum all the moments resulting from moving each force component to A.
3) Shift F_{RA} to a distance d such that $d = M_{RA}/F_{Ry}$

EXAMPLE (continued)



$\rightarrow \Sigma F_{Rx} = 150\,(3/5) + 50 - 100\,(4/5)$
$= 60$ N

$\downarrow \Sigma F_{Ry} = 150\,(4/5) + 100\,(3/5)$
$= 180$ N

$\circlearrowleft M_{RA} = 100\,(4/5)\,1 - 100\,(3/5)\,6$
$- 150(4/5)\,3 = -640$ Nm

$F_R = (\,60^2 + 180^2\,)^{1/2} = 190$ n

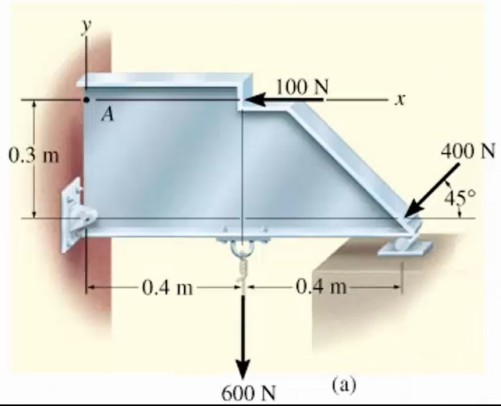
$\theta = \tan^{-1}(\,180/60) = 71.6\,^\circ$

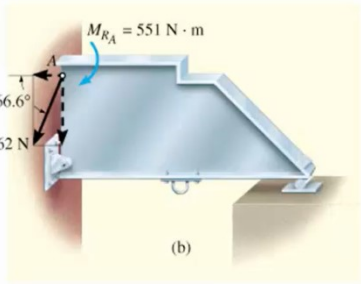
The equivalent single force F_R can be located at a distance d measured from A.

$d = M_{RA}/F_{Ry} = 640 / 180 = 3.56$ m.

Example

- Replace the forces acting on the brace shown in Figure by an equivalent resultant and couple moment acting at point A.





+ $F_{Rx} = \Sigma F_x$; F_{Rx} = -100 N - 400 cos 45° = - 382.8 N = 382.8 N

+ $F_{Ry} = \Sigma F_y$; F_{Ry} = -600 N - 400 sin 45° = - 882.8 N = 882.8 N

F_R has a magnitude of

$$F_R = \sqrt{(F_{Rx})^2 + (F_{Ry})^2} = \sqrt{(382.8)^2 + (882.8)^2} = 962 \text{ N}$$

and a direction of $\theta = \tan^{-1}\left(\frac{F_{Ry}}{F_{Rx}}\right) = \tan^{-1}\left(\frac{882.8}{382.8}\right) = 66.6°$

- The resultant couple moment M_{RA} is determined by summing the moments of the forces about point A. Assuming that positive moments act clockwise, we have

+ $M_{RA} = \Sigma M_A$

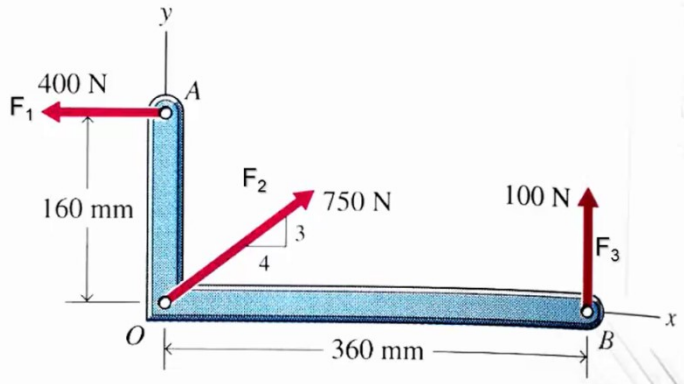
M_{RA} = 100 N (0) + 600 N (0.4m) + (400 sin 45°) (0.8 m) + (400 cos 45°) (0.3 m)

= 551 Nm

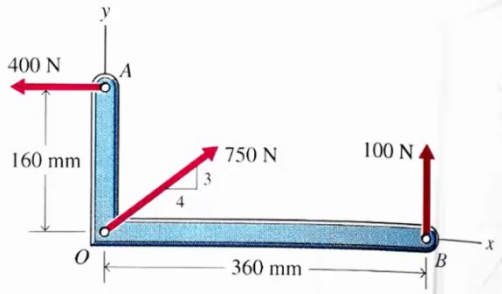
Lecture 30

Example

Replace the three forces shown with an equivalent force-couple system at A.



To find the equivalent set of forces at A.



$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ$$

$$R_x = \sum F_x$$

$$= -400\ \text{N} + 750\ \text{N}\cos\left(36.87^\circ\right)$$

$$= 200\ \text{N}$$

$$R_y = \sum F_y = 750\ \text{N}\sin\left(36.87^\circ\right) + 100\ \text{N}$$

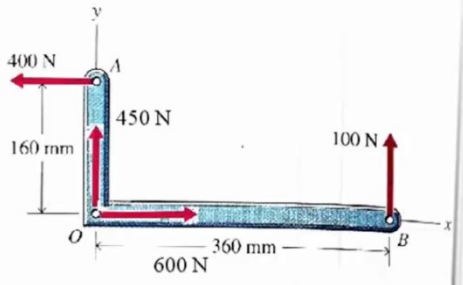
$$= 550\ \text{N}$$

$$R = \sqrt{200^2 + 550^2} = 585N \quad \alpha = \tan^{-1}\left(\frac{550}{200}\right) = 70^0$$

Find the moments about point A.

The force at O can be broken up into its two components in the x and y direction

$$F_x = 750\ \text{N}\cos\left(36.87^\circ\right)$$
$$= 600\ \text{N}$$
$$F_y = 750\ \text{N}\sin\left(36.87^\circ\right)$$
$$= 450\ \text{N}$$



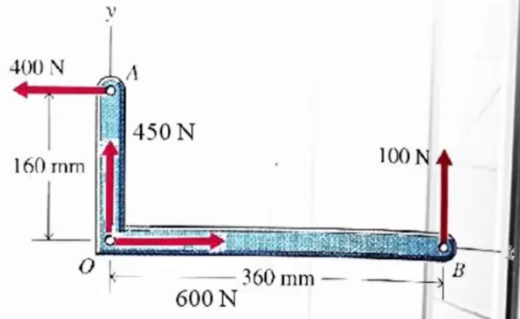
Using the line of action for each component, their moment contribution can be determined.

Find the moments about point A.

Using the line of action for F_x component d is 160 mm.

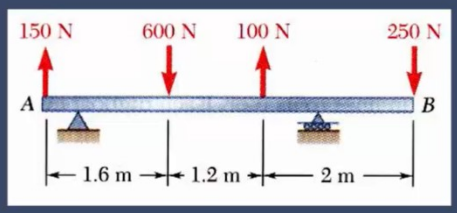
$$\left|\overrightarrow{M_2}\right| = \left|\overrightarrow{F_{Ox}}\right| d$$
$$= 600\ \text{N}(160\ \text{mm})$$
$$= 96000\ \text{N-mm}$$

Fy component is 0 since in line with A.



$$M_A = \sum M_i$$
$$= M_1 + M_2 + M_3$$
$$= 36000\ \text{N-mm} + 96000\ \text{N-mm} + 0\ \text{N-mm}$$
$$= 132000\ \text{N-mm} = 132\text{N-m}$$

Example

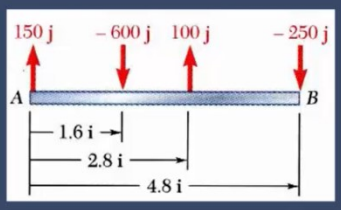


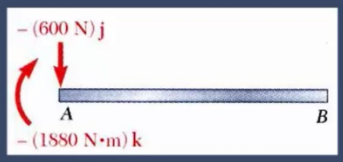
For the beam given in the fig.

a) Compute the resultant force for the forces shown and the resultant couple for the moments of the forces about A.

b) Find an equivalent force-couple system at B based on the force-couple system at A.

Example





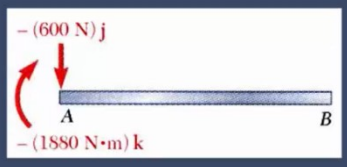
SOLUTION:

a) Compute the resultant force and the resultant couple at A.

$$\vec{R} = \sum \vec{F}$$
$$= (150\text{ N})\vec{j} - (600\text{ N})\vec{j} + (100\text{ N})\vec{j} - (250\text{ N})\vec{j}$$
$$\vec{R} = -(600\text{ N})\vec{j}$$

$$\vec{M}_A^R = \sum (\vec{r} \times \vec{F})$$
$$= (1.6\vec{i}) \times (-600\vec{j}) + (2.8\vec{i}) \times (100\vec{j}) + (4.8\vec{i}) \times (-250\vec{j})$$
$$\vec{M}_A^R = -(1880\text{ N}\cdot\text{m})\vec{k}$$

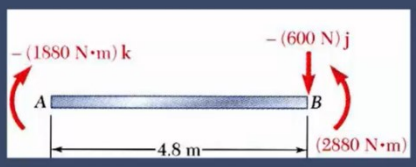
Example



b) Find an equivalent force-couple system at B based on the force-couple system at A.

The force is unchanged by the movement of the force-couple system from A to B.

$$\vec{R} = -(600\ \text{N})\vec{j}$$

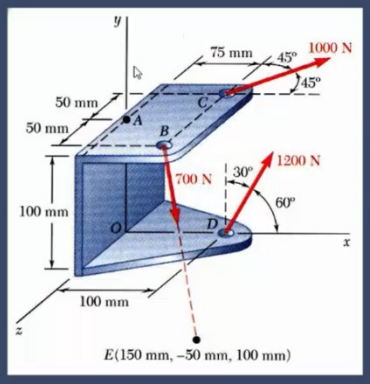


The couple at B is equal to the moment about B of the force-couple system found at A.

$$\vec{M}_B^R = \vec{M}_A^R + \vec{r}_{B/A} \times \vec{R}$$
$$= -(1880\ \text{N}\cdot\text{m})\vec{k} + (-4.8\ \text{m})\vec{i} \times (-600\ \text{N})\vec{j}$$
$$= -(1880\ \text{N}\cdot\text{m})\vec{k} + (2880\ \text{N}\cdot\text{m})\vec{k}$$
$$\vec{M}_B^R = +(1000\ \text{N}\cdot\text{m})\vec{k}$$



Example

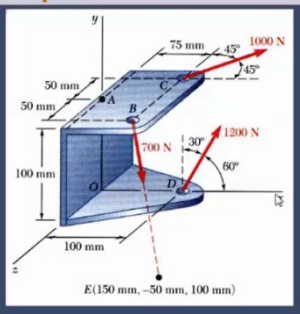


Three cables are attached to the bracket as shown. Replace the forces with an equivalent force-couple system at A.

SOLUTION:

- Determine the relative position vectors for the points of application of the cable forces with respect to A.
- Resolve the forces into rectangular components.
- Compute the equivalent force,
  $\vec{R} = \sum \vec{F}$
- Compute the equivalent couple,
  $\vec{M}_A^R = \sum (\vec{r} \times \vec{F})$

Example



SOLUTION:

- Determine the relative position vectors with respect to A.

$$\vec{r}_{B/A} = 0.075\vec{i} + 0.050\vec{k}\ (\text{m})$$
$$\vec{r}_{C/A} = 0.075\vec{i} - 0.050\vec{k}\ (\text{m})$$
$$\vec{r}_{D/A} = 0.100\vec{i} - 0.100\vec{j}\ (\text{m})$$

- Resolve the forces into rectangular components.

$$\vec{F}_B = (700\ \text{N})\vec{\lambda}$$
$$\vec{\lambda} = \frac{\vec{r}_{E/B}}{r_{E/B}} = \frac{75\vec{i} - 150\vec{j} + 50\vec{k}}{175}$$
$$= 0.429\vec{i} - 0.857\vec{j} + 0.289\vec{k}$$
$$\vec{F}_B = 300\vec{i} - 600\vec{j} + 200\vec{k}\ (\text{N})$$

$$\vec{F}_C = (1000\ \text{N})(\cos 45\vec{i} - \cos 45\vec{j})$$
$$= 707\vec{i} - 707\vec{j}\ (\text{N})$$

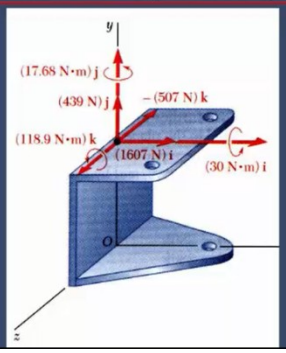
$$\vec{F}_D = (1200\ \text{N})(\cos 60\vec{i} + \cos 30\vec{j})$$
$$= 600\vec{i} + 1039\vec{j}\ (\text{N})$$

Example

- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$
$$= (300 + 707 + 600)\vec{i}$$
$$+ (-600 + 1039)\vec{j}$$
$$+ (200 - 707)\vec{k}$$

$$\vec{R} = 1607\vec{i} + 439\vec{j} - 507\vec{k}\ (\text{N})$$



- Compute the equivalent couple,

$$\vec{M}_A^R = \sum (\vec{r} \times \vec{F})$$

$$\vec{r}_{B/A} \times \vec{F}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\vec{i} - 45\vec{k}$$

$$\vec{r}_{C/A} \times \vec{F}_C = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\vec{j}$$

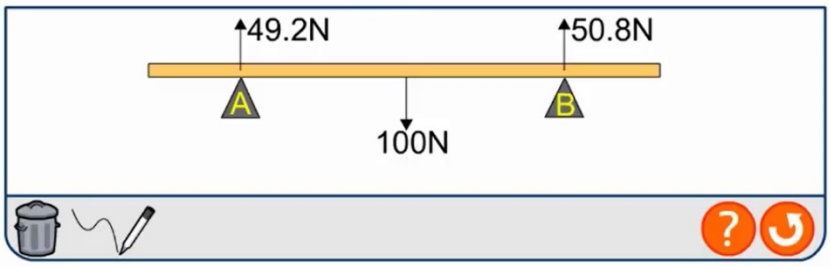
$$\vec{r}_{D/A} \times \vec{F}_D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\vec{k}$$

$$\vec{M}_A^R = 30\vec{i} + 17.68\vec{j} + 118.9\vec{k}$$

Lecture 31

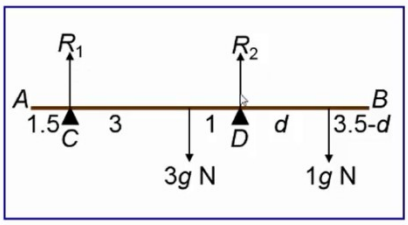
Tilting

If a rod is resting on two supports at *A* and *B* and is on the point of tilting about *A*, then the reaction force at *B* is 0.



Tilting questions

1. A 9 m uniform rod *AB* is resting on two supports at *C* and *D*, where *C* is 1.5 m from *A* and *D* is 3.5 m from *B*. The mass of the rod is 3 kg. A mass of 1 kg is attached to the rod so that the rod is on the point of tilting about *D*. Find the distance from *A* where this mass should be placed.



When the rod is about to tilt:

$R_1 = 0$

Taking moments about *D*:

$3g \times 1 = 1g \times d$

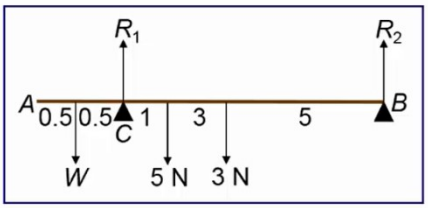
$3 = 1d$

$d = 3$ m

Therefore the 1 kg mass should be placed **8.5 m** from *A*.

Tilting questions

2. A uniform rod AB of length 10 m and weight 3 N rests on two supports at B and C, where C is 1 m from A. A weight of 5 N is attached to the rod 2 m from A. Given that when a load is attached 0.5 m from A, the rod is on the point of tilting about C, find the weight of this load in Newtons.



Taking moments about *C*:

$0.5W = 5 \times 1 + 3 \times 4$

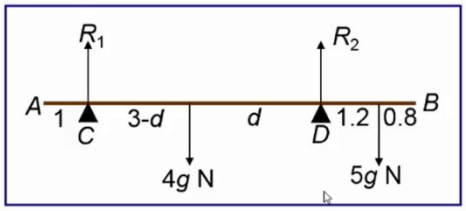
$0.5W = 17$

$W = 34$ N

Therefore the load must have a weight of **34 N**.

Tilting questions

3. A non-uniform rod AB of length 6 m and mass 4 kg is resting on two supports C and D, where C is 1 m from A and D is 2 m from B. When a mass of 5 kg is attached at a point 0.8 m from B, the rod is on the point of tilting about D. Calculate the distance between A and the centre of mass of the rod.



Taking moments about *D*:

$4g \times d = 5g \times 1.2$

$4d = 6$

$d = 1.5$ m

Therefore the centre of mass of the rod is **2.5 m** from A.

Example

A log AB of length 10 m is placed on two smooth supports at C and D where C is 1 m from A and D is 2 m from B. When a force of 600 N is applied to the log vertically upwards at A, the log is on the point of tilting about D.

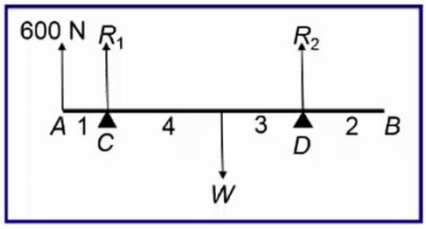
Initially, the log is modelled as a uniform rod.

a) Calculate an estimate of the weight of the log.

The force at A is now removed and a force of 750 N is applied to the log vertically upwards at B. The log is now on the point of tilting about C.

0:00:05 0:05:55

Solution



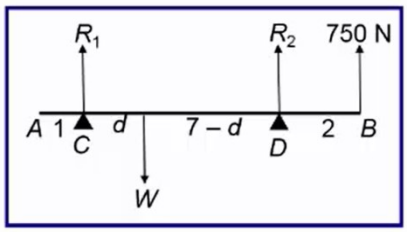
Taking moments about *D*:

Remember: $R_1 = 0$

$$600 \times 8 = 3W$$
$$3W = 4800$$
$$W = 1600 \text{ N}$$

Therefore an estimate of the weight of the log is **1600 N**.

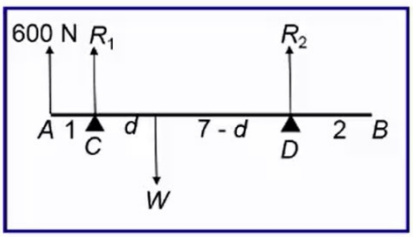
Solution



Taking moments about *C*:

$W \times d = 750 \times 9$

$Wd = 6750$



Taking moments about *D*:

$600 \times 8 = W \times (7 - d)$

$7W - Wd = 4800$

Solution

Solving simultaneously gives:

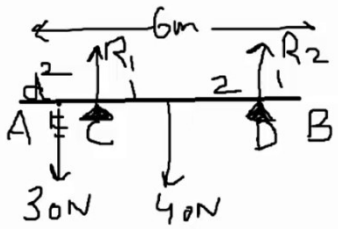
$$\begin{array}{r} 7W - Wd = 4800 \\ + \quad Wd = 6750 \\ \hline 7W = 11550 \end{array}$$

So, $W = 1650$

Therefore, a revised estimate of the weight of the log is **1650 N**.

Example

A uniform bar AB of length 6 m and weight 40 N is resting in a horizontal position on supports at points C and D where AC = 2 m and AD = 5 m. When a particle of weight 30 N is attached to the bar at point E the bar is on the point of tilting about C. Calculate the distance AE.



Sol:-

$R_2 = 0$ (on the pt. of tilting) $\longleftarrow \bar{s} \longrightarrow$

Moments about C

$$30(2-d) = 40(1)$$

$$60 - 30d = 40 \Rightarrow 30d = 20 \Rightarrow d = \frac{2}{3}$$

Lecture 32

Example

A uniform rod AB has length 4 m and mass 8 kg. It is resting in a horizontal position on supports at points C and D were AC = 1m and AD = 2.5m. A particle of mass m kg is placed at point E where AE = 3.3m. Given that rod is about to tilt about D, calculate the value of m.

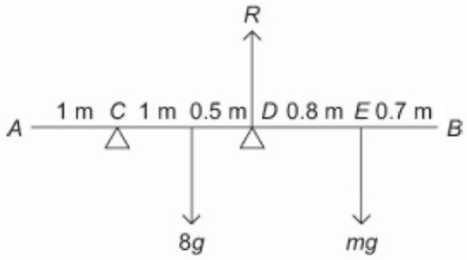
Solution

If the rod is about to turn about D then the reaction at C is zero.

Taking moments about point D:

$8g \times 0.5 = m\ g \times 0.8$

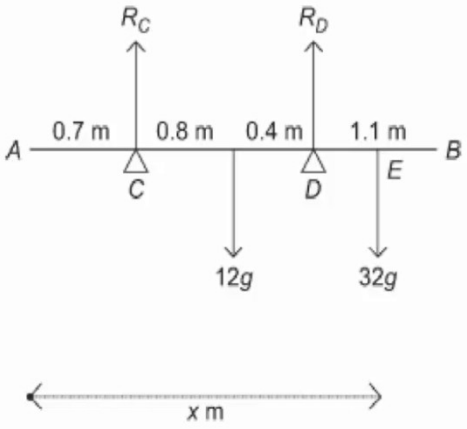
$\Rightarrow m = 5$



Question:

A plank AB of mass 12 kg and length 3 m is in equilibrium in a horizontal position resting on supports at C and D where AC = 0.7 m and DB = 1.1 m. A boy of mass 32 kg stands on the plank at point E. The plank is about to tilt about D. By modelling the plank as a uniform rod and the boy as a particle, calculate the distance AE.

Solution:



Let the distance AE be x m.

If the plank is about to tilt about D then $R_C = 0$.

Taking moments about D:

$12g \times 0.4 = 32g \times (x - 1.9)$

$12 \times 0.4 = 32x - 32 \times 1.9$

$32x = 4.8 + 60.8 = 65.6$

$\Rightarrow x = 65.6 \div 32 = 2.05$ m

E is 2.05 m from A

Question:

A uniform rod AB has length 5 m and weight 20 N. The rod is resting on supports at points C and D where $AC = 2$ m and $BD = 1$ m.

a Find the magnitudes of the reactions at C and D.

A particle of weight 12 N is placed on the rod at point A.

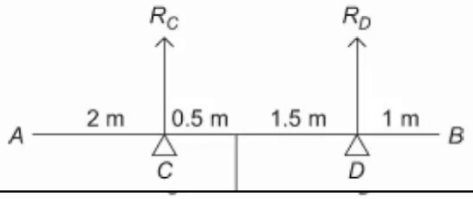
b Show that this causes the rod to tilt about C.

A second particle of weight 12 N is placed on the rod at E to hold it in equilibrium.

c How far must E be from A?

Solution:

a



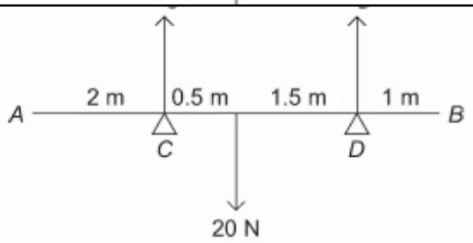
R(↑)

$R_C + R_D = 20$

Taking moments about C:

$20 \times 0.5 = R_D \times 2$, $R_D = 5$ N

$R_C = 15$ N



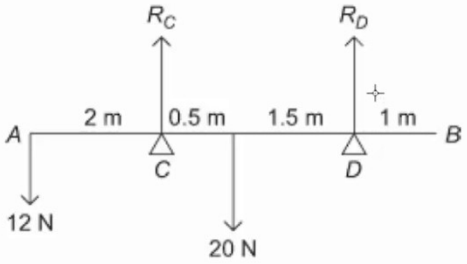
$R_C + R_D = 20$

Taking moments about C:

$20 \times 0.5 = R_D \times 2$, $R_D = 5$ N

$R_C = 15$ N

b Adding the weight of 12 N:



Taking moments about C:

$20 \times 0.5 = 12 \times 2 + R_D \times 2$

$10 = 24 + 2R_D$

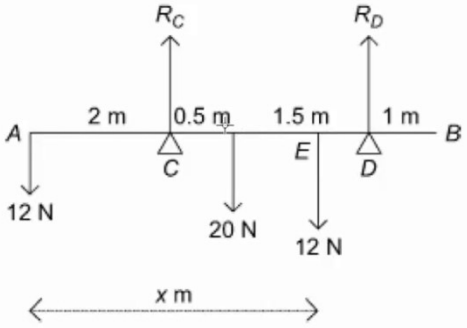
$\Rightarrow R_D$ is negative, which is impossible, therefore there is an anticlockwise moment about C – the rod will tilt.

c Adding the second particle:



will tilt.

c Adding the second particle:



Let the distance AE be x m.
If the system is just about to tilt about C, taking moments about C and $R_D = 0$

$$12 \times (x - 2) + 20 \times 0.5 = 12 \times 2$$
$$12x - 24 + 10 = 24$$
$$12x = 38$$
$$\Rightarrow x = 38 \div 12 \approx 3.17$$

The second particle needs to be 3.17 m from A to prevent tilting.

Question:

A non-uniform rod AB of length 5 m and mass 15 kg rests horizontally suspended from the ceiling by two vertical strings attached to C and D, where $AC = 1$ m and $AD = 3.5$ m.

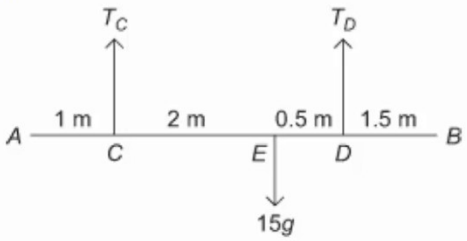
a Given that the centre of mass is at E where $AE = 3$ m, find the magnitudes of the tensions in the strings.

When a particle of mass 10 kg is attached to the rod at F the rod is just about to rotate about D.

b Find the distance AF.

Solution:

a

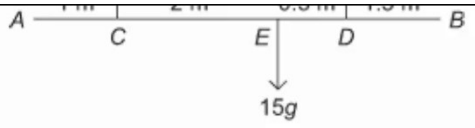


Taking moments about C:

$$T_D \times 2.5 = 15g \times 2$$
$$2.5T_D = 30g$$
$$\Rightarrow T_D = 12g = 118 \text{ N (3 s.f.)}$$

R(↑)

$T_C + T_D = 15g$, $T_C = 3g = 29.4$ N

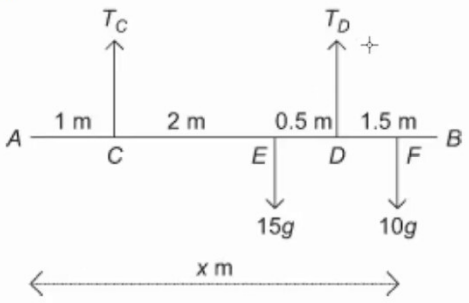


$\Rightarrow T_D = 12g = 118$ N (3 s.f.)

R(↑)

$T_C + T_D = 15g$, $T_C = 3g = 29.4$ N

b



With the particle attached at F, T_C is zero because the rod is about to rotate about D.

Let the distance $AF = x$ m.

Taking moments about D:

$15g \times 0.5 = 10g \times (x - (1 + 2 + 0.5))$

$= 10g \times (x - 3.5)$

$7.5g = 10gx - 35g$

$42.5 = 10x$

$\Rightarrow x = 4.25$

The distance AF is 4.25 m

Lecture 33

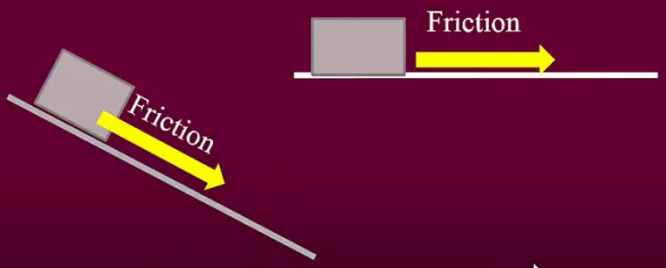
Friction



0:00:03 0:11:35

Friction

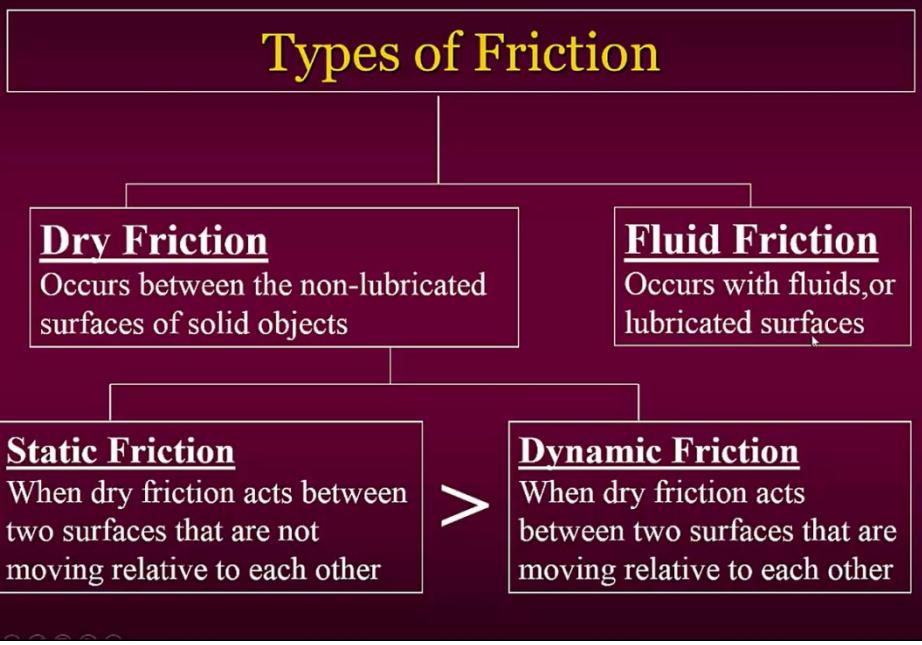
Friction is a force that opposes motion and occurs when two surfaces are in contact.



It will therefore act in the opposite direction to motion.

What is Friction

- Friction is a force
- A frictional force can exist when two substances contact each other.
- The molecules of each surface interact according to Newton's Laws of Motion.
- Friction always opposes motion, i.e., it is opposite to the direction of velocity.
- If there is no motion, then friction opposes the sum of all the other forces which are parallel to the surfaces in contact.



Laws of Friction

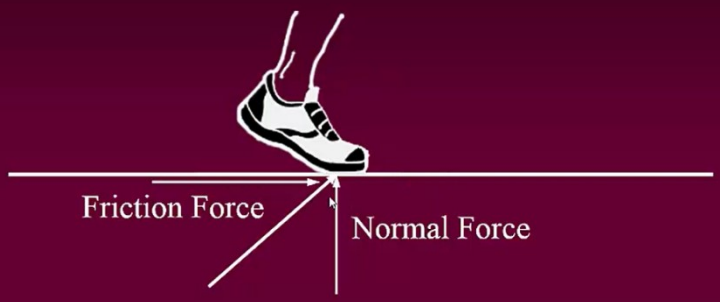
- When two surfaces are in contact, the force of friction opposes the motion.
- If two surfaces are in equilibrium, the force of friction is just sufficient to prevent motion and can be found by resolving the forces parallel to the surface.
- Friction force will not be larger than necessary to prevent motion.
- The size of friction force is limited. Once the limited friction is reached and the acting force is great enough to overcome friction then motion will occur. At such point equilibrium is broken. That is when:

Limiting Friction $f = \mu R$

Contact Force

- Force that occurs between objects that are in contact with each other.
- Contact forces can be resolved into components that are perpendicular and parallel to the surfaces in contact.
- The perpendicular component is called the **normal force**.
- The parallel component is called **friction**.

Contact Force in Running



During the push off phase in running, the **normal** force acts upward on the runner, while the **friction** force acts forward on the runner. The friction force is the only force capable of moving the runner horizontally down the track. The normal force can only accelerate the runner upwards.

Friction and the Normal Force

- The maximum frictional force is proportional to the normal contact force.
- An increase in the normal force results in an increase in the maximum friction.
- This is because the molecules on the two surfaces are pushed together more, thus increasing their interactions.

The maximum value of friction is calculated by using the formula:

$$f_{max} = \mu R$$

Reaction

Coefficient of friction

dependant upon the material of the surface and object.

The friction acting on the particle is Fmax when :

1. The particle is in 'limiting equilibrium' (on the point of slipping).
2. The particle is moving.

The following equation is used for friction when an object is at rest and is not moving.

$$f \leq \mu R$$

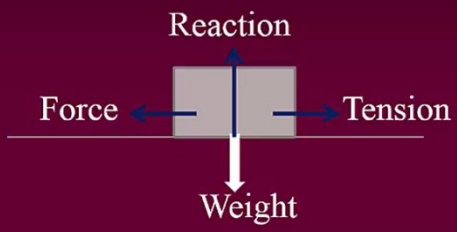
NOTE : Friction can be less than or equal to it's maximum value.

The magnitude of the frictional force can change to ensure equilibrium.



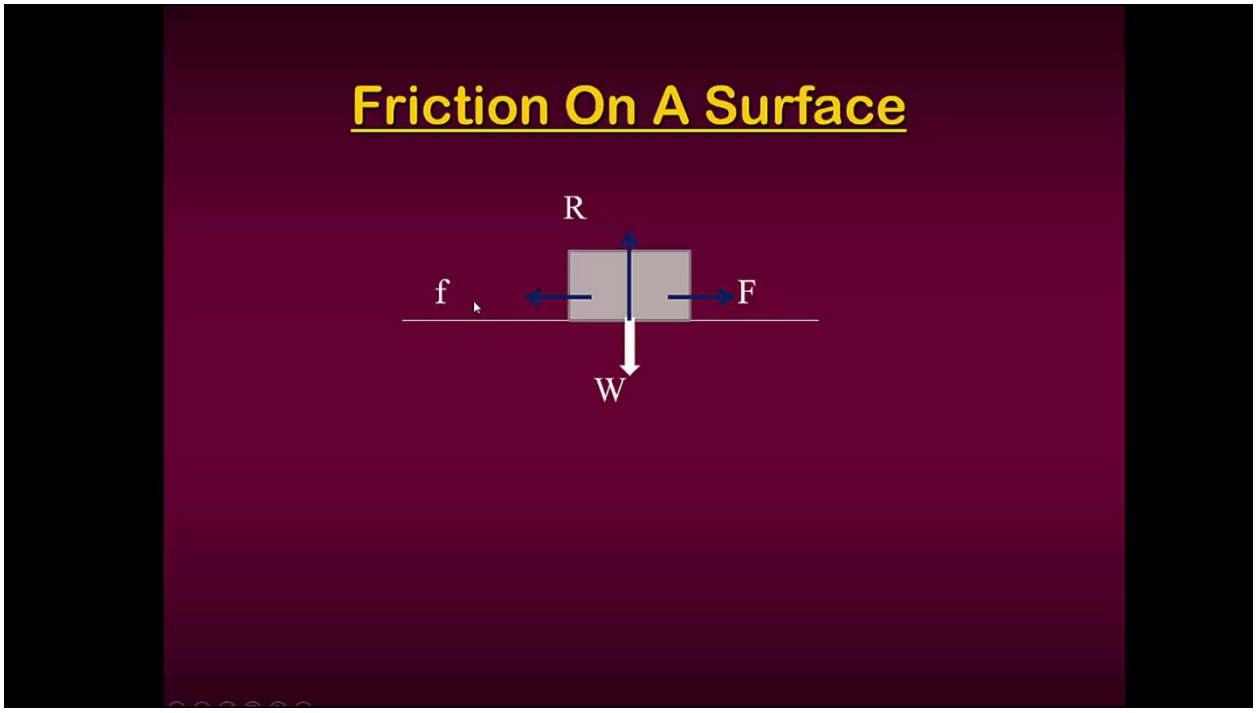
As the force pulling the sleigh increases, so will the friction to ensure equilibrium.
If the force increases to a greater magnitude than the maximum value of friction then the sleigh will move. (limiting friction)

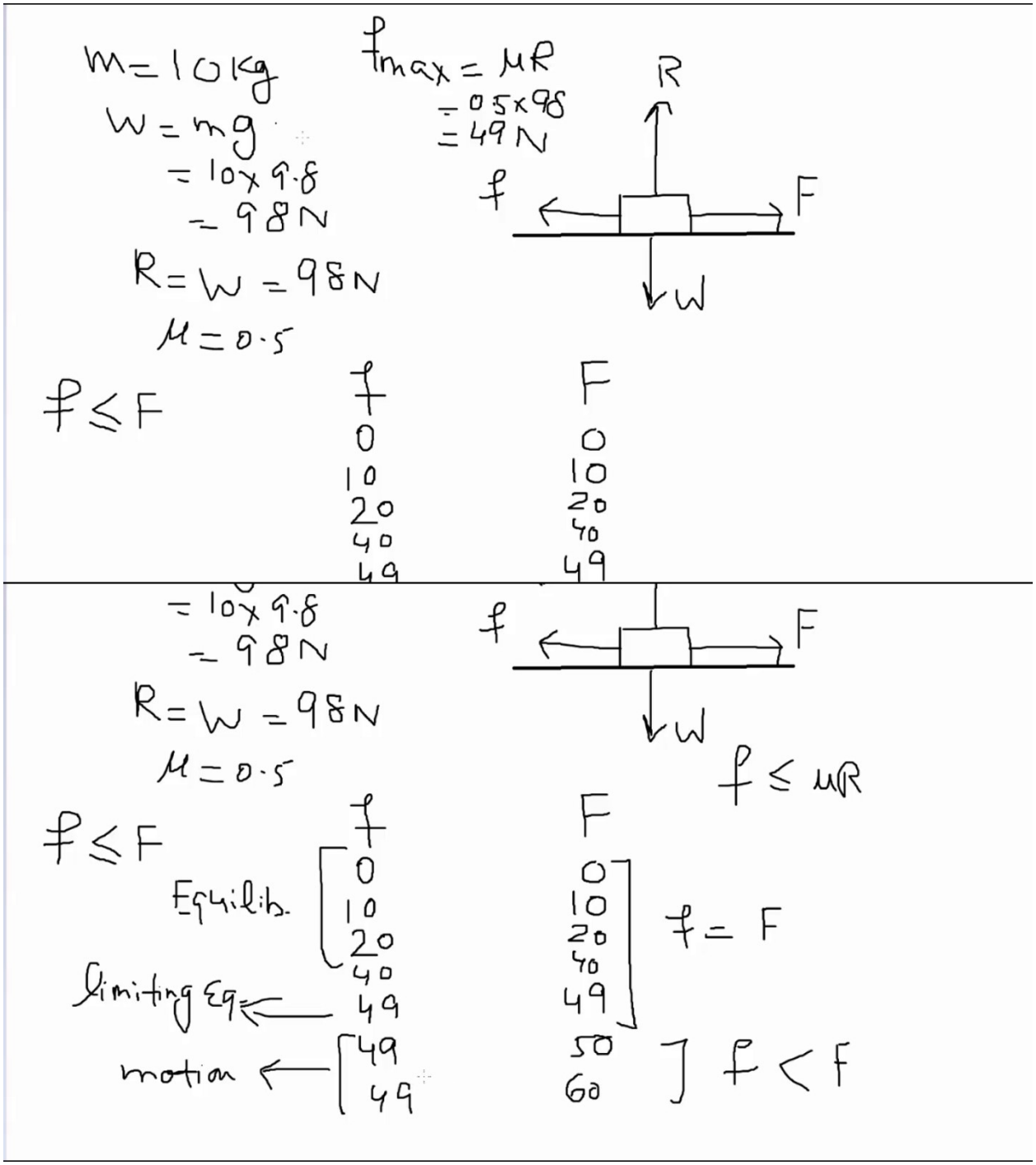
Friction On A Surface



- Resolving Vertically, $R = mg$. Where $g = 9.8\ m/s^2$
- Resolving Horizontally, $f = T$ for equilibrium at rest.
- If $T > f$ there will be a resultant force, which will produce an acceleration in the positive direction.

$\Rightarrow$Force $= ma = T - f$ according to Newton's Second Law of Motion. Where $f = \mu R$.


Friction On A Surface
R
f
F
W

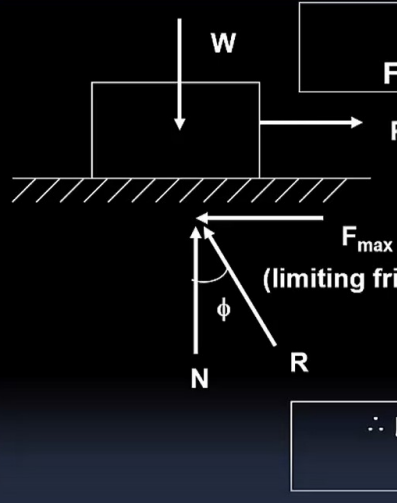

m = 10kg
W = mg
= 10 × 9.8
= 98N
R = W = 98N
μ = 0.5
f ≤ F
f_max = μR
= 0.5 × 98
= 49N
R
f
F
W
f
F
0
10
20
40
49
0
10
20
40
49
= 10 × 9.8
= 98N
R = W = 98N
μ = 0.5
f ≤ μR
f ≤ F
f
F
Equilib.
0
10
20
40
49
0
10
20
40
49
f = F
limiting Eq
motion
49
49
50
60
f < F

Lecture 34

Angle of Friction

Angle of limiting friction is defined as the angle between the resultant reaction (of limiting friction and normal reaction) and the normal to the plane on which the motion of the body is impending.

Angle of Friction



$F \alpha N$

$F_{max} = \mu N$

Where F_{max} = Limiting Friction

N= Normal Reaction between the contact surfaces

μ =Coefficient of friction

$$\therefore \mu = \frac{F_{max}}{N}$$

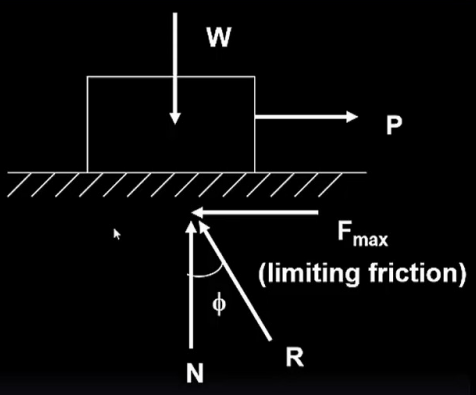
Note : Static friction varies from zero to a maximum value. Dynamic friction is fairly a constant.

0:01:16

0:03:13

Angle of Friction

The angle between **N** & **R** depends on the value of **F**. This angle **θ**, between the resultant **R** and the normal reaction **N** is termed as angle of friction. As **F** increases, **θ** also increases and will reach to a maximum value of **ϕ** when **F** is F_{max} (limiting friction)

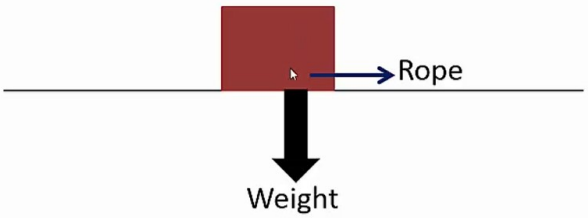


$$\text{i.e. } \tan\phi = (F_{max})/N = \mu$$

Angle **ϕ** is known as **Angle of limiting Friction**.

Example :

A horizontal rope is attached to a crate of mass 70 kg at rest on a flat surface. The coefficient of friction between the floor and the crate is 0.6. Find the maximum force that the rope can exert on the crate without moving it.



Solution:

Given:

m = 70 kg and μ = 0.6

F < μR and R = mg

Find:

F = ? R = ?

R = mg

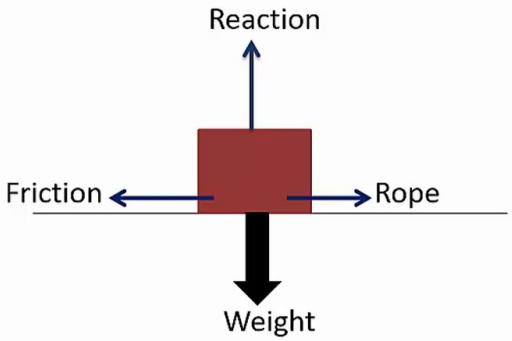
R = 70 x 9.8 = 686 N

F < μR

F < 0.6 x 686

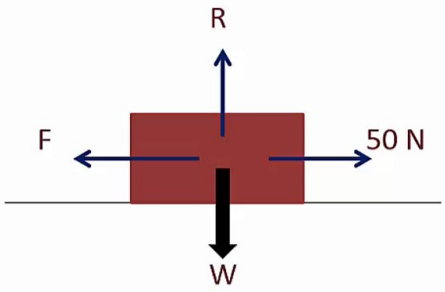
F < 411.6 N

Fmax=411.6



Example :

A block of mass 15 kg rests on a rough horizontal surface. Coefficient of friction between block and the plane is 0.25. Calculate the friction force acting on the block when a horizontal force of 50 N acts on the block. State whether the block will move.



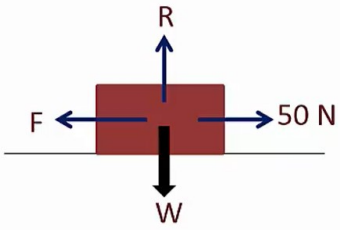
Solution:

Given: m = 15 kg and μ = 0.25 with Horizontal Force = 50 N

Using: R = mg

R = 15 x 9.8

= 147 N



Using: F = μR

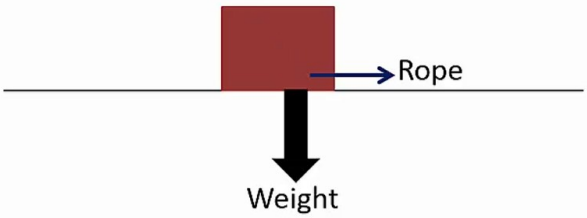
= 0.25 x 147

= 36.75 N Since F < 50N So the block moves.

0:02:36 0:00:05

Example :

A horizontal rope is attached to a crate of mass 50 kg at rest on a flat surface. The coefficient of friction between the floor and the crate is 0.4. Tension in the rope is 100N. Find the frictional force exerted on the crate.



Solution:

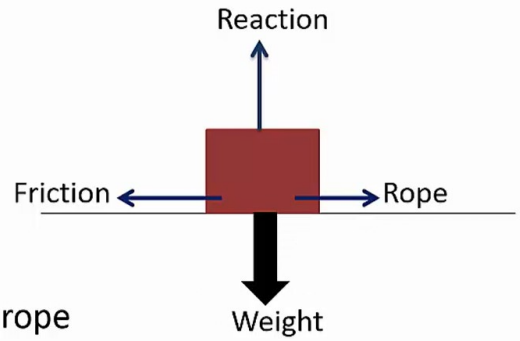
Given:

m = 50 kg and μ = 0.4

R = mg

Find:

F = ? R = ?



Let T be the tension in the rope

T=100N

R = mg

R = 50 x 9.8 = 490 N

μR=0.4 x 490=196

Frictional force f=T=100 N

Example

A block of mass 400 kg rests in limiting equilibrium on horizontal ground. A force of magnitude 2000N acts at angle of 15 degrees to the upward vertical. Find the coefficient of friction between the block and the ground, correct to 2 significant figures.

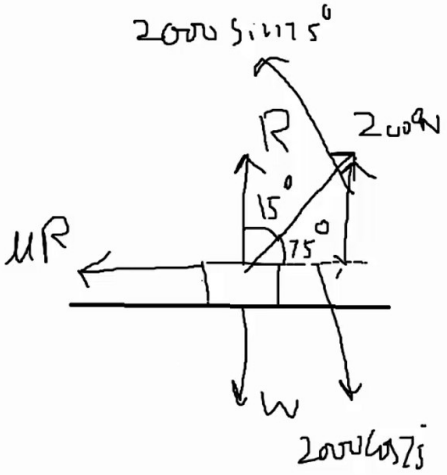
Sol:-

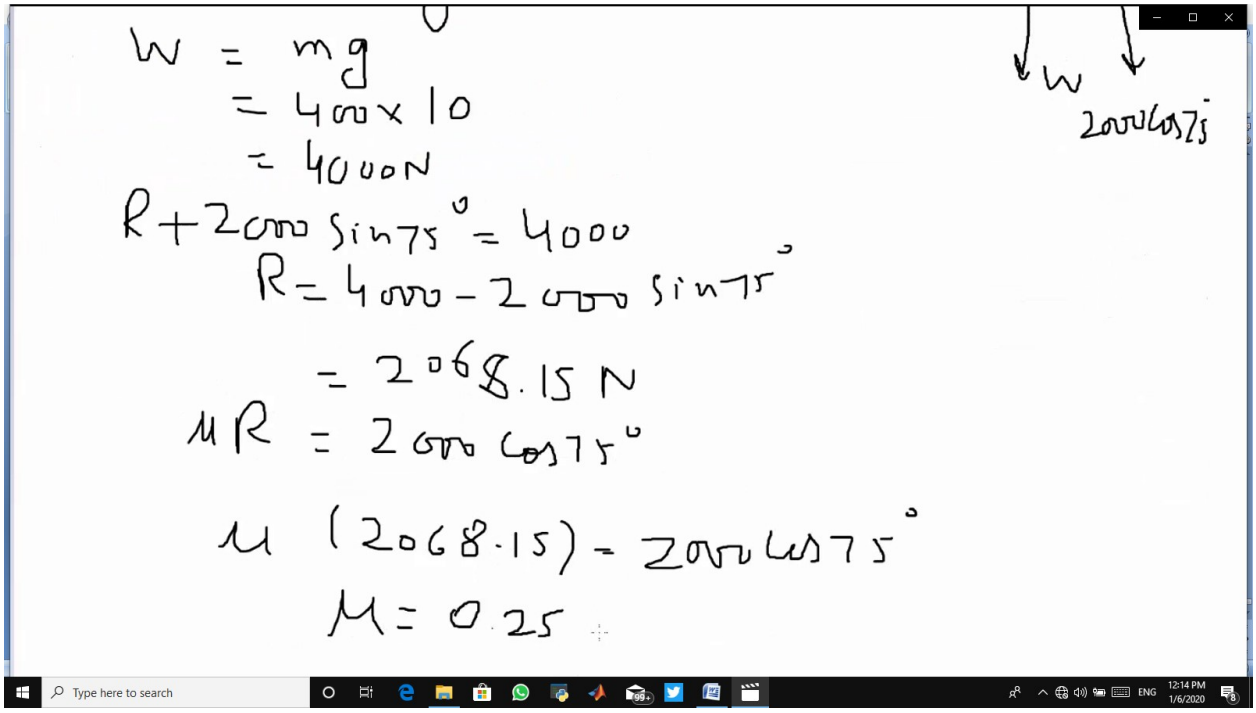
$m = 400kg$

$W = mg$

$= 400 \times 10$

$= 4000N$




W = mg
= 400 x 10
= 4000N
R + 2000 sin75° = 4000
R = 4000 - 2000 sin75°
= 2068.15 N
μR = 2000 cos75°
μ (2068.15) = 2000 cos75°
μ = 0.25
W
2000 cos75°
Type here to search
ENG
12:14 PM
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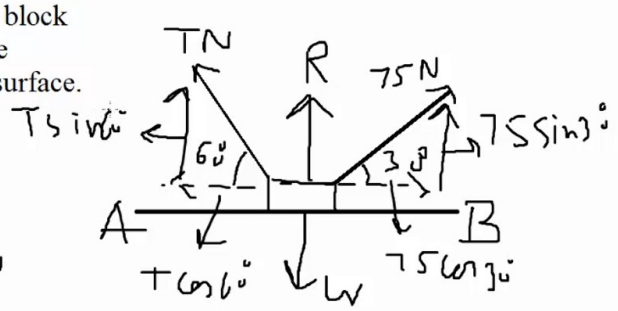
Lecture 35

Example

Two light strings are attached to a block of mass 20kg. The block is in equilibrium on a horizontal surface AB with the string taught. The strings make angles of 60 and 30 degrees with the horizontal, on either side of the block, and the tensions in the strings are TN and 75N resp. as shown in the diagram.

(i) Given that the surface is smooth, find the value of T and the magnitude of the contact force acting on the block.

(ii) It is given instead that the surface is rough and block is on the point of slipping. The friction force on the block has magnitude 25 N and acts towards A. Find the coefficient of friction between the block and the surface.

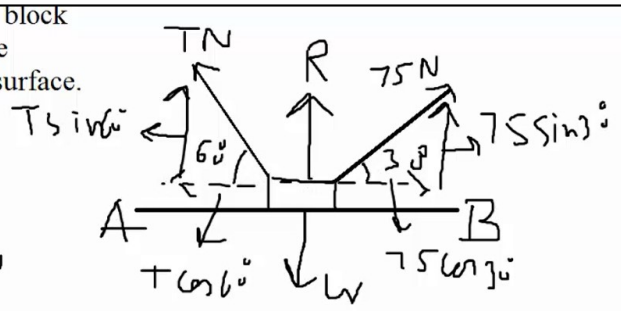


Sol.

(i) $W = mg$

$= 20(10) = 200\ N$

on the point of slipping. The friction force on the block has magnitude 25 N and acts towards A. Find the coefficient of friction between the block and the surface.



Sol.

(i) $W = mg$

$= 20(10) = 200\ N$

$\rightarrow \quad T\cos 60^\circ = 75\cos 30^\circ$

$T\left(\frac{1}{2}\right) = 75 \times \frac{\sqrt{3}}{2}$

$T = 75\sqrt{3}\ N$

$\uparrow \quad R + 75\sin 30^\circ + T\sin 60^\circ = 200$

$R + 75 \times \frac{1}{2} + 75\sqrt{3} \times \frac{\sqrt{3}}{2} = 200$

$R + 75\sin 30 + 75\sin 60 = 200$

$R + 75 \times \frac{1}{2} + 75\sqrt{3} \times \frac{\sqrt{3}}{2} = 200$

$R + \frac{75}{2} + \frac{225}{2} = 200$

$R = 50\,N$

$(\uparrow) \rightarrow \quad T\cos 60^\circ + 25 = 75\cos 30^\circ$

$T = \left(75\frac{\sqrt{3}}{2} - 25\right) \times 2$

$T = 75\sqrt{3} - 50\,N$

0:06:24 0:03:11

$$T = 75\sqrt{3} - 50 \text{ N}$$

$\uparrow$ $T\sin 60 + R + 75\sin 30^\circ = 200$

$$(75\sqrt{3} - 50)\times\frac{\sqrt{3}}{2} + R + 75\left(\frac{1}{2}\right) = 200$$

$$R = 50 + 25\sqrt{3} \text{ N}$$

$$\mu R = F$$

$$\mu(50 + 25\sqrt{3}) = 25$$

$$\mu =$$

$R = 50 + 25\sqrt{3}$ N

$$\mu R = F$$

$$\mu(50 + 25\sqrt{3}) = 25$$

$$\mu = \frac{25}{50 + 25\sqrt{3}}$$

$$= \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$= \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$$

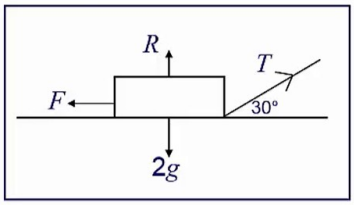
$$\mu = 0.268$$

Example

A particle of mass 2 kg is resting on a rough horizontal surface. The coefficient of friction between the particle and the surface is 0.1. A light inextensible string is attached to the right of the particle at an angle of 30° to the horizontal.

If the particle is on the point of moving to the right, find the tension in the string.

Solution



Resolving vertically: $R + T\cos60° = 2g$

Resolving horizontally: $F = T\cos30°$

$F = \mu R$

$\therefore T\cos30° = 0.1 \times (2g - T\cos60°)$

$\therefore T(\cos30° + 0.1\cos60°) = 0.2g$

$\therefore T = 2.14$ (to 3 s.f.)

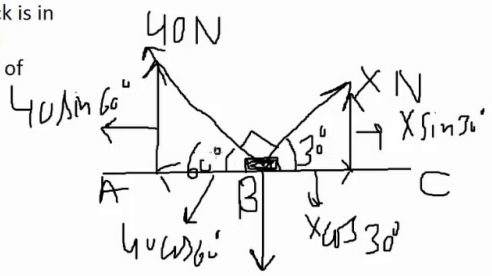
Therefore the tension in the string is **2.14 N**.

Example

Forces of magnitude X N and 40 N act on a block B of mass 15 kg, which is in equilibrium in contact with a horizontal surface between points A and C on the surface. The forces act in the same vertical plane and in the directions shown in the diagram.

(i) Given that the surface is smooth, find the value of X.

(ii) It is given instead that the surface is rough and that the block is in limiting equilibrium. The frictional force acting on the block has magnitude 10 N in the direction towards A. Find the coefficient of friction between the block and the surface.



Sol:-

$m = 15\,kg$

$W = mg$

$= 15g\ N$

$\rightarrow \quad X\cos 30^\circ = 40\cos 60^\circ$

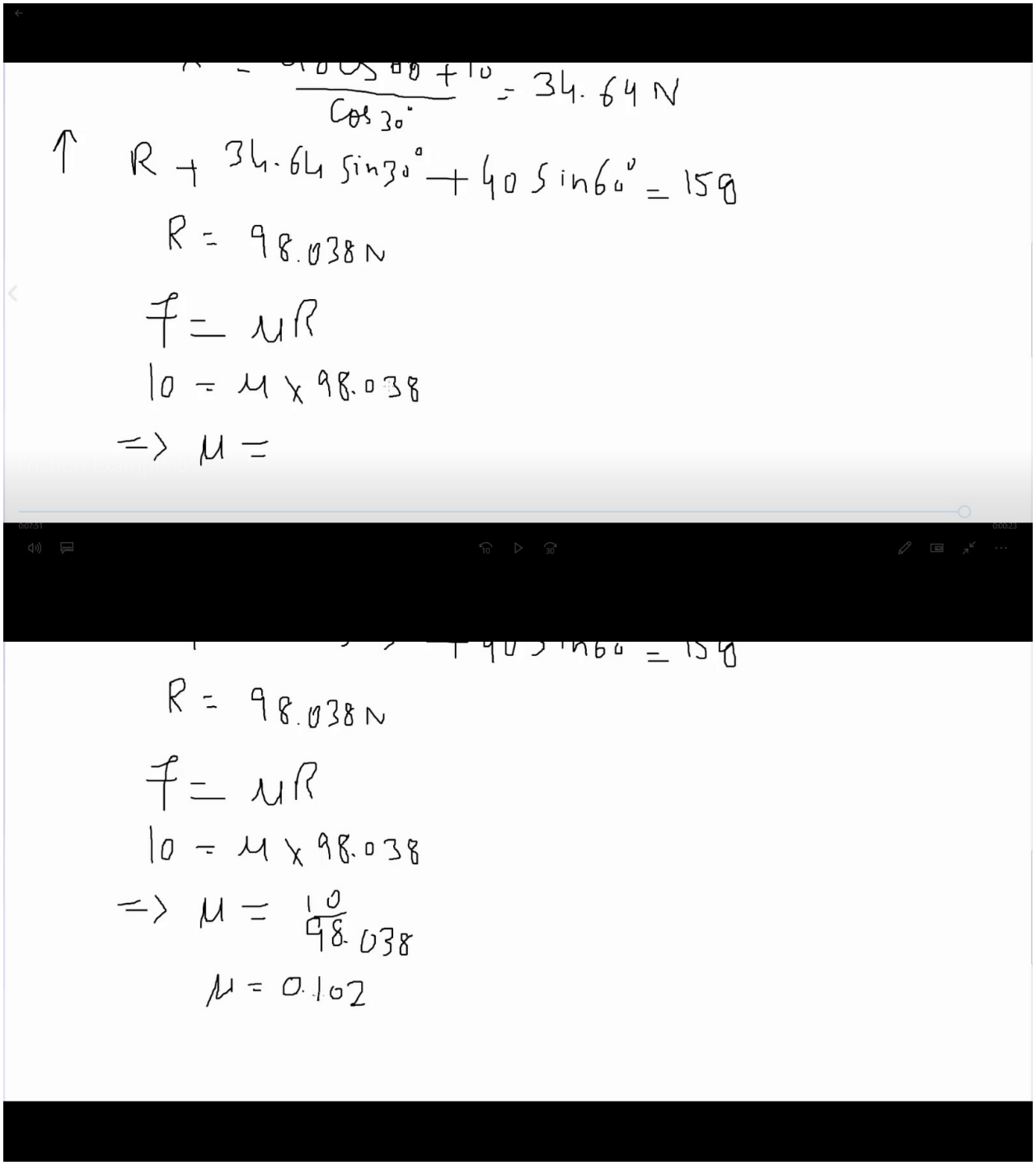
$X = \dfrac{40\cos 60^\circ}{\cos 30^\circ} = 23.1\,N$

(ii) $X\cos 30^\circ = 40\cos 60^\circ + 10$

$X = \dfrac{40\cos 60^\circ + 10}{\cos 30^\circ} = 34.64\,N$

Friction Example 8

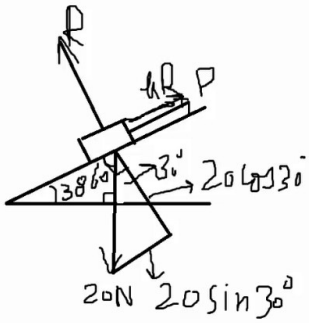
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$\frac{\cos 30 + 10}{\cos 30^\circ} = 34.64\ N$
$\uparrow \quad R + 34.64 \sin 30^\circ + 40 \sin 60^\circ = 150$
$R = 98.038\ N$
$f = \mu R$
$10 = \mu \times 98.038$
$\Rightarrow \mu =$
$R = 98.038\ N$
$f = \mu R$
$10 = \mu \times 98.038$
$\Rightarrow \mu = \frac{10}{98.038}$
$\mu = 0.102$

Lecture 36

Example

A box of mass 2 kg is placed on a plane which is inclined at an angle of 30degrees with the horizontal. The plane is rough and the coefficient of friction between the box and the plane is 0.5. The box is kept in equilibrium by applying a horizontal force P Newton to it as shown in the diagram. Find the least and maximum value of the force P which enable the box to remain in equilibrium.



Sol

$m = 2kg$

$W = mg = 2 \times 10 = 20N$

$\mu = 0.5$

$R = 20\cos 30^\circ$

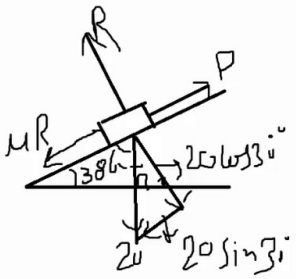
$R = 20\cos 30^\circ$

$P + \mu R = 20\sin 30^\circ$

$P + 0.5(20\cos 30^\circ) - 20\sin 30^\circ$

$P = 20\sin 30^\circ - 0.5(20\cos 30^\circ)$

$= 1.34N$



max force

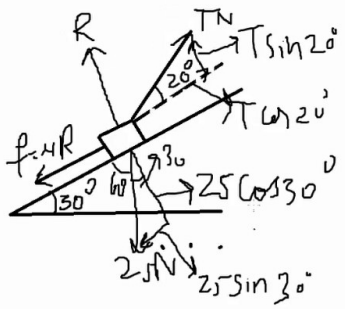
$P = \mu R + 20\sin 30^\circ$

$= 0.5(20\cos 30^\circ) + 20\sin 30^\circ$

$= 18.7N$

Example

A block of mass 2.5 kg is placed on a plane which is inclined at angle of 30 degrees to the horizonta. The block is kept in equilibrium by a light string making an angle of 20 degree above a line of greatest slope. The tension in the string is T N, as shown in the diagram. The coefficient of friction between the block and the plane is 1/4. The block is in limiting equilibrium and is about to move up the plane. Find the value of T.



Sol.

$m = 2.5\,kg$

$W = mg = 2.5 \times 10 = 25\,N$

$\mu = \frac{1}{4} = 0.25$

$T\cos 20 = 25\sin 30^\circ + \mu R$

$= 25\sin 30^\circ + \frac{1}{4}R$ — (1)

$R + T\sin 20^\circ = 25\cos 30^\circ$

$R = 25\cos 30^\circ - T\sin 20^\circ$ — (2)

$T\cos 20^\circ = 25\sin 30^\circ + \frac{1}{4}(25\cos 30^\circ - T\sin 20^\circ)$

$T\cos 20^\circ + \frac{1}{4}T\sin 20^\circ = 25\sin 30^\circ + \frac{25}{4}\cos 30^\circ$

$T(1.025) = 17.913$

$\frac{1}{4}R$

$\uparrow \quad R + T\sin 20^\circ = 25\cos 30^\circ$

$R = 25\cos 30^\circ - T\sin 20^\circ \quad \text{—(2)}$

$T\cos 20^\circ = 25\sin 30^\circ + \frac{1}{4}(25\cos 30^\circ - T\sin 20^\circ)$

$T\cos 20^\circ + \frac{1}{4}T\sin 20^\circ = 25\sin 30^\circ + \frac{25}{4}\cos 30^\circ$

$T(1.025) = 17.913$

$T = \frac{17.913}{1.025}$

$T = 17.5\,N$

Condition of Equilibrium for a particle on a rough inclined plane

mass = m kg

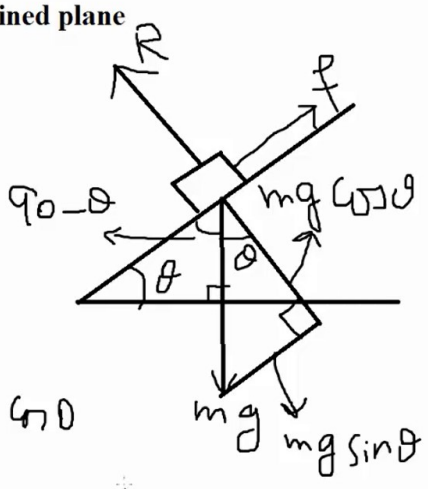
$W = mg\,N$

μ = coefficient of friction

$f_{max} = \mu R \qquad R = mg\cos\theta$

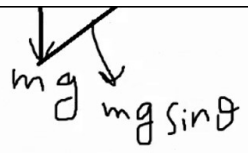
$f_{max} = \mu(mg\cos\theta) = \mu mg\cos\theta$

$f = mg\sin\theta$ (equilibrium)



friction

$f_{max} = \mu R$ $\quad R = mg\cos\theta$



$f_{max} = \mu (mg\cos\theta) = \mu mg\cos\theta$

$f = mg\sin\theta$ (equilibrium)

for limiting Eq.

$f_{max} = mg\sin\theta$

$\mu mg\cos\theta = mg\sin\theta$

$\mu = \tan\theta$

$f_{max} = \mu (mg\cos\theta) = \mu mg\cos\theta$

$f = mg\sin\theta$ (equilibrium)

for limiting Eq.

$f_{max} = mg\sin\theta$

$\mu mg\cos\theta = mg\sin\theta$

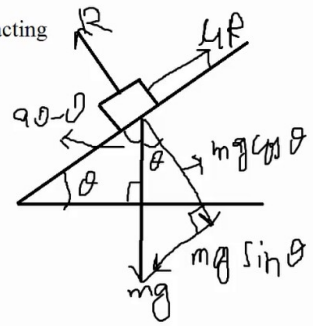
$\mu = \tan\theta$

$\tan\lambda = \tan\theta$

$\lambda = \theta$

Sliding of Particle

An object will slide down a plane if the component of its weight acting down the plane is larger than the maximum friction.



$mg\sin\theta > \mu R$

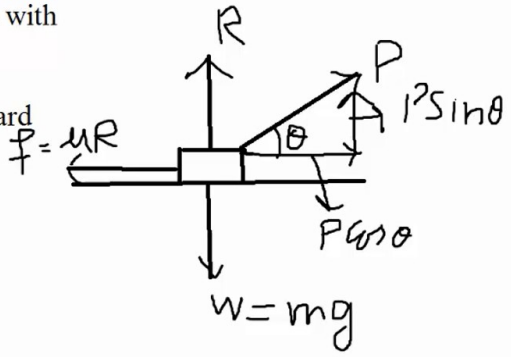
$mg\sin\theta > \mu(mg\cos\theta)$

$\sin\theta > \mu\cos\theta$

$\tan\theta > \mu$

Least force to Slide a particle on a rough Horizontal plane

Let P be the magnitude of the force to drag a particle on a rough horizontal plane and let it make an angle theta with the plane as shown in the fig. Other forces are:
weight of the particle acting downward
Normal reaction R of the plane acting vertically upward
friction of the plane acting along the plane



$W = mg$

$\rightarrow \quad P\cos\theta = \mu R \quad \text{—— (1)}$

$\uparrow \quad R + P\sin\theta = mg$

$R = mg - P\sin\theta \quad \text{— (2)}$

$R + P\sin\theta = mg$

$R = mg - P\sin\theta \quad \text{---(2)}$

$P\cos\theta = \mu(mg - P\sin\theta)$

$P\cos\theta = \tan\lambda\,(mg - P\sin\theta)$

$P\cos\theta = \frac{\sin\lambda}{\cos\lambda}(mg - P\sin\theta)$

$P\cos\theta\cos\lambda = \sin\lambda\, mg - \sin\lambda\, P\sin\theta$

$P(\cos\theta\cos\lambda + \sin\lambda\sin\theta) = mg\sin\lambda$

$P\cos(\theta - \lambda) = mg\sin\lambda$

$P\cos\theta\cos\lambda = \sin\lambda\, mg - \sin\lambda\, P\sin\theta$

$P(\cos\theta\cos\lambda + \sin\lambda\sin\theta) = mg\sin\lambda$

$P\cos(\theta - \lambda) = mg\sin\lambda$

$P = \frac{mg\sin\lambda}{\cos(\theta - \lambda)}$

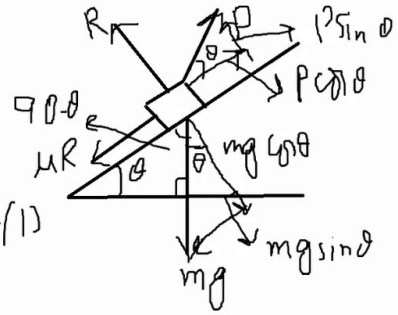
$P_{least} \Rightarrow \max\cos(\theta - \lambda) = 1$

$\Rightarrow \theta - \lambda = 0$

$\theta = \lambda$

$P_{least} = mg\sin\lambda$

Least force to slide a particle on rough inclined plane



μ

$ma_x = m$

$P\cos\theta = \mu R + mg\sin\theta$ ——(1)

$R + P\sin\theta = mg\cos\theta$

$R = mg\cos\theta - P\sin\theta$ —— (2)

$P\cos\theta = \mu(mg\cos\theta - P\sin\theta) + mg\sin\theta$

$P\cos\theta = \mu mg\cos\theta - \mu P\sin\theta + mg\sin\theta$

$P\cos\theta = \frac{\sin\lambda}{\cos\lambda} mg\cos\theta - \frac{\sin\lambda}{\cos\lambda} P\sin\theta + mg\sin\theta$

$P\cos\theta\cos\lambda = \sin\lambda\, mg\cos\theta - P\sin\lambda\sin\theta + mg\sin\theta\cos\lambda$

$P(\cos(\theta-\lambda)) = mg(\sin\theta\cos\lambda + \cos\theta\sin\lambda)$

$P = \frac{mg\sin(\theta+\lambda)}{\cos(\theta-\lambda)}$

P_{least} $\cos(\theta-\lambda) = 1$

$\theta - \lambda = 0$

$\theta = \lambda$

$P_{least} = mg\sin(\theta+\lambda)$

Lecture 38

Example

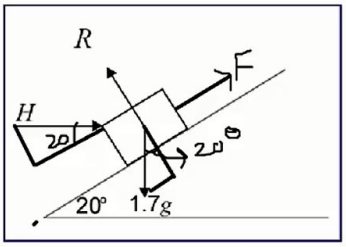
A particle of mass 1.7 kg rests on a rough plane inclined at an angle of 20° to the horizontal. A horizontal force, H, is applied to the particle. The coefficient of friction between the particle and the plane is 0.15.

Find H if

a) The particle is on the point of sliding down the plane.

b) The particle is on the point of sliding up the plane.

Solution



(a) Resolving perpendicular to the plane:

$R = 1.7g\cos20° + H\cos70°$

Resolving parallel to the plane:

$F + H\cos20° = 1.7g\cos70°$

$\therefore F = 1.7g\cos70° - H\cos20°$

$F = \mu R$

$\therefore\ 1.7g\cos70° - H\cos20° = 0.15(1.7g\cos20° + H\cos70°)$

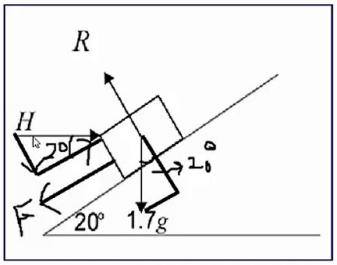
$\therefore 0.15H\cos70° + H\cos20° = 1.7g\cos70° - 0.15 \times 1.7g\cos20°$

$\therefore H(0.15\cos70° + \cos20°) = 1.7g\cos70° - 0.15 \times 1.7g\cos20°$

$$\therefore H = \frac{1.7g\cos70° - 0.15 \times 1.7g\cos20°}{0.15\cos70° + \cos20°} = 3.38 \text{ (to 3 s.f.)}$$

Therefore the horizontal force is **3.38 N**.

Solution



(b)Resolving perpendicular to the plane:

$R = 1.7g\cos20° + H\cos70°$

Resolving parallel to the plane:

$F + 1.7g\cos70° = H\cos20°$

$\therefore F = H\cos20° - 1.7g\cos70°$

$F = \mu R$

$\therefore H\cos20° - 1.7g\cos70° = 0.15(1.7g\cos20° + H\cos70°)$

$\therefore H\cos20° - 0.15H\cos70° = 0.15 \times 1.7g\cos20° + 1.7g\cos70°$

$\therefore H(\cos20° - 0.15\cos70°) = 0.15 \times 1.7g\cos20° + 1.7g\cos70°$

$$\therefore H = \frac{0.15 \times 1.7g\cos20° + 1.7g\cos70°}{\cos20° - 0.15\cos70°} = 9.06 \text{ (to 3 s.f.)}$$

Therefore the horizontal force is **9.06 N**.

Sliding of a particle Example

A particle of mass 10 kg is held in equilibrium on a smooth plane inclined at an angle of 30° to the horizontal by means of a light inextensible string acting at an angle of 10° to the plane.

a) Draw a diagram showing all the forces acting on the particle.

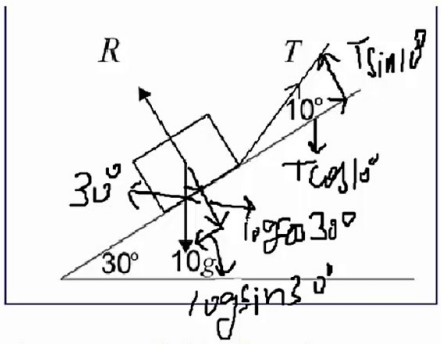
b) Calculate the tension in the string.

A horizontal force of 30 N is applied to the particle to try to make it slide up the slope. The tension in the string is adjusted so that the particle remains in equilibrium.

c) Calculate the new tension in the string.

Solution

a)



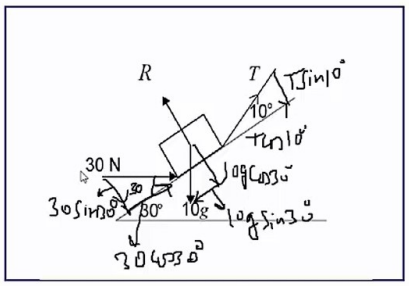
b) Resolving parallel to the plane:

$T\cos 10° = 10g\cos 60°$

$T = \dfrac{10g\cos 60°}{\cos 10°} = 49.8$ (to 3sf)

Therefore the tension in the string is **49.8 N**.

Solution



c) Resolving parallel to the plane:

$T\cos 10° + 30\cos 30° = 10g\cos 60°$

$T = \dfrac{10g\cos 60° - 30\cos 30°}{\cos 10°}$

$T = 23.4$ (to 3 s.f.)

Therefore the tension in the string is **23.4 N**.

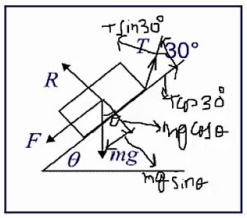
Sliding of a particle Example

A particle of mass m kg lies on a rough plane inclined at an angle of θ° to the horizontal. The particle is held in equilibrium by means of a light inextensible string held at an angle of 30° to the plane.

The coefficient of friction between the plane and the particle is 0.25 and the particle is about to slide up the plane.

Show that the tension in the string is $2mg\left(\frac{\cos\theta+4\sin\theta}{4\sqrt{3}+1}\right)$

Solution



Resolving perpendicular to the plane,

$$R + T\cos 60 = mg\cos\theta$$

$$R = mg\cos\theta - \frac{1}{2}T$$

Using $F = \mu R$, $\quad F = \frac{1}{4}mg\cos\theta - \frac{1}{8}T$

Resolving parallel to the plane,

$$F + mg\cos(90 - \theta) = T\cos 30$$

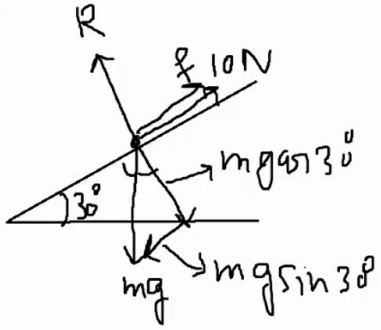
$$\frac{1}{4}mg\cos\theta - T + mg\sin\theta = T\cos 30$$

$$\frac{1}{4}mg\cos\theta - T + mg\sin\theta = \frac{\sqrt{3}}{2}T + \frac{1}{8}T = \left(\frac{4\sqrt{3}+1}{8}\right)T$$

$$\therefore T = 2mg\left(\frac{\cos\theta + 4\sin\theta}{4\sqrt{3}+1}\right) \text{ as required.}$$

Example

A particle of mass mkg is resting on a rough plane inclined at 30 degrees to the horizontal. A force of magnitude 10 N applied to the particle up a line of greatest slope of the plane us just sufficient to stop the particle sliding down the plane. When a force of 75 N is applied to the particle up a line of greatest slope of the plane, the particle is on the point of sliding up the plane. Find m and the coefficient of friction between the particle and the plane.



Sol

$R = mg\cos 30^\circ$ —(1)

$f + 10 = mg\sin 30^\circ$ —(2)

$f = mg\sin 30^\circ - 10$ —(3)

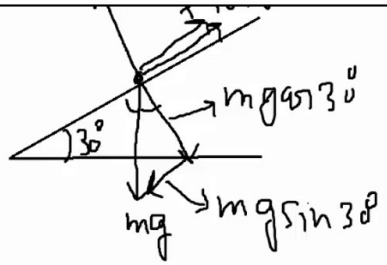
$f + mg\sin 30^\circ = 75$

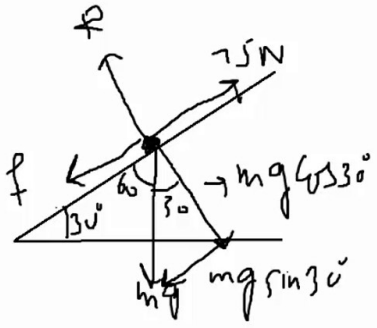
$f = 75 - mg\sin 30^\circ$ —(4)

$mg\sin 30^\circ - 10 = 75 - mg\sin 30^\circ$

$2mg\sin 30 = 75 + 10$

$m = \frac{85}{10}$





$m = 8.5\ kg$

$R = mg\cos 30°$

$= 8.5 \times 10 \times \frac{\sqrt{3}}{2}$

$= 73.61$

$f = mg\sin 30° - 10$

$= 8.5 \times 10 \times \frac{1}{2} - 10$

$= 32.5$

$f = \mu R$

$\mu = \frac{f}{R} = \frac{32.5}{73.6}$

$= 8.5 \times 10 \times \frac{\sqrt{3}}{2}$

$= 73.61$

$f = mg\sin 30° - 10$

$= 8.5 \times 10 \times \frac{1}{2} - 10$

$= 32.5$

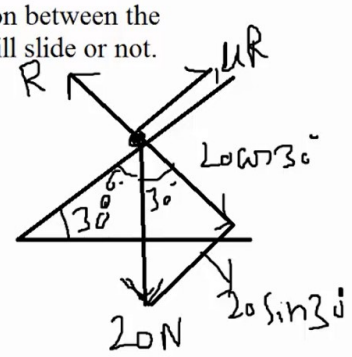
$f = \mu R$

$\mu = \frac{f}{R} = \frac{32.5}{73.61}$

$\mu = 0.442$

Example

A particle of mass 2kg is placed on a rough inclined plane which is at an angle of 30 derees with the horizontal. The coefficient of friction between the particle and the plane is 0.5. Determine whether the particle will slide or not.



Sol

$20\sin 30^\circ > \mu R$

$20\sin 30^\circ > 0.5 \times 20\cos 30^\circ$

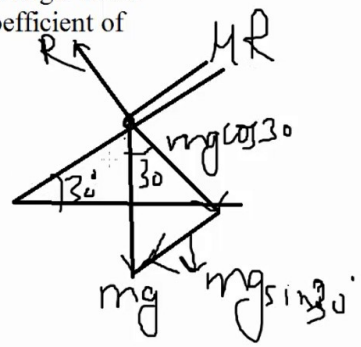
$10 > 10 \times \frac{\sqrt{3}}{2}$

$10 > 10 \times 0.86$

$10 > 8.66$ which is True

Example

A paricle is placed on a rough inclined plane whichis at angle of 30 degrees with the horizontal. Find the set of values of coefficient of friction for the particle to slide.



Sol

For sliding

$\mu R < mg\sin 30^\circ$

$\mu mg\cos 30^\circ < mg\sin 30^\circ$

$\mu < \frac{\sin 30}{\cos 30^\circ}$

Sol

For sliding

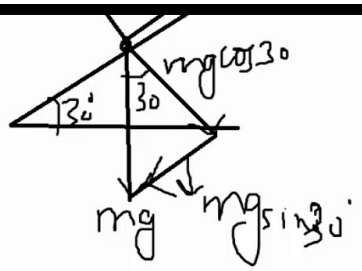
$\mu R < mg\sin 30^\circ$

$\mu \cancel{mg}\cos 30^\circ < \cancel{mg}\sin 30^\circ$

$\mu < \frac{\sin 30}{\cos 30^\circ}$

$\mu < \tan 30^\circ$

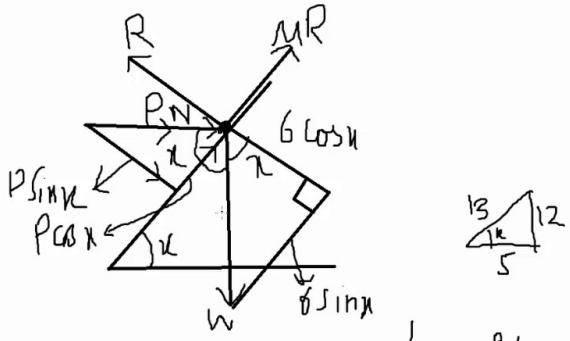
$\mu < 0.577$



Example

A rough inclined plane is inclined at an angle x to the horizontal, where tanx =2.4. A small block of mass 0.6kg is held at rest on the plane by a horizontal force of magnitude P N. This force acts in a vertical plane through a line of greatest slope as shown in the diagram.

The coefficient of friction between the block and the plane is 0.4. The block is on the point of sliding down the plane. By resolving force parallel to and perpendicular to the inclined plane, or otherwise, find the value of P.



Sol:-

$m = 0.6$

$W = mg = 0.6 \times 10 = 6N$

$\sin x = \frac{12}{13}$

$\cos x = \frac{5}{13}$

$\tan x = 2.4$

$= \frac{24}{10}$

$= \frac{12}{5}$

$W = mg = 0.6 \times 10 = 6N$

$\cos x = \frac{5}{13}$ $= \frac{12}{5}$

$\mu = 0.4$

$\uparrow \quad \mu R + P\cos x = 6\sin x,$

$R = P\sin x + 6\cos x$

$= P \times \frac{12}{13} + 6 \times \frac{5}{13}$

$= \frac{12P}{13} + \frac{30}{13}$

$0.4\left(\frac{12P}{13} + \frac{30}{13}\right) + \frac{5P}{13} = 6 \times \frac{12}{13}$

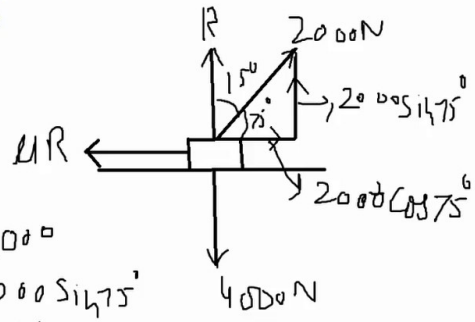
$4.8P + 12 + 5P = 72$

$9.8P = 60$

$P = \frac{60}{9.8} \Rightarrow P = 6.12$

Example

A block of mass 400kg rests in limiting equilibrium on horizontal ground. A force of magnitude 2000N acts on the block at an angle of 15 degrees to the upward vertical. Find the coefficient of friction between the block and the ground, correct to 2 significant figures.



Sol

$m = 400kg$

$W = mg$

$= 400 \times 10$

$= 4000N$

$\uparrow \quad R + 2000\sin 75° = 4000$

$R = 4000 - 2000\sin 75°$

$= 2068.15N$

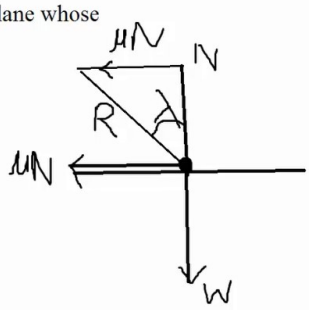
$\rightarrow \quad \mu R = 2000\cos 75°$

$\mu(2068.15N) \Rightarrow \mu = \frac{2000\cos 75°}{2068.15} \Rightarrow \mu = 0.25$

Angle of friction

Example

A particle of mass 2 kg is placed on a rough horizontal plane whose coefficient of friction is 0.4. Find the angle of friction.



Sol

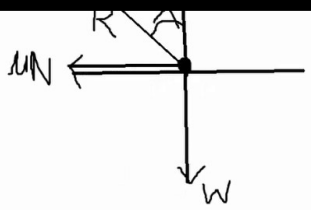
$$\tan\lambda = \frac{\mu N}{N}$$

$$\Rightarrow \mu = \tan\lambda$$

$$\Rightarrow \tan\lambda = \mu$$

$$\Rightarrow \lambda = \tan^{-1}(\mu)$$

$$\lambda = \tan^{-1}(0.4)$$



$$\tan\lambda = \frac{\mu N}{N}$$

$$\Rightarrow \mu = \tan\lambda$$

$$\Rightarrow \tan\lambda = \mu$$

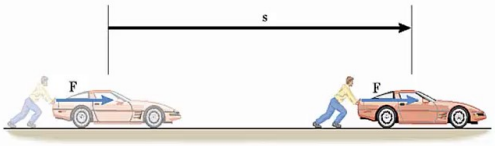
$$\Rightarrow \lambda = \tan^{-1}(\mu)$$

$$\lambda = \tan^{-1}(0.4)$$

$$= 21.8^{\circ}$$

Lecture 40

Work Done by a Constant Force



Definition of Work done

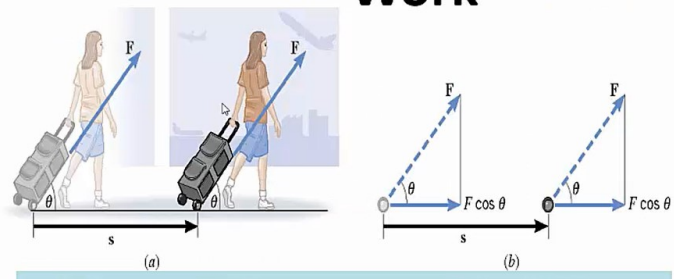
The work done by a constant force acting on an object is equal to the displacement times the component of the force parallel to the displacement.

Force *F* points in the same direction as the resulting displacement *s*

$$W=Fs$$

1

Work



- The work done on an object by an external constant force:

$$W = F\,s\cos\theta$$

- θ is the angle between the **displacement, s, and the force F**.

Products of two Vectors A and B

- **Dot Product:**
  - $\mathbf{A \cdot B} = A\,B\cos\theta$ ***Scalar***
- **Cross Product:**
  - $\mathbf{AxB = AB}\sin\theta$ ***Vector***

Work is a dot product

- Work is the dot product of Force and displacement.

$$\mathbf{W = F \cdot s}$$

Units of Work

- **Is a scalar quantity.**
- **Unit: Joule 1 J = 1 N·m**
- **James Prescott Joule, English, 19thCentury**

- **Other units of work:**
  - **British system:**
    - **foot-pound**
  - **cgs system:**
    - **erg=dyne ·cm**
  - **Sub-atomic level:**
    - **electron-Volt (eV)**

Meaning of Sign of Work

When the work done by an external force onto the system is:

\+ : the system gains energy

\- : the system loses energy

- When the force is in the direction of the displacement, $\theta=0^{o}$:

cos 0= 1 => **W= F s and work is positive**

When the force acts opposite to the direction of the displacement, $\theta=180^{o}$:

$\cos 180^{o} = -1 \Rightarrow$ **W= F s and work is negative**

When the force acts **perpendicular** to the displacement,

then the work done by that force is **zero**.

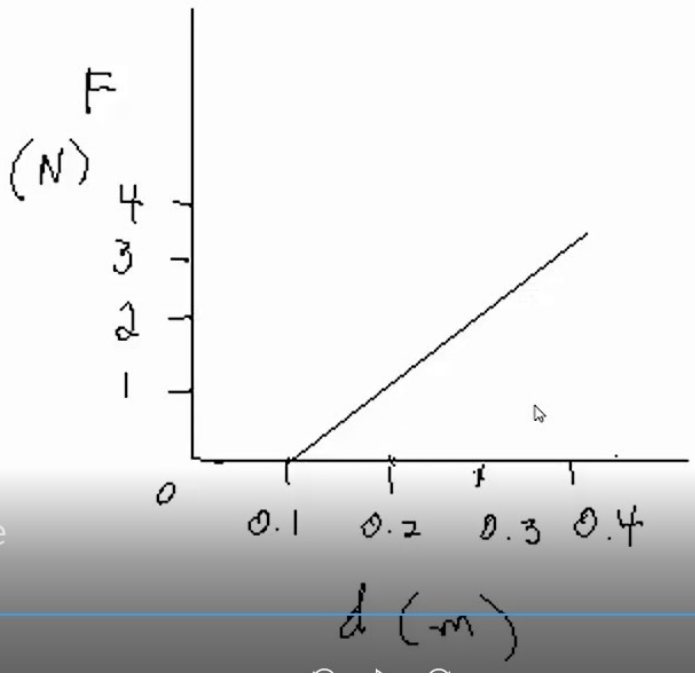
Work done by Resultant

$$W = W_1 + W_2 + W_n$$

$$W_{\text{net}} = \sum W_{\text{by individual forces}}$$

Since work is a scalar, the net work done by a resultant force is equal to the sum of the individual works done by each individual force acting on the object.

Work is the area under an F vs d graph.



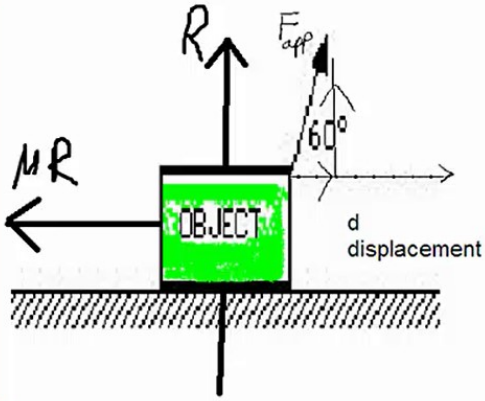
constant force

Example

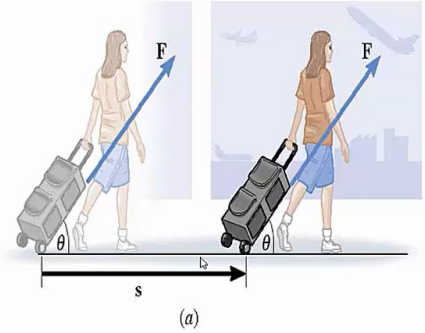
- How much work was done by the 30N applied tension force to move the 6kg box along the floor by 20m?
  Work done=(30cos60)*20=300J
- How much work did the weight do in the same case?
- Work=6*10*cos90=0
- If μ=0.2, how much work did the friction do?
  R=6*10-30sin60=34.02

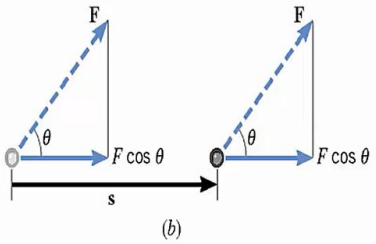
$$f = \mu R = 0.2 * 34.02 = 6.8$$

$$W = f * d = 6.8 * 20 = 136J$$



Example Pulling a Suitcase-on-Wheels





Find the work done by a 45.0 N force in pulling the suitcase at an angle $\theta = 50.0°$ for a distance s=75.0 m

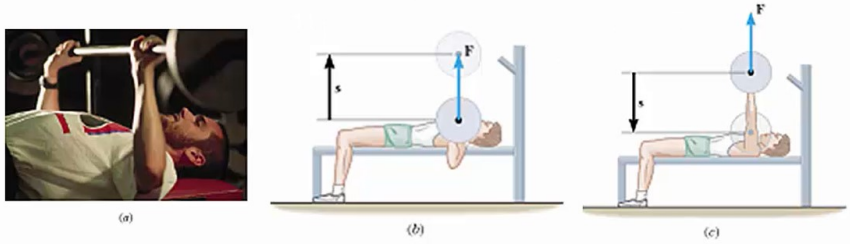
$$W = (F\cos\theta)s = (45.0N)\cos 50.0°(75.0m) = 2170J$$

No work done due to Fsin θ

F and s in the same direction ⟶ positive work

F and s in the opposite direction ⟶ negative work

Example Bench-Pressing



The weight lifter is bench-pressing a barbell whose weight is 710N. In part (b) of the figure, he raises the barbell a distance of 0.65m above his chest, and in part (c) he lowers it the same distance.

The weight is raised and lowered at a constant velocity. Determine the work done on the barbell by the weight lifter during (a) the lifting phase and (b) the lowering phase.

(a) $W = (F\cos\theta)s = (710N)\cos 0°(0.65m) = 460J$

(b) $W = (F\cos\theta)s = (710N)\cos 180°(0.65m) = -460J$

$\cos 180^0 = -1$

4

Example Work as dot product of vectors

A particle is pushed by a constant force of F=(4i +5j)N and undergoes a displacement of Δr= (3i+ 2j)m.

Find the work done by the force.

W=F. Δr

= (4i +5j)N *(3i+ 2j)m=(4*3+5*2)Nm

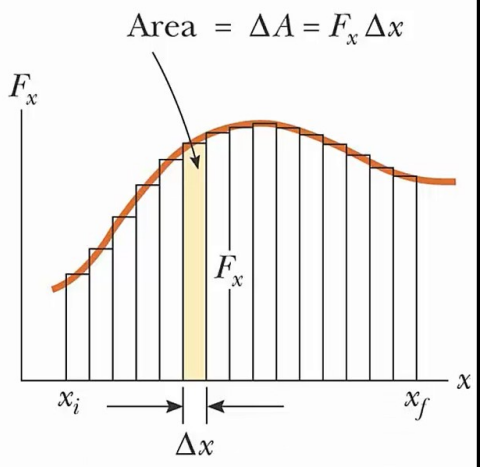
=(12 +10)J= 22J (Scalar Quantity)

Work Done by a Variable Force

- For a small displacement, Δx, $W \sim F\,\Delta x$

- Over the total displacement:

$$W \approx \sum_{x_i}^{x_f} F_x \Delta x$$



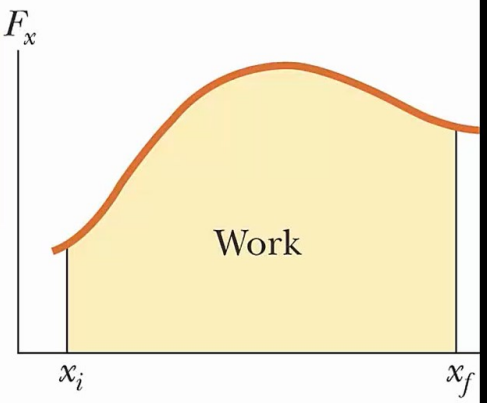
(a)

Work Done by a Variable Force

$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x dx$$

$$W = \int_{x_i}^{x_f} F_x dx$$

- The work done is equal to the area under the curve.



(b)

Example – Work by a Variable Force

The force acting on a particle is $F(x)= (9x^2 +3)$ N.

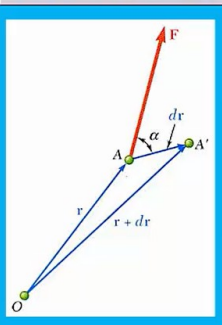
Find the work done by the force on the particle as it moves it from $x_1 =1.5$m to $x_2=4$m.

$$W = \int_{1.5}^{4} (9x^2 + 3) = \left| \frac{9x^3}{3} + 3x \right|_{1.5}^{4} = \left| 3x^3 + 3x \right|_{1.5}^{4}$$

$$= (192 + 12) - (10.125 + 4.5) = 204 - 14.625 = 189.375J$$

Lecture 41

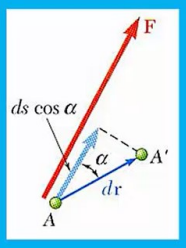
Work of a Force

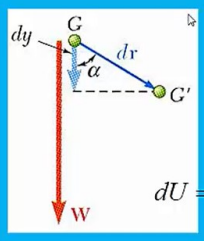


$dU = \vec{F} \cdot d\vec{r}$ = work of the force $\vec{F}$ corresponding to the displacement $d\vec{r}$

$$dU = F\,dr\cos\alpha$$

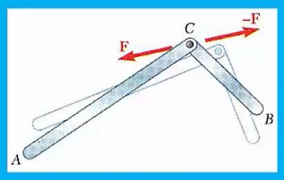
$\alpha = 0,\ dU = +F\,ds$ $\quad$ $\alpha = \pi,\ dU = -F\,ds$ $\quad$ $\alpha = \frac{\pi}{2},\ dU = 0$

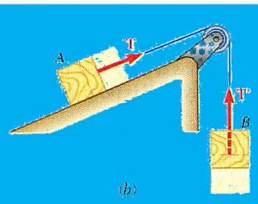


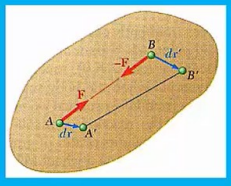


$dU = Wdy$

Work of a Force







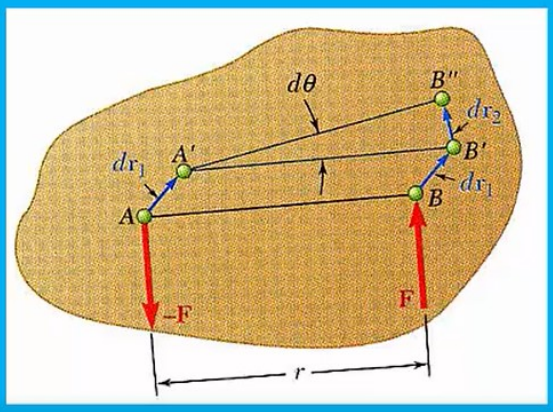
Forces which do no work:

- reaction at a frictionless pin due to rotation of a body around the pin
- reaction at a frictionless surface due to motion of a body along the surface
- weight of a body with cg moving horizontally
- friction force on a wheel moving without slipping

Sum of work done by several forces may be zero:

- bodies connected by a frictionless pin
- bodies connected by an inextensible cord
- internal forces holding together parts of a rigid body

Work of a Couple



Small displacement of a rigid body:

- translation to $A'B'$
- rotation of B' about A' to B''

$$W = -\vec{F} \cdot d\vec{r}_1 + \vec{F} \cdot (d\vec{r}_1 + d\vec{r}_2)$$
$$= \vec{F} \cdot d\vec{r}_2 = F\, ds_2 = F\, r d\theta$$
$$= M\, d\theta$$

Types of forces

Particles and rigid bodies are generally subject to two types of forces.

- Internal forces
- External forces

Internal Forces

Internal forces are those forces which the different parts of a system exert on each other and such forces obey Newton's Third Law of Motion.

External Forces

External forces are those which are not due to any part of the system but which are due to some external agency

If we regard the earth and the moon as a system, then mutual attraction will be an internal force but the attraction of the sun exerted on each of them will be an external force.

External forces are further classified into two types:

- Applied or active forces
- Reactive force

Forces of Constraint

When a particle or body is made to move along or rest on a curve or surface, the force exerted by such curve or surface is called a force of constraint or reactive force.

Applied Force

An external force which is not due to any constraint is called active force or applied force.

If a particle rests on or moves along an inclined plane, the reaction of the plane is a reactive force but the weight of the particle is an active force

Virtual displacement and Virtual Work

Virtual displacement

- If a set of particles or a body in equilibrium is supposed to undergo a displacement, the forces acting thereon being regarded as constant during the displacement, then such a displacement is called a virtual displacement and work done by the forces during such displacement is called Virtual Work.

Virtual Work

Virtual work is the total work done by the applied forces and the inertial forces of a mechanical system as it moves through a set of virtual displacements.

Virtual displacement

Note

It may be noted that virtual displacement is only hypothetical displacement involving no passage of time and is quite different from the actual displacement of a moving body taking place in course of time.

Types of Virtual displacement

In the case of body subject to constraints,

Virtual displacement may be of two types:

1. Virtual displacement consistent with the constraints.
2. Virtual displacement violating the constraints.

If a particle rests on a plane, any displacement along the plane satisfies the constraint(contact with the plane) and the displacement away from the plane violates the constraint.

Workless Constraints

In the case of displacement consistent with the constraints of a system, the forces of constraint generally do no work. For example, if a particle in contact with a smooth plane, is displaced along the plane, the reactive force, being perpendicular to the displacement, does zero work.

Workless Constraints

Definition:

Constraints in which the sum of the virtual works of all the forces of constraints in any virtual displacement of a system is zero are called workless constraints or ideal constraints. It is the assumption that the constraints that have to be dealt are workless.

Examples of Workless Constraints

1.

Reaction of a smooth surface with which the body is in contact.

In such a case, the reaction of the surface at a point of contact is entirely normal to the surface and, therefore, does zero work.

Examples of Workless Constraints

2.

Reaction at a fixed point or fixed axis of a body.

In this case, displacement of the point of application of any reactive force is zero, and so the work done by it vanishes.

Examples of Workless Constraints

3.

Reaction at the point of contact of a fixed surface on which a body rolls without sliding

The point of contact is instantaneously at rest, so its displacement in the direction of the reaction is zero.

Examples of Workless Constraints

4.

The mutual action and reaction of two bodies which roll upon each other.

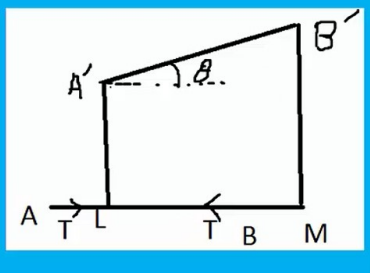
The bodies when regarded as a system, will exert equal and opposite forces on each other. The algebraic sum of the works done by such forces in any displacement, therefore, vanishes.

Examples of Workless Constraints

5.Tension of an inextensible string

Suppose a string AB is displaced to a nearby position $A'B'$, its length remaining unchanged. Suppose $A'B'$ makes an angle of measure θ with the direction of AB. θ being small . Let A′L,B′M be the perpendicular to the line AB.

The work done by the tension
=T*AL-T*AB
=T*(AB-LB)-T.(LM-LB)
=T.(AB-LM)
=T.AB$(1-\cos\theta)$
=T.AB(1-(1-$\frac{\theta^2}{2}$+....)=0



Important points about the Virtual Work Method

- It is not necessary to dismember the systems in order to establish relations between the active forces.
- Relations between active forces can be determined directly without reference to the reactive forces.
- The method is particularly useful in determining the position of equilibrium of a system under known loads (This is in contrast to determining the forces acting on a body whose equilibrium position is known).
- The method requires that internal frictional forces do negligible work during any virtual displacement.
- If internal friction is appreciable, work done by internal frictional forces must be included in the analysis.

Principle of Virtual Work

(i) Principle of Virtual Work for a single particle.

A particle subject to workless constraints is in equilibrium if only if zero virtual work is done by the applied forces in any arbitrary infinitesimal displacement consistent with the constraints.

Proof:

Let the total applied force on the particle be $\vec{F}_a$ and the total force of constraint $\vec{F}_c$. The particle is in equilibrium if and only if

$$\vec{F}_a + \vec{F}_c = 0$$

If $\delta\vec{r}$ is any infinitesimal displacement of the particle consistent with the constraints, the total virtual work will be zero if and only if

$$(\vec{F}_a + \vec{F}_c).\delta\vec{r} = 0$$

$$\vec{F}_a.\delta\vec{r} + \vec{F}_c.\delta\vec{r} = 0$$

$$\vec{F}_a.\delta\vec{r} + 0 = 0 \quad \because \vec{F}_c.\delta\vec{r} = 0 \ \text{workless constrints}$$

$$\vec{F}_a.\delta\vec{r} = 0$$

Principle of Virtual Work

(ii) Principle of Virtual work for a set of particles

A set of particles, subject to workless constraints, is in equilibrium if and only if zero virtual work is done by applied forces in any arbitrary infinitesimal displacement consistent with the constraints.

Proof

Let the particles of the set be $m_1, m_2, m_3, \ldots\ldots m_n$ Let the total applied force on m_1 be $\vec{F}_{1a}$ and the total force of constraint on it be F_{1c}. The set of particles will be in equilibrium if and only if the total force on each particle vanishes. i.e. if and only if

$$\vec{F}_{ia} + \vec{F}_{ic} = 0 \quad i = 1, 2, 3, \ldots, n \text{ ----------(1)}$$

Internal forces do not occur in this equation because they occur as pairs of equal and opposite forces, so their vector sum vanishes.

Principle of Virtual Work

Let $\delta\vec{r}$ be an arbitrary infinitesimal displacement of m_i. .The total work done by all the forces on the particles of the set will vanish if only if

$$\sum_{i=1}^{n} (\vec{F}_{ia} + \vec{F}_{ic}).\delta\vec{r} = 0$$

$$\sum_{i=1}^{n} \vec{F}_{ia}.\delta\vec{r} + \sum_{i=1}^{n} \vec{F}_{ic}.\delta\vec{r} = 0$$

$$\sum_{i=1}^{n} \vec{F}_{ia}.\delta\vec{r} + 0 = 0 \quad \because \sum_{i=1}^{n} \vec{F}_{ic}.\delta\vec{r} = 0 \ \textit{workless constrain}$$

$$\sum_{i=1}^{n} \vec{F}_{ia}.\delta\vec{r} = 0 \text{ ---- -(2}$$

If Equation (1) holds, then the equation (2) holds. Conversely, if equation (2) hold, then equation (1) holds since $\delta\vec{r}_i$ are arbitrary and

Principle of Virtual Work

(iii) Principle of Virtual Work for a Rigid body or a Set of Rigid Bodies

A rigid body or a set of rigid bodies, subject to workless constraints, is in equilibrium if and only if zero virtual work is done by the applied forces and applied torques in any arbitrary infinitesimal displacement consistent with the constraints.

Proof

A general system of forces acting on a rigid body is equivalent to a force $\sum \vec{F}_i$ passing through any point O, together with the a couple

$\vec{G}=\sum \vec{r}_i \times \vec{F}_i$

Where $\vec{F}_i$ is the force acting at a point of the body whose position vector relative to O is $\vec{r}_i$. Suppose that in the present case

$\vec{R}=\vec{R}_a+\vec{R}_c$

Principle of Virtual Work

where $\vec{R}_a$ is the vector sum of applied forces and $\vec{R}_c$ is the vector sum of forces of constraints and $\vec{G}=\vec{G}_a+\vec{G}_c$ where $\vec{G}_a$ is the sum of applied torques and $\vec{G}_c$ is the sum of torques of constraints.

The body will be in equilibrium if only if

$$\vec{R} = \vec{R}_a + \vec{R}_c = 0$$
$$\vec{G} = \vec{G}_a + \vec{G}_c = 0 \quad \text{---------(1)}$$

An arbitrary virtual displacement of the body will consist of a translation $\delta\vec{s}$ and a rotation $\delta\Omega$ about O. The vanishing of the total virtual work, therefore, implies

$$(\vec{R}_a + \vec{R}_c).\delta\vec{s} + (\vec{G}_a + \vec{G}_c).\delta\vec{\Omega} = 0 \quad ---(2)$$
$$\because \vec{R}_c.\delta\vec{s} + \vec{G}_c.\delta\vec{\Omega} = 0$$
$$\therefore \vec{R}_a.\delta\vec{s} + \vec{G}_a.\delta\vec{\Omega} = 0 \quad --(3)$$

If equation (1) hold then (2) holds and therefore (3) holds.

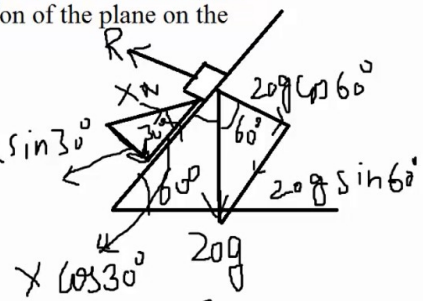
Lecture 43

Example

A particle of mass 20kg is supported on a smooth plane inclined at 60 degrees to the horizontal by a force of magnitude X Newton which makes an angle of 30 degrees with the plane. Find X and also the reaction of the plane on the particle.

Sol:

Principle of virtual work



$X\cos 30^\circ \cdot \delta s - 20g\sin 60^\circ \cdot \delta s = 0$

$X\cos 30^\circ = 20g\sin 60^\circ$

$X = 20g$

$R\cdot \delta y - X\sin 30^\circ \delta y - 20g\cos 60^\circ \cdot \delta y$

$R = X\sin 30^\circ + 20g\cos 60^\circ$

$= 20g \times \frac{1}{2} + 20g \times \frac{1}{2} = 20g$

Example

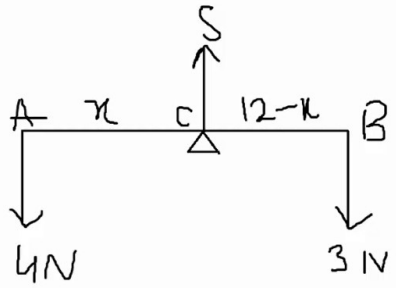
A light thin rod, 12m long, can turn in a vertical plane about one of its points which is attached to a pivot. If wights of 3N and 4N are suspended from its ends; it rests in a horizontal position. Find the position of the pivot and its reaction on the rod.

Sol:

eq. of V.W.

$\delta\Omega$

$4x \quad -3(12-x)$



$4x\cdot\delta\Omega - 3(12-x)\cdot\delta\Omega = 0$

$4x - 36 + 3x = 0 \Rightarrow 7x - 36 = 0 \Rightarrow x = \frac{36}{7} = 5\frac{1}{7}$

$\delta\Omega$

$4x \qquad -3(12-x) \qquad 4N \qquad 3N$

$4x \cdot \delta\Omega - 3(12-x)\cdot\delta\Omega = 0$

$4x - 36 + 3x = 0 \Rightarrow 7x - 36 = 0 \Rightarrow x = \frac{36}{7} = 5\frac{1}{7}$

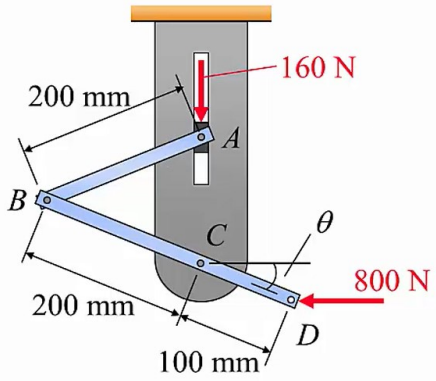
$AC = 5\frac{1}{7}\,m, \quad BC = 12 - \frac{36}{7} = \frac{48}{7} = 6\frac{6}{7}\,m$

δy

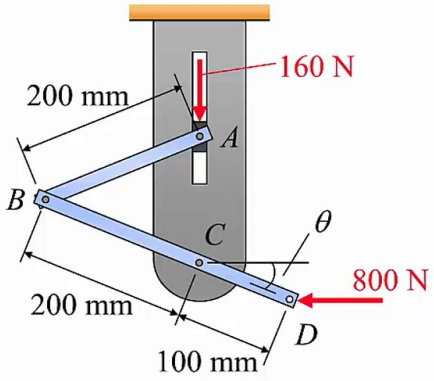
$5\delta y - 4\delta y - 3\delta y = 0$

$S - 7 = 0 \Rightarrow S = 7\,N$

Example



Rod AB is attached to a block at A that can slide freely in the vertical slot shown. Neglecting the effect of friction and the weights of the rods, determine the value of θ corresponding to equilibrium.

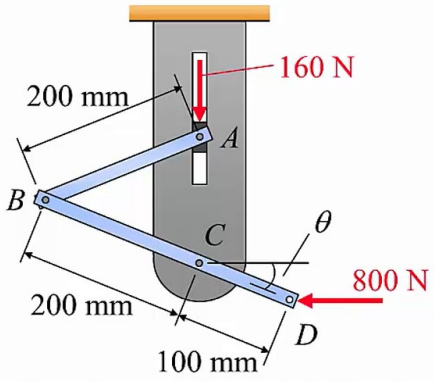


Solution

Rod AB is attached to a block at A that can slide freely in the vertical slot shown. Neglecting the effect of friction and the weights of the rods, determine the value of θ corresponding to equilibrium.

1. *Apply the principle of virtual work.* If a system of connected rigid bodies is in equilibrium, the total virtual work of the external forces applied to the system is zero for any virtual displacement of the system.

1a. Define a virtual displacement. By using a single variable, define a virtual displacement of all the forces that do work.



Solution

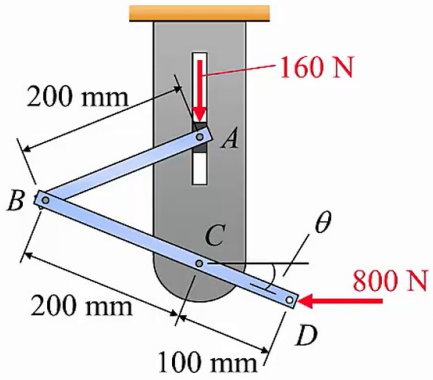
Rod AB is attached to a block at A that can slide freely in the vertical slot shown. Neglecting the effect of friction and the weights of the rods, determine the value of θ corresponding to equilibrium.

1b. Express the total virtual work done by the forces and couples during the virtual displacement.

$$\delta U = \mathbf{F} \cdot \delta \mathbf{r} \quad \text{and} \quad \delta U = M\,\delta\theta$$

where δU is the virtual work, $\mathbf{F}$ is a force undergoing a virtual displacement $\delta \mathbf{r}$, and M is a couple undergoing a virtual rotation $\delta\theta$.

1c. Set the virtual work to zero and solve for the variable.



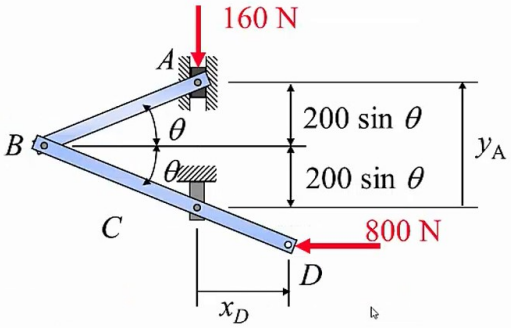
Define a virtual displacement.

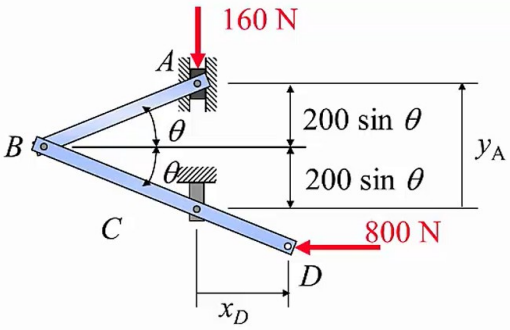
$y_A = 400 \sin \theta$

$\delta y_A = 400 \cos \theta \, \delta\theta$

$x_D = 100 \cos \theta$

$\delta x_D = -100 \sin \theta \, \delta\theta$





Solution

Express the total virtual work done by the forces.

Set the virtual work to zero and solve for the variable.

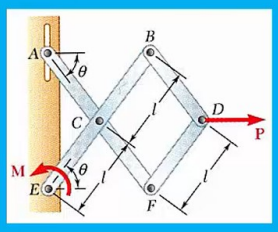
Virtual work: $\delta U = -(160 \text{ N}) \, \delta y_A - (800 \text{ N}) \, \delta x_D = 0$

$-(160)(400 \cos \theta \, \delta\theta) - (800)(-100 \sin \theta \, \delta\theta) = 0$

$\frac{\sin \theta}{\cos \theta} = 0.8 ; \quad \tan \theta = 0.8$ $\boxed{\theta = 38.7^\circ}$

Lecture 44

Example

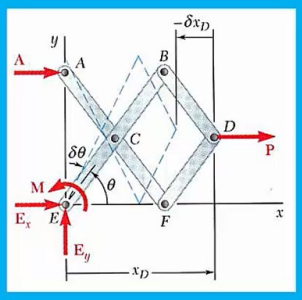


Determine the magnitude of the couple M required to maintain the equilibrium of the mechanism.

SOLUTION:

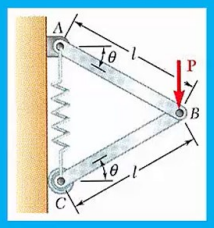
- Apply the principle of virtual work

$$\delta U = 0 = \delta U_M + \delta U_P$$
$$0 = M\delta\theta + P\delta x_D$$



$$x_D = 3l\cos\theta$$
$$\delta x_D = -3l\sin\theta\delta\theta$$
$$0 = M\delta\theta + P\left(-3l\sin\theta\delta\theta\right)$$
$$0 = M - 3P\sin\theta$$
$$M = 3P\sin\theta$$

Example

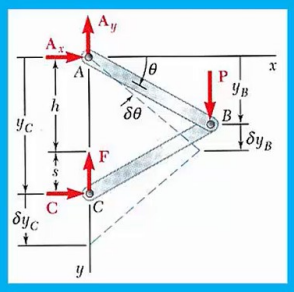


Determine the expressions for θ and for the tension in the spring which correspond to the equilibrium position of the spring. The unstretched length of the spring is h and the constant of the spring is k. Neglect the weight of the mechanism.

SOLUTION:

- Apply the principle of virtual work

$$\delta U = \delta U_B + \delta U_F = 0$$
$$0 = P\delta y_B - F\delta y_C$$

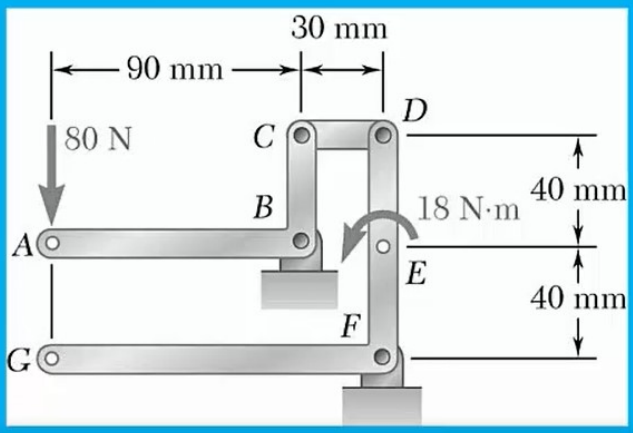


$$y_B = l\sin\theta \qquad y_C = 2l\sin\theta \qquad F = ks$$
$$\delta y_B = l\cos\theta\delta\theta \qquad \delta y_C = 2l\cos\theta\delta\theta \qquad = k\left(y_C - h\right)$$
$$= k\left(2l\sin\theta - h\right)$$
$$0 = P\left(l\cos\theta\delta\theta\right) - k\left(2l\sin\theta - h\right)\left(2l\cos\theta\delta\theta\right)$$

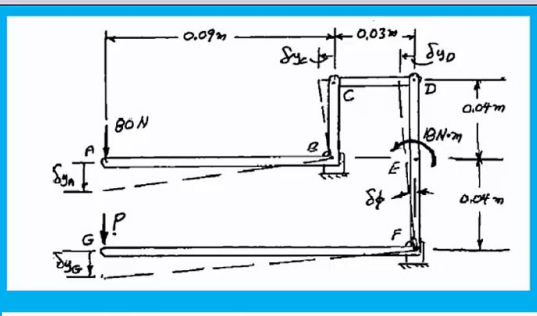
$$\sin\theta = \frac{P + 2kh}{4kl}$$
$$F = \frac{1}{2}P$$

Example

Determine the vertical force **P** that must be applied at *G* to maintain the equilibrium of the linkage.



Solution



$$\delta y_C = \frac{0.04}{0.09}\delta y_A = \frac{4}{9}y_A \leftarrow, \quad \delta y_D = \delta y_C = \frac{4}{9}\delta y_A \leftarrow$$

$$\delta y_G = \frac{0.12\text{ m}}{0.08\text{ m}}\delta y_D = 1.5\left(\frac{4}{9}\delta y_A\right) = \frac{2}{3}\delta y_A \downarrow$$

$$\delta\phi = \frac{\delta y_D}{0.08} = \frac{4}{9}\delta y_A/0.08 = \frac{4}{0.72}\delta y_A = \frac{50}{9}\delta y_A \circlearrowright$$

Solution

Virtual Work:

$$\delta U = 0: \quad (80\ \text{N})\delta y_A + (18\ \text{N}\cdot\text{m})\delta\phi + P\delta y_G = 0$$

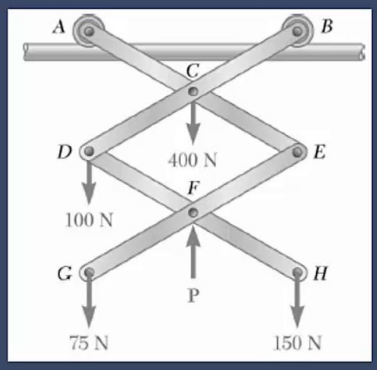
$$80\delta y_A + 18\left(\frac{50}{9}\delta y_A\right) + P\left(\frac{2}{3}\delta y_A\right) = 0$$

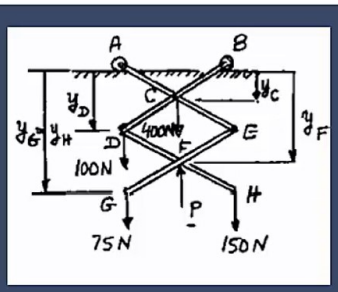
$$80 + 100 + \frac{2}{3}P = 0$$

$$P = -270\ \text{N}$$

Example

Determine the force **P** required to maintain the equilibrium of the linkage shown. All members are of the same length and the wheels at *A* and *B* roll freely on the horizontal rod.





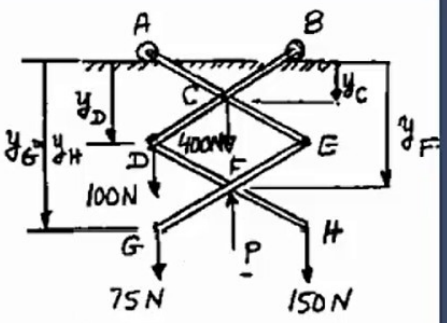
Solution

$$y_D = 2y_C \qquad \delta y_D = 2\delta y_C$$

$$y_F = 3y_C \qquad \delta y_F = 3\delta y_C$$

$$y_G = y_H = 4y_C$$

$$\delta y_G = \delta y_H = 4\delta y_C$$



$$\delta U = (400\text{ N})\delta y_C + (100\text{ N})\delta y_D - P\delta y_F + (75\text{ N})\delta y_G + (150\text{ N})\delta y_H = 0$$

$$400\delta y_C + 100(2\delta y_C) - P(3\delta y_C) + (75+150)(4\delta y_C) = 0$$

$$3P = 400 + 200 + 900 \qquad P = +500\text{ N}$$