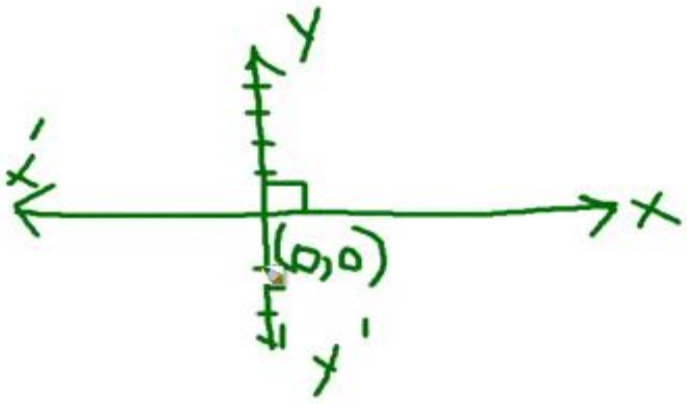
Lecture 23:





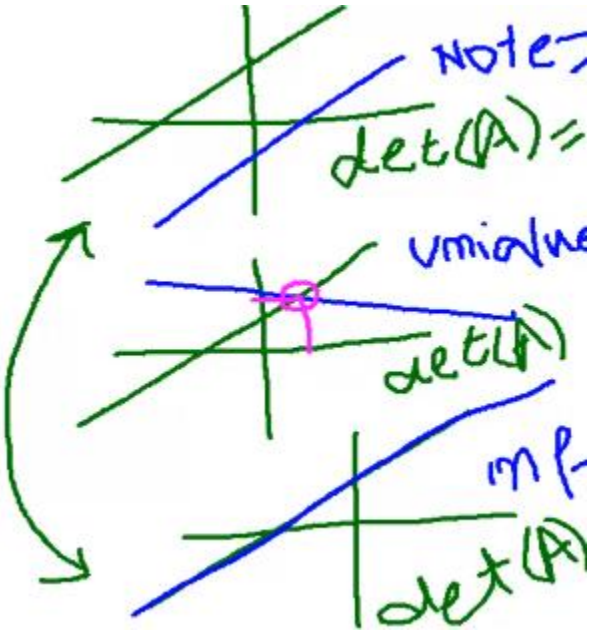
1

$a_1x + b_1y = 0$
$a_2x + b_2y = 0$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = A$$

$y = x^2 \quad x \in \mathbb{R}$

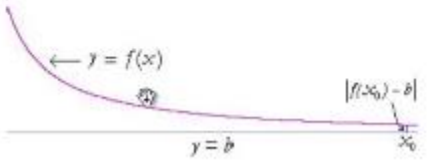
0	1	-1	2	-2
0	1	1	4	4



- Graphs and its applications
- Preliminary Definitions
  - Asymptote.
  - Horizontal Asymptote.
  - Vertical Asymptote.

Asymptote

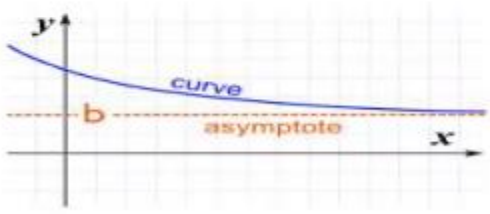
An asymptote is a line to which the graph gets arbitrarily close.That means that for any distance named, no matter how small, the graph will get within that distance and stay within that distance for some section of the graph with infinite length.More precisely



Horizontal Asymptote

The graph of $y = f(x)$ has a horizontal asymptote of $y = b$, if and only if, either

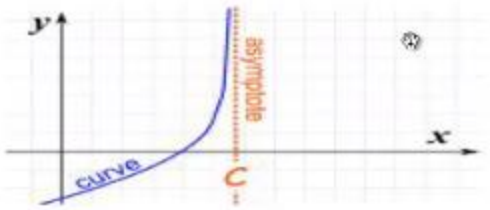
$$\lim_{x\to\infty} f(x) = b \quad or \quad \lim_{x\to-\infty} f(x) = b. \tag{1}$$



Vertical Asymptote

The graph of $y = f(x)$ has a vertical asymptote of $x = c$, if and only if, either

$$\lim_{x \to c^-} f(x) = \pm\infty \quad or \quad \lim_{x \to c^+} f(x) = \pm\infty. \tag{2}$$



Asymptotes of a Rational functions

Let r(x) be a rational function with polynomial
$p(x) = a_n x_n + ... + a_0$ of degree n in the numeratorand polynomial
$q(x) = b_m x_m + ... + b_0$ of degree m in the denominator

- If $n < m$, then $r(x)$ has a horizontal asymptote of $y = 0$.
- If $n > m$, then $r(x)$ becomes unbounded for large values of x (positive or negative).
- If $n = m$, then $r(x)$ has a horizontal asymptote of $y = \frac{a_n}{b_n}$.

Asymptotes of a Rational functions

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}$$

(1) if $n = m$, then the line $y = \frac{a_n}{b_m}$

(2) if $n < m$, then $y = 0$

(3) if $n > m$, then $r(x)$ becomes unbounded for large values of x.

Asymptotes of a Rational functions

Example(1):

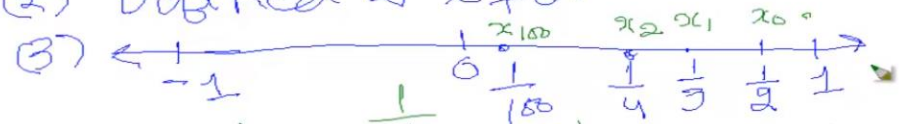
Discuss the Asymptotes of the following rational function

$$r(x) = \frac{1}{x}, \tag{3}$$

(1) $y=0$ is horizontal asymptote of $f(x)$.

(2) Defined $\forall\ x \neq 0$.

(3)



$f(x_0) = \frac{1}{\frac{1}{2}} = 2$ | $f(x_2) = 4$

$f(x_1) = 3$ | $f(x_{150}) = 150$

Asymptotes of a Rational functions

Example(1):

Discuss the Asymptotes of the following rational function

$$r(x) = \frac{1}{x}, \tag{3}$$

- $f(x_n) = \frac{1}{\frac{1}{n}} = n$

$\lim_{n \to \infty} f(x_n) = \lim_{n \to 0} (n) = \infty$

Thus at $x=0$, $f(x)$ has a vertical Asymptote.

Asymptotes of a Rational functions

Example(1):

Discuss the Asymptotes of the following rational function

$$r(x) = \frac{1}{x}, \tag{3}$$

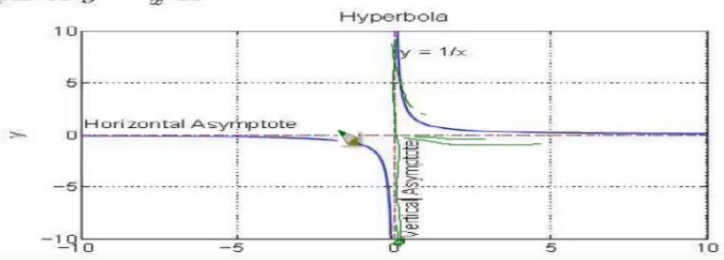
- This is defined $\forall x \neq 0$.
- Consider the sequence of numbers $x_n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{k}$ for $k = 1, 2, \cdots, n$.
- These numbers are getting closer and closer to zero.
- Since $r(x_n) = \frac{1}{x_n} = \frac{1}{\frac{1}{n}} = n$.
- So, $r(x_n) = 2, 3, 4, \cdots, k, \cdots$ for $n = 2, 3, 3, \cdots, k, \cdots$, which is getting larger and larger, so approaching the vertical line $x = o$.

Asymptotes of a Rational functions

Example(1):

- Thus there is a vertical asymptote at $x = 0$.

The graph of $y = \frac{1}{x}$ is



ASYMPTOTES OF A RATIONAL FUNCTIONS

EXAMPLE(2):

$$g(x) = \frac{10x}{x+2}$$

(1) $\frac{10}{1} = 10$

$g(x) \sim \frac{10x}{x}$ as $x \to \infty$

$= 10$

$$g(x) = \frac{10x}{x+2} = 10 - \frac{20}{x+2}$$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left(10 - \frac{20}{x+2}\right)$$

ASYMPTOTES OF A RATIONAL FUNCTIONS

EXAMPLE(2):

$$g(x) = \frac{10x}{x+2}$$

(1) $\Rightarrow \lim_{x\to\infty} g(x) = 10 - 0$

$= 10$

Asymptotes of a Rational functions

Example(2):

Discuss the Asymptotes of the following rational function

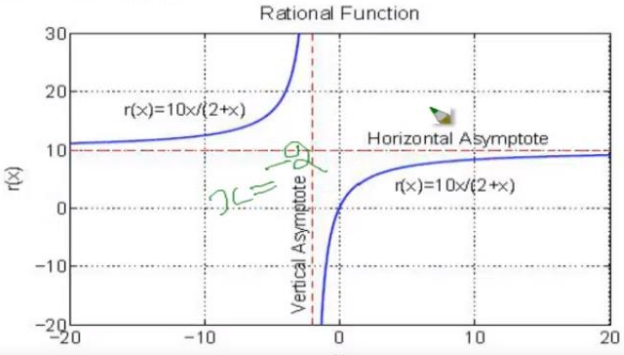
$$r(x) = \frac{10x}{2+x} \quad (4)$$

- Defined $\forall x \neq -2$
- The numerator and denominator are linear functions (degree of polynomials are the same).
- we can see that as x get large then 2 in the denominator becomes insignificant.
- Thus $r(x) \approx \frac{10x}{x} = 10$
- Thus horizontal asymptote occurs at $x = 10$.

Asymptotes of a Rational functions

Example(2):

The graph of $y = \frac{10x}{2+x}$ is



Asymptotes of a Rational functions

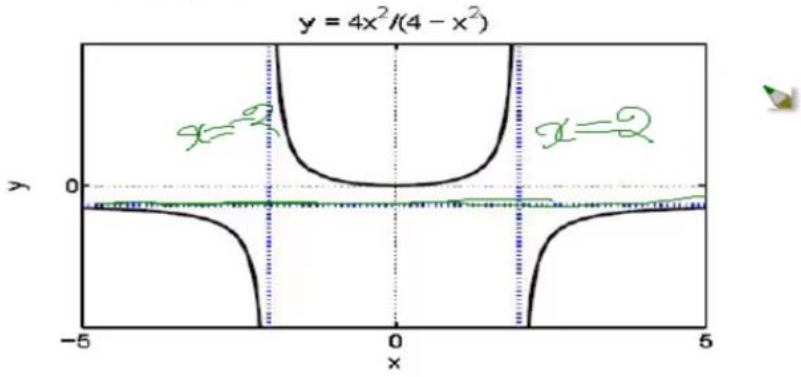
Example(3):

Discuss the Asymptotes of the following rational function

$$r(x) = \frac{4x^2}{4 - x^2}, \tag{5}$$

- Defined $\forall x \neq \pm 2$.
- Clearly the function passes through the origin, so the x and y-intercept is $(x, y) = (0, 0)$.
- Note that this function is an even function, the edge of the domain is $x = 2$, so we see there are vertical asymptotes at $x = 2$.
- Thus, for x large $r(x) \approx \frac{4x^2}{-x^2} = -4$.
- Thus, a horizontal asymptote occurs at $y = -4$.

The graph of $y = \frac{4x^2}{4-x^2}$ is



Lecture 24:

Asymptotes of a Rational functions

Oblique Asymptote:

- $r(x) = \frac{p(x)}{q(x)} = f(x) + \frac{g(x)}{q(x)}$.
- Obviously $deg(g(x)) < deg(q(x))$.
- Implies $\frac{g(x)}{q(x)} \to 0$ as $x \to \pm\infty$.
- Thus $r(x) \to f(x)$ as $x \to \pm\infty$.
- Particularly if $deg(p(x))$ is one more less than $deg(q(x))$, then $f(x)$ is a linear function whose graph is a non-horizontal line in the plane.
- That line is called oblique asymptote.

Asymptotes of a Rational functions

Example(1):

$$r(x) = \frac{x^2 - 3}{2x+1} = \frac{P(x)}{q(x)}$$

$$= \left(\frac{1}{2}x - 1\right) + \frac{1}{2x+1} \to 0 \text{ as } x \to \pm\infty$$

$$= f(x) + \frac{g(x) \to \text{Remainder}}{q(x) \to \text{divisor}}$$

$f(x) \to$ quotient

ASYMPTOTES OF A RATIONAL FUNCTIONS

EXAMPLE(1):

$100 \rightarrow f(x) = \left(\frac{1}{2}x - 1\right)$

↓

Slant ASymptote

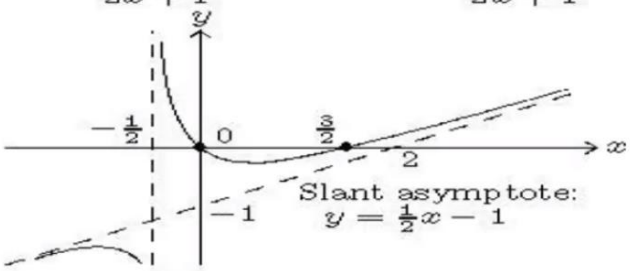
whenever $x \rightarrow \pm\infty$ and

$\frac{1}{2x+1} \rightarrow 0$

ASYMPTOTES OF A RATIONAL FUNCTIONS

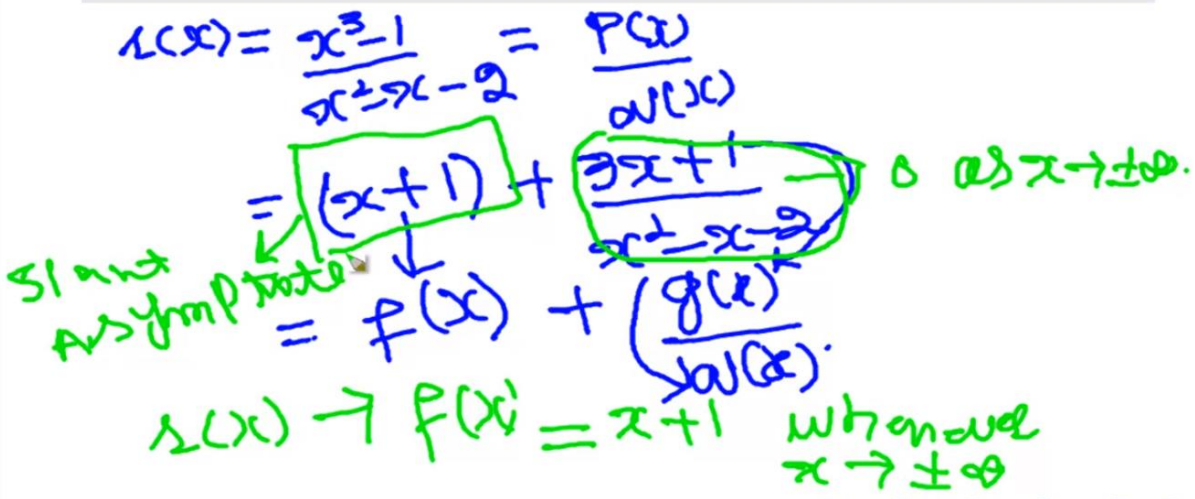
OBLIQUE ASYMPTOTE:

$$\frac{x^2 - \frac{3}{2}x}{2x+1} = \left(\frac{1}{2}x - 1\right) + \frac{1}{2x+1}$$



Asymptotes of a Rational functions

Example(2):



Asymptotes of a Rational functions

Example(2):

Discuss the Asymptotes of the following rational function

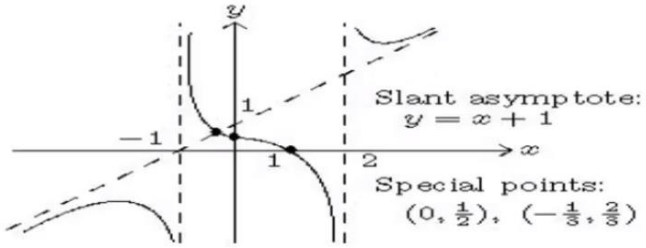
$$\begin{aligned} r(x) &= \frac{x^3-1}{x^2-x-2}, \\ &= x+1+\frac{3x+1}{x^2-x-2}. \end{aligned} \tag{7}$$

- Clearly the quotient is a linear function.
- Thus it is our oblique asymptote.
- For more understanding see the graph.

Asymptotes of a Rational functions

Oblique Asymptote:

$$\frac{x^3-1}{x^2-x-2} = (x+1)+\frac{3x+1}{(x-2)(x+1)}$$



Asymptotes in Polar Coordinates

Polar Equation of a Straight Line:

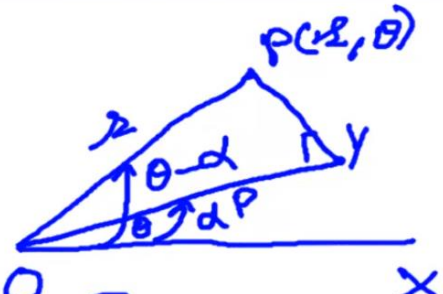
$p = r\cos(\theta-\alpha)$

Then

$\cos(POX) = \frac{p}{r}$

$\Rightarrow \cos(\theta-\alpha) = \frac{p}{r}$

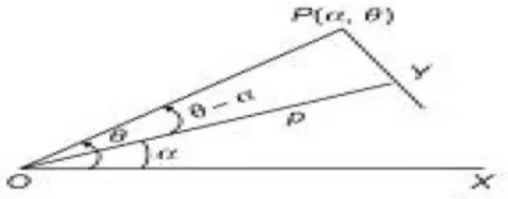
$\Rightarrow \boxed{p = r\cos(\theta-\alpha)}$



Asymptotes in Polar Coordinates

Polar Equation of a Straight Line:

- The polar equation of any line is $p = r\cos(\theta - \alpha)$.
- where p is the length of the perpendicular from pole to the line.
- α is the angle which this perpendicular makes with the initial line.



Asymptotes in Polar Coordinates

Polar Equation of a Straight Line (Proof):

- Let OX be initial line and P be any point on the line whose polar coordinates are (r, θ).
- Therefore, $OP = r$ and $\angle POX = \theta$.
- Let us draw a perpendicular OY on the line from P produced.
- Now using $\triangle OPY$, we have
-

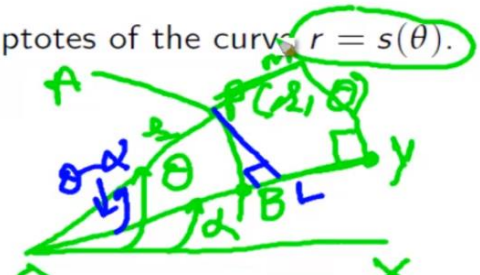
$$\cos POY = \frac{OY}{OP} = \frac{p}{r}$$
$$p = r\cos POY = r\cos(\theta - \alpha), \qquad (8)$$

Lecture 25:

Asymptotes in Polar Coordinates

- Now we determine the asymptotes of the curve $r = s(\theta)$.

$P = s\cos(\theta - \alpha)$



$\Rightarrow PM = LY$

$= OY - OL$

$= P - OP\cos(\theta - \alpha)$

$PM = P - OP\cos(\theta - \alpha)$

$= P - s\cos(\theta - \alpha)$

Asymptotes in Polar Coordinates

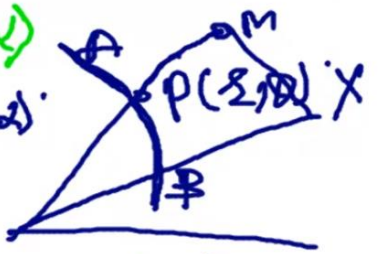
- Now we determine the asymptotes of the curve $r = s(\theta)$.

$PM = P - s\cos(\theta - \alpha)$

$\Rightarrow \frac{PM}{s} = \frac{P}{s} - \cos(\theta - \alpha)$.

As $s \to \infty$ then

$PM \cdot \frac{1}{s} \to 0$ and $\frac{P}{s} \to 0$



Asymptotes in Polar Coordinates

- Now we determine the asymptotes of the curve $r = s(\theta)$.

$\cos(\theta - \alpha) = 0$ As $\xi \to \infty$

$\Rightarrow \cos(\theta_1 - \alpha) = 0$ $\theta_1 \to \theta_1$

$\Rightarrow \cos(\theta_1 - \alpha) = \cos(\frac{\pi}{2})$

$\Rightarrow \theta_1 - \alpha = \pi/2.$

$\Rightarrow \theta_1 = \alpha + \pi/2.$

$\Rightarrow \alpha = \theta_1 - \pi/2.$

Asymptotes in Polar Coordinates

- Now we determine the asymptotes of the curve $r = s(\theta)$.

$PM = p - \xi \cos(\theta - \alpha)$ As $\xi \to \infty$

$\Rightarrow 0 = p - \xi \cos(\theta - \alpha)$ then $PM \to 0$

$\Rightarrow p = \xi \cos(\theta - \alpha).$

$= \lim_{\xi \to \infty} [\xi \cos(\theta - \alpha)].$

$= \lim_{\xi \to \infty} [\xi \cos\theta(\theta - \theta_1 + \frac{\pi}{2})]$

Asymptotes in Polar Coordinates

- Now we determine the asymptotes of the curve $r = s(\theta)$.

$$p = \lim_{s \to \infty} \left[s \cos(\theta - \theta_1 + \pi/2) \right]$$

$$= \lim_{\theta \to \theta_1} \left[\frac{\cos(\theta - \theta_1 + \pi/2)}{\frac{1}{s}} \right]$$

$$= \left[\lim_{\theta \to \theta_1} \frac{\cos(\theta - \theta_1 + \pi/2)}{\lim_{\theta \to \theta_1} \left(\frac{1}{s}\right)} \right]$$

Asymptotes in Polar Coordinates

- Now we determine the asymptotes of the curve $r = s(\theta)$.

$$p = \frac{\lim_{\theta \to \theta_1} \sin(\theta_1 - \theta)}{\lim_{\theta \to \theta_1} \left(\frac{1}{s}\right)} = \frac{0}{0}$$

$$= \frac{\lim_{\theta \to \theta_1} \cos(\theta_1 - \theta)}{\lim_{\theta \to \theta_1} \left(-\frac{1}{s^2} \frac{ds}{d\theta}\right)} = \frac{1}{\lim_{\theta \to \theta_1} \left(-\frac{1}{s^2} \frac{ds}{d\theta}\right)}$$

Asymptotes in Polar Coordinates

- Now we determine the asymptotes of the curve $r = s(\theta)$.

$$p = \frac{1}{\lim_{\theta \to \theta_1} \left(-\frac{1}{s^2} \frac{ds}{d\theta} \right)}$$

$$= \frac{1}{\lim_{\theta \to \theta_1} \left(\frac{du}{d\theta} \right)}$$

$$\Rightarrow p = \lim_{\theta \to \theta_1} \left(\pm \frac{d\theta}{du} \right)$$

Let $u = \frac{1}{s}$

$$\frac{du}{d\theta} = -\frac{1}{s^2} \frac{ds}{d\theta}$$

Asymptotes in Polar Coordinates

- Now we determine the asymptotes of the curve $r = s(\theta)$.

$p = s \cos(\theta - \alpha)$

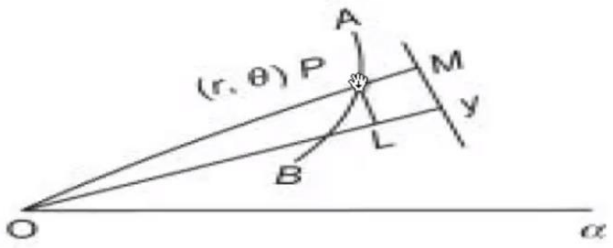
Hence Asymptotes in Polar-coordinates are

$$\lim_{\theta \to \theta_1} \left(\frac{d\theta}{du} \right) = -s \cos(\theta - \theta_1 \mp \pi/2)$$

$$\boxed{\lim_{\theta \to \theta_1} \left(\frac{d\theta}{du} \right) = \pm s \sin(\theta_1 - \theta)}$$

Asymptotes in Polar Coordinates

- Now we determine the asymptotes of the curve $r = s(\theta)$.
- For this we have to determine the constant p and α.



Asymptotes in Polar Coordinates

Construction:

- Let $P(r, \theta)$ be any point on the curve AB whose equation is $r = p(\theta)$.
- Let us draw a perpendicular OY on the line $p = r\cos(\theta - \alpha)$ which is MY.
- Therefore

$$\begin{aligned} PM = LY &= OY - OL = p - OP\cos(\theta - \alpha) \\ &= p - r\cos(\theta - \alpha). \end{aligned} \tag{9}$$

Asymptotes in Polar Coordinates

Construction:

- Now $r \to \infty$ as the point recedes to infinity along the curve.Let $\theta \to \theta_1$ where $r \to \infty$.
- Therefore we have $\frac{PM}{r} = \frac{p}{r} - \cos(\theta - \alpha)$ by (9). when $r \to \infty$, $PM \to 0$.
- So that

$$\frac{PM}{r} = PM.\frac{1}{r} \to 0 \text{ and } \frac{p}{r} \to 0. \tag{10}$$

Asymptotes in Polar Coordinates

Construction:

- Let

$$\begin{aligned} \cos(\theta - \alpha) &= 0 \\ \cos(\theta - \alpha) &= \cos(\frac{\pi}{2}) \\ \theta_1 - \alpha &= \frac{\pi}{2} \quad (\because \theta \to \theta_1 \text{ as } r \to \infty) \\ \theta_1 &= \alpha + \frac{\pi}{2} \\ \alpha &= \theta_1 - \frac{\pi}{2}. \end{aligned} \tag{11}$$

Asymptotes in Polar Coordinates

Construction:

- Again from (9), we have when $r \to \infty$

$$\begin{aligned} p &= \lim_{r\to\infty}[r\cos(\theta-\alpha)] \\ &= \lim_{r\to\infty}[r\cos(\theta-\theta_1+\frac{\pi}{2})] \\ &= \lim_{r\to\infty}[r\sin(\theta_1-\theta)] \\ &= \lim_{\theta\to\theta_1}\left[\frac{\sin(\theta_1-\theta)}{\frac{1}{r}}\right]\cdots\frac{0}{0} \end{aligned}$$

Asymptotes in Polar Coordinates

Construction:

-

$$\begin{aligned} p &= \lim_{\theta\to\theta_1}\left[\frac{\cos(\theta_1-\theta)}{-\frac{1}{r^2}\frac{dr}{d\theta}}\right] \\ &= \frac{\lim_{\theta\to\theta_1}(\cos(\theta_1-\theta))}{\lim_{\theta\to\theta_1}(-\frac{1}{r^2}\frac{dr}{d\theta})}. \end{aligned} \tag{13}$$

- Now let $u = \frac{1}{r}$,this implies $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$. Therefore (13) becomes

Asymptotes in Polar Coordinates

Construction:

- $$p = \lim_{\theta \to \theta_1} -\frac{d\theta}{du}. \tag{14}$$
- Hence the asymptotes is

$$\begin{aligned} \lim_{\theta \to \theta_1} -\frac{d\theta}{du} &= r\cos(\theta - \alpha) \\ &= r\cos(\theta - \theta_1 + \frac{\pi}{2}) \\ &= r\sin(\theta_1 - \theta), \end{aligned} \tag{15}$$

Asymptotes in Polar Coordinates

Working Rules:

- We have to change r to $\frac{1}{u}$ in the given equation and find out the limit of θ as $u \to 0$.
- Let θ_1 be any one of the angle possible limits of θ.
- We have to determine then $-\frac{d\theta}{du}$ and its limit as $u \to 0$ and $\theta \to \theta_1$.
- Let this limit be p.
- Then $p = r\sin(\theta_1 - \theta)$ is the required asymptote.

ASYMPTOTES IN POLAR COORDINATES

EXAMPLE(1):

$$-\lim_{\theta \to \theta_1} \frac{d\theta}{du} = \frac{1}{r} = 2\sin(\theta_1 - \theta)$$

$$2\sin\theta = 2\cos(2\theta)$$

Step 1:- $\Rightarrow r = \dfrac{2\cos(2\theta)}{\sin\theta}$

$$\Rightarrow \frac{1}{r} = \frac{\sin\theta}{2\cos(2\theta)}$$

ASYMPTOTES IN POLAR COORDINATES

EXAMPLE(1):

Step 1:- $\dfrac{1}{r} = \sin\theta / 2\cos 2\theta$

Step 2 $r \to \infty$ when $\theta \to \theta_1$

$\Rightarrow \sin\theta = 0$

$\Rightarrow \theta = n\pi$

$n \in \mathbb{Z}$

$\theta \to \theta_1 = n\pi$

Let $\dfrac{1}{r} = u = \dfrac{\sin\theta}{2\cos 2\theta}$

As $r \to \infty$ when $\dfrac{1}{r} \to 0$

$\Rightarrow u = 0$

$\Rightarrow \dfrac{\sin\theta}{2\cos 2\theta} = 0$

Asymptotes in Polar Coordinates

Example(1):

Steps:- $\boxed{p = -\lim_{\theta \to \theta_1} \frac{d\theta}{du}}$

$\Rightarrow -\lim_{\theta \to n\pi} \frac{du}{d\theta} = \left[\frac{d}{d\theta}\left(\frac{\sin\theta}{2\cos 2\theta}\right)\right]_{\lim \theta \to n\pi}$

$= \lim_{\theta \to n\pi} \left[\frac{\cos 2\theta \cos\theta - \sin\theta(-2\sin 2\theta)}{(\cos 2\theta)^2}\right]$

$= \frac{1}{2}\left[\frac{\cos(2n\pi)\cos(n\pi) + \sin(n\pi)(2\sin(4n\pi))}{\cos^2(2n\pi)}\right]$ ($\sin(n\pi) \to 0$)

Asymptotes in Polar Coordinates

Example(1):

$-p = \frac{1}{2}\left[\frac{\cos(2n\pi)\cos(n\pi)}{\cos^2(2n\pi)}\right] = \frac{1 \times (-1)^n}{2}$ $n \in \mathbb{Z}$

$= \frac{(-1)^n}{2}$

$\Rightarrow -\lim_{\theta \to n\pi}\left(\frac{du}{d\theta}\right) = \boxed{\frac{2}{(-1)^n} = p}$

Asymptotes in Polar Coordinates

Example(1):

$$p = 2\sin(\theta_1 - \theta)$$

$$\Rightarrow \frac{-2}{(-1)^n} = 2\sin(n\pi - \theta); \quad n \in \mathbb{Z}$$

$$= 2\left[\sin(n\pi)\cos\theta - \cos(n\pi)\sin\theta\right]$$

$$\pm\frac{2}{(-1)^n} = 2\left[0 \mp (-1)^n \sin\theta\right]$$

Asymptotes in Polar Coordinates

Example(1):

Hence required Asymptotes are

$$\boxed{2\sin\theta = 2}$$

Asymptotes in Polar Coordinates

Example(1):

Find the asymptotes of the curve

$$r\sin(\theta) = 2\cos(2\theta) \tag{16}$$

- **Step1**:(16) implies

$$\begin{aligned} r &= \frac{2\cos 2\theta}{\sin\theta} \\ \frac{1}{r} &= \frac{\sin\theta}{2\cos 2\theta}, \end{aligned} \tag{17}$$

Asymptotes in Polar Coordinates

Example(1):

- according to working rules , we can write (17) as

$$u = \frac{\sin\theta}{2\cos 2\theta}. \tag{18}$$

- **Step2:** As $u \to 0$ then $\theta \to n\pi,\ n \in Z$.
- **Step3**: Now we determine $-\lim_{\theta\to n\pi}(\frac{d\theta}{du})$

Asymptotes in Polar Coordinates

Example(1):

- $$\begin{aligned} -\lim_{\theta \to n\pi}\left(\frac{du}{d\theta}\right) &= \lim_{\theta \to n\pi} \frac{\cos 2\theta \cos\theta - \sin\theta(-2\sin 2\theta)}{\cos^2 2\theta} \\ &= \frac{\cos^2(2n\pi)}{\cos(2\pi)\cos(\pi) - \sin(\pi)(-2\sin(2\pi)} \\ &= \frac{2}{\cos(n\pi)} \end{aligned}$$

Asymptotes in Polar Coordinates

Example(1):

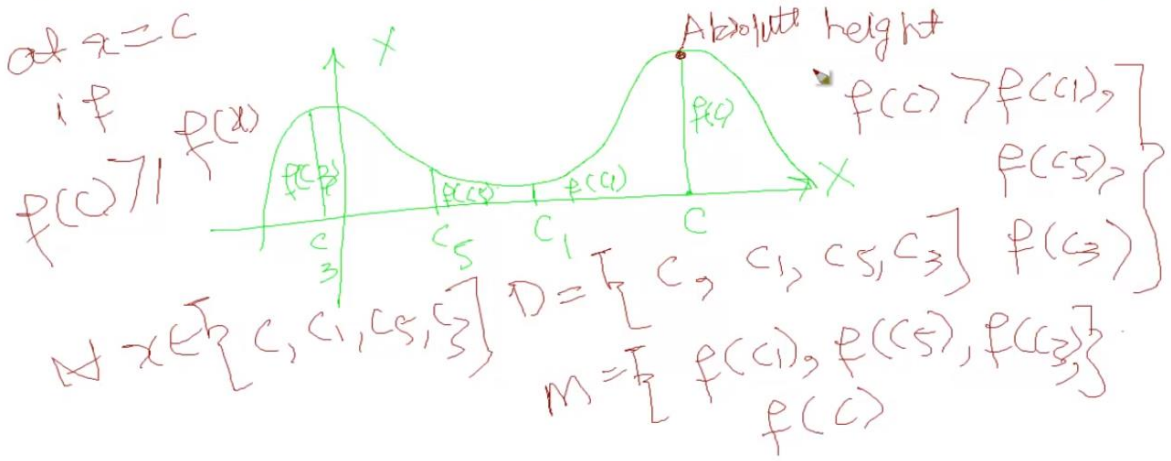
- **Step4:** This limit be p. That is $p = -\frac{2}{(-1)^n}$. Hence, the required asymptotes are

$$\begin{aligned} r\sin(n\pi - \theta) &= -\frac{2}{(-1)^n} \\ r[(-1)^{n-1}\sin\theta] &= \frac{2}{(-1)^n} \\ r\sin(\theta) &= 2. \end{aligned} \tag{20}$$

Lecture 26:

Maxima and Minima

Absolute Maximum:



Maxima and Minima

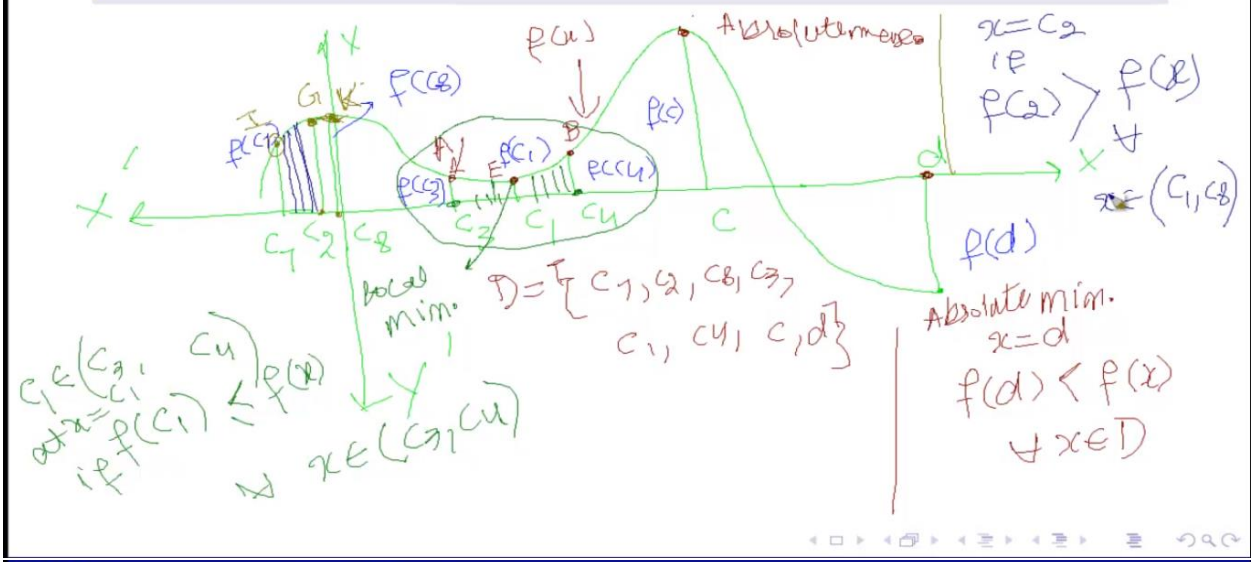
Absolute Maximum:

- A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D.
- Where D is the domain of f.
- The number $f(c)$ is called the maximum value of f on D.



Maxima and Minima

Absolute Minima:



Maxima and Minima

Absolute Minima:

- A function f has an absolute minimum (or global minimum) at d if $f(d) \leq f(x)$ for all x in D.
- Where D is the domain of f.
- The number $f(d)$ is called the minimum value of f on D.



Maxima and Minima

Local Maximum or Minimum:

- A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ when x is near c.
- This means that $f(c) \geq f(x)$ for all x in some open interval containing c.
- Similarly, f has a local minimum (or relative minimum) at d if $f(d) \leq f(x)$ when x is near ~~c~~ d.



Maxima and Minima

Examples:

$f(x) = x^2$; $x \in \mathbb{R}$.

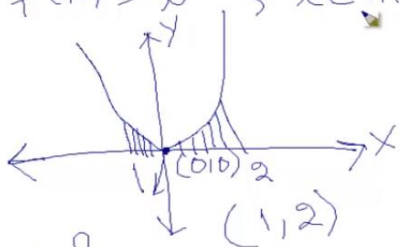
$x \to \infty$ or $x \to -\infty$

$f(x) \to \infty$

Having no Absolute max.

$\forall\, x \in \mathbb{R}$

$\exists\, f(0) \leq f(x)$



$(1,2)$

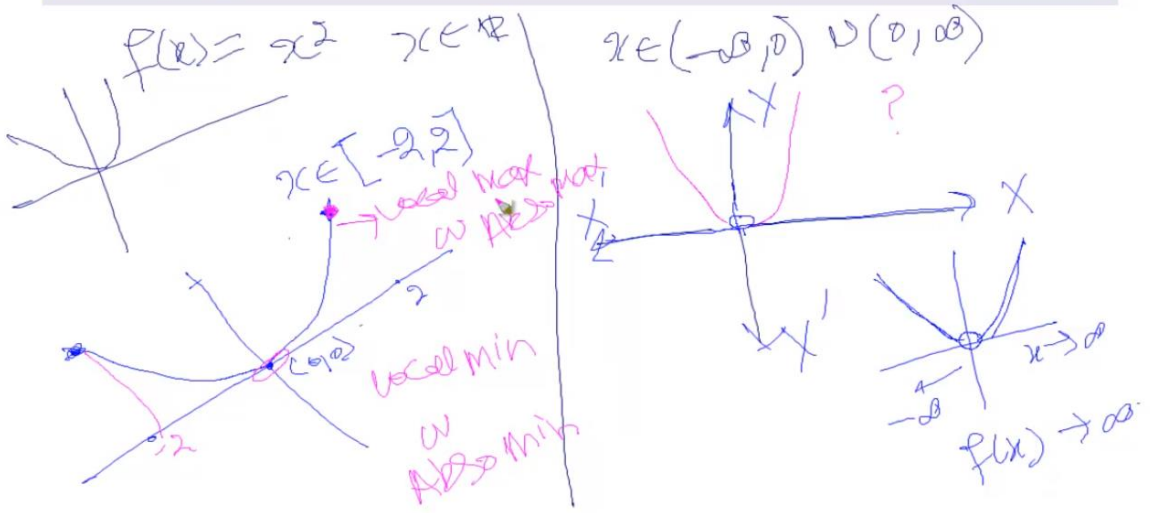
$x = 0$

local min. at $x = 0$

$f(0) \leq f(x)$ $\forall\, x \in (1,2)$ $f(x) = 0$

Maxima and Minima

Examples:

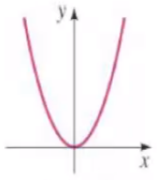


Maxima and Minima

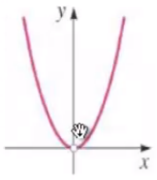
Examples:

Let $f(x) = x^2, \quad x \in \mathbb{R}$

- $f(x)$ has absolute and local minimum at $x = 0$.
- Having no absolute or local maximum.
- The right most diagram showing the function behavior when $x \in (-\infty, 0) \cup (0, \infty)$.



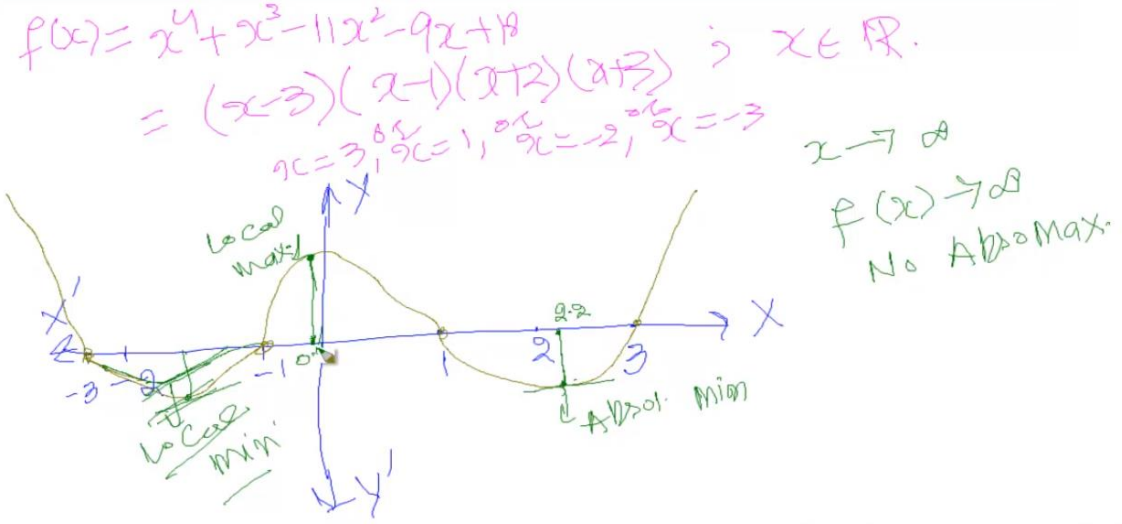
$f(x) = x^2$



$f(x) = x^2, \ x \in (-\infty, 0) \cup (0, \infty)$

Maxima and Minima

Local Examples:



Maxima and Minima

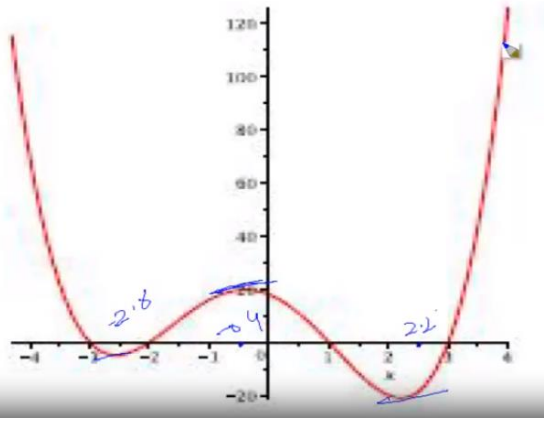
Local Examples:

Let $f(x) = f(x) = x^4 + x^3 - 11x^2 - 9x + 18 =$
$(x-3)(x-1)(x+2)(x+3), \quad x \in \mathbb{R}$

- $f(x)$ has the absolute minimum at $x \approx 2.2$ and has no absolute maximum.
- It has two local minima at $x \approx -2.6$ and $x \approx 2.2$.
- Having local maximum at $x \approx -0.4$.

Maxima and Minima

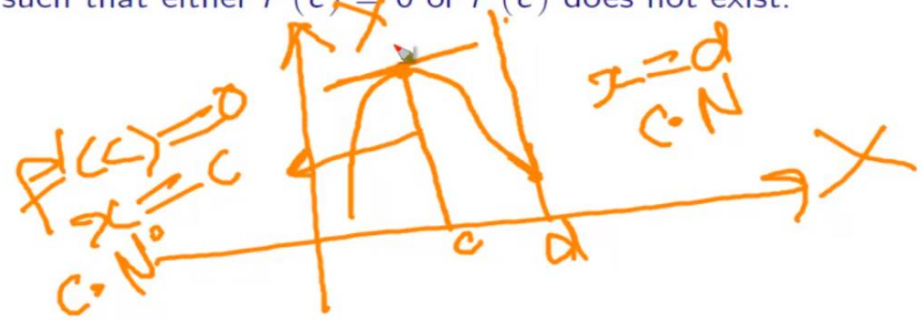
Local Examples:



Lecture 27:

Critical Number

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.



Critical Number

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Remark(3):

From Fermat's Theorem it follows that if f has a local maximum or minimum at c, then c is a critical number of f.

CRITICAL NUMBER

EXAMPLES:

- If $f(x) = 2x^2 + 5x - 1, \quad x \in \mathbb{R}$, then $f'(x) = 4x + 5$. Hence the only critical number of f is $x = -\frac{5}{4}$.

$f(x) = (x^2)^{1/3}$

$f'(x) = \frac{2}{3} x^{-1/3} \Rightarrow$ at $x = 0$

C.N

CRITICAL NUMBER

EXAMPLES:

- If $f(x) = 2x^2 + 5x - 1, \quad x \in \mathbb{R}$, then $f'(x) = 4x + 5$. Hence the only critical number of f is $x = -\frac{5}{4}$.

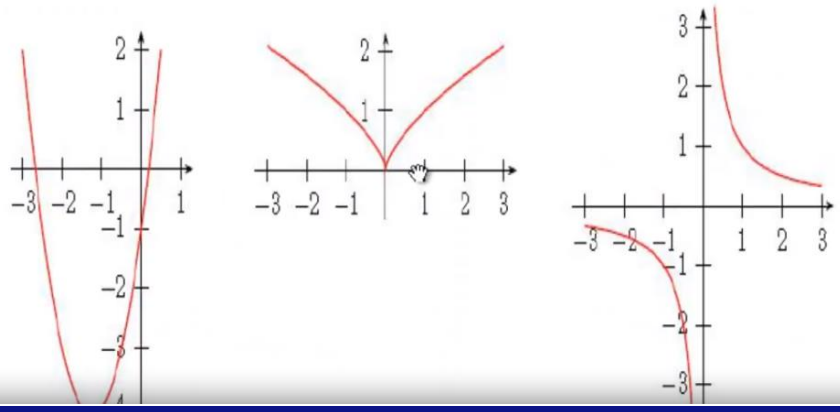
$f(x) = \frac{1}{x}$; $x \in \mathbb{R} - \{0\}$

$f'(x) = -\frac{1}{x^2}$ at $x = 0$

$\neq$ C.N

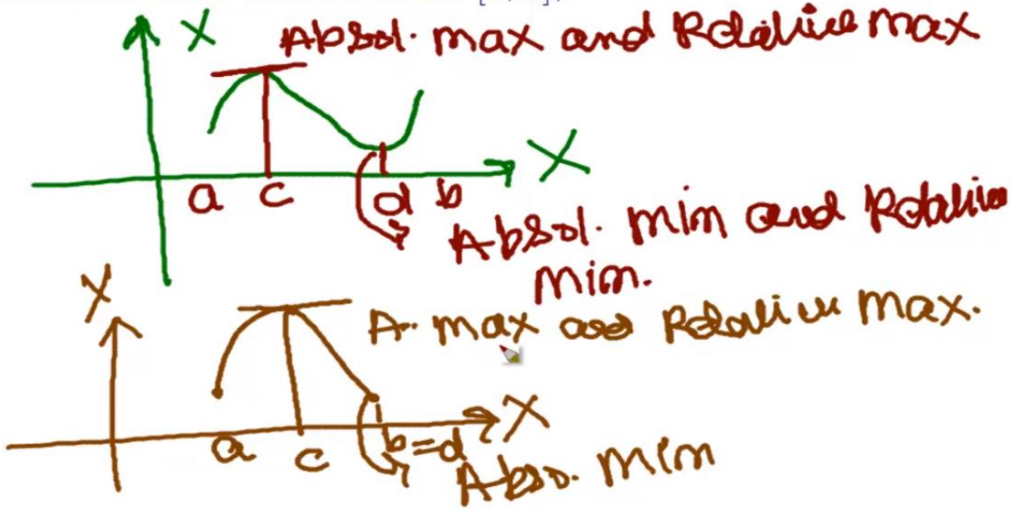
Critical Number

Examples



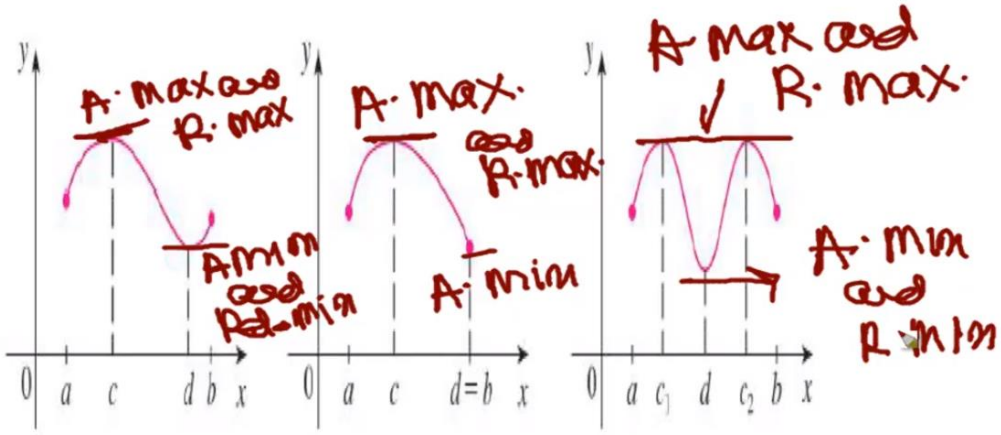
Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$,



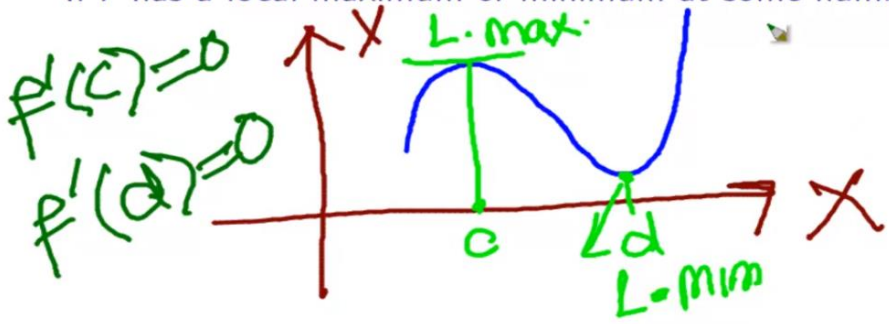
Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$,then f attains an absolute maximum value $f(c)$ at some number c and an absolute minimum value $f(d)$ at some number d.



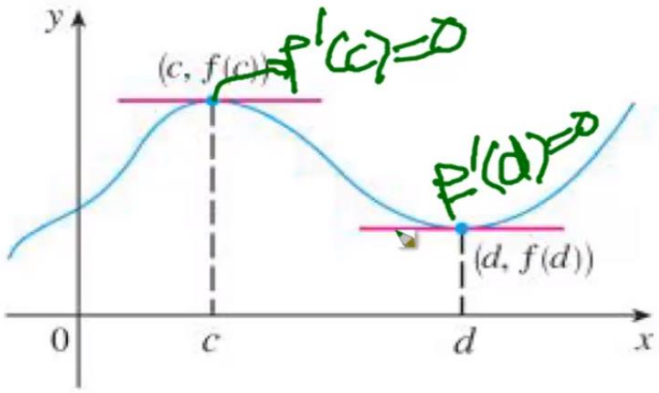
Fermats Theorem

If f has a local maximum or minimum at some number c ,



FERMATS THEOREM

If f has a local maximum or minimum at some number c , and if $f'(c)$ exists, then $f'(c) = 0$.



FERMATS THEOREM

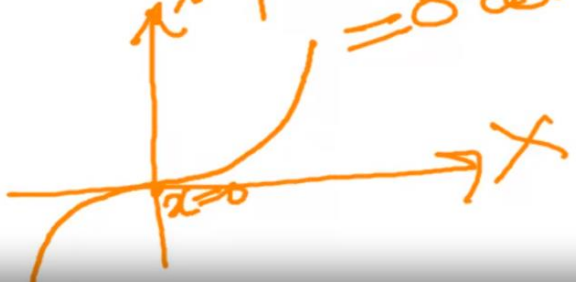
If f has a local maximum or minimum at some number c ,

$f(x) = x^3 ; x \in \mathbb{R}$

(•) Op $f'(x) = 0$

$f'(x) = 3x^2$

$= 0$ at $x = 0$



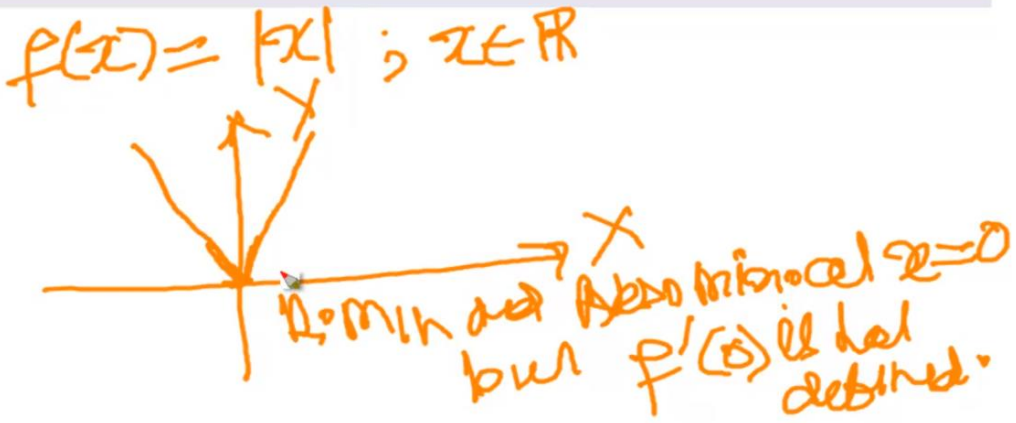
FERMATS THEOREM

REMARK(1):

- The converse of this theorem is not true.
- In other words, when $f'(c) = 0$, f does not necessarily have a local maximum or minimum.
- For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ equals 0 at $x = 0$.

FERMATS THEOREM

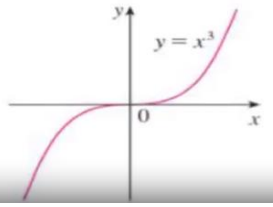
REMARK(1):

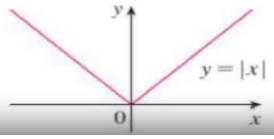


FERMATS THEOREM

REMARK(2):

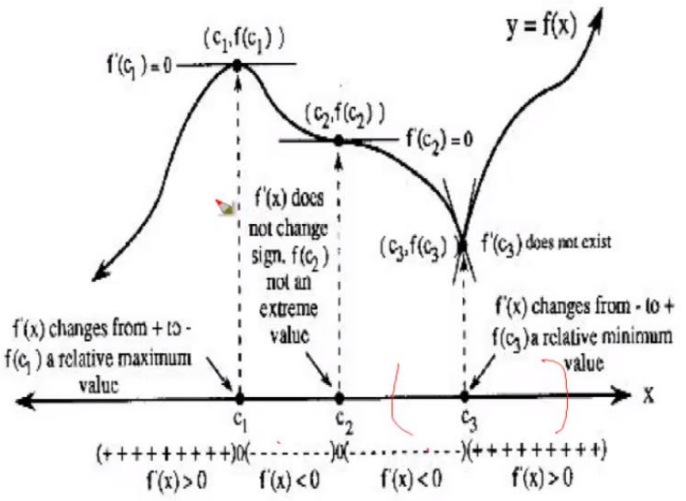
- Sometimes $f'(c)$ does not exist, but some number $f(c)$ is a point of a local maximum or minimum.
- For example, if $f(x) = |x|$, then $f'(0)$ does not exist.
- But $f(x)$ has its local (and absolute) minimum at $x = 0$.





Lecture 28:

Relative Maxima and Minima



Relative Maxima and Minima

First Derivative Teat

- If $f'(x) > 0$ on an open interval extending left from c and $f'(x) < 0$ on an open interval extending right from c, then f has a relative maximum at c.
- If $f'(x) < 0$ on an open interval extending left from c and $f'(x) > 0$ on an open interval extending right from c, then f has a relative minimum at c.
- If $f'(x)$ has the same sign on both an open interval extending left from c and an open interval extending right from c, then f does not have a relative extremum at c.

Relative Maxima and Minima

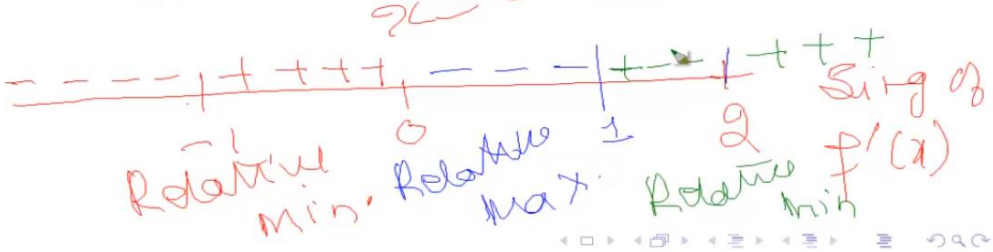
- Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$.

$x \in [-2, 3]$;

$f'(x) = 12x^3 - 12x^2 - 24x$

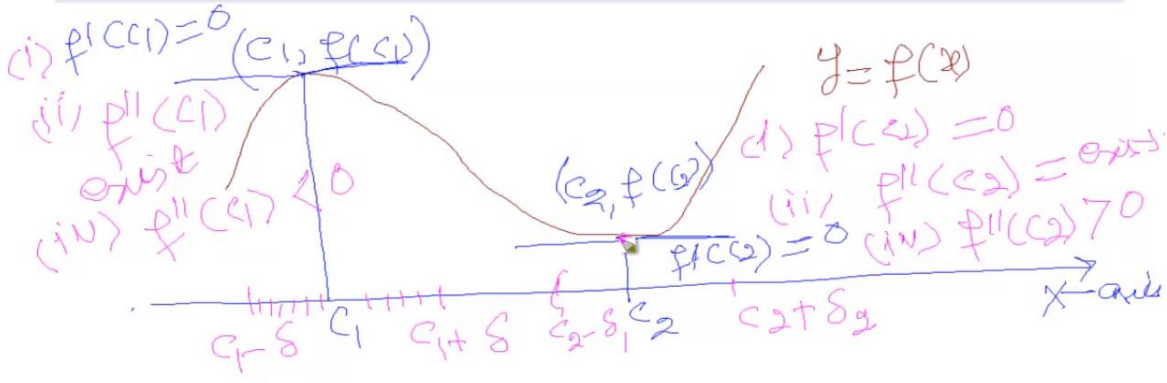
$0 = 12x^3 - 12x^2 - 24x$

$\Rightarrow x = -1, 0$ and $x = 2$



Relative Maxima and Minima

Second Derivative Teat



Relative Maxima and Minima

Second Derivative Teat

- Let c is a critical point at which $f'(c) = 0$.
- Let $f'(x)$ exists in a neighborhood of c.
- Let $f''(c)$ exists.
- Then f has a relative maximum value at c if $f''(c) < 0$,
- and a relative minimum value at c if $f''(c) > 0$.
- If $f''(c) = 0$, the test is not informative.

Relative Maxima and Minima

Second Derivative Teat

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

(i) $0 = 4x^3 - 12x^2$

$x = 0, \quad x = 3$



(ii) $f'(x)$ changes sign at the both of $x = 0$ and $x = 3$

(iii) $f''(x) = 12x^2 - 24x$

$f''(3) = 12(3)^2 - 24(3) > 0$

Relative min.

Relative Maxima and Minima

Second Derivative Teat

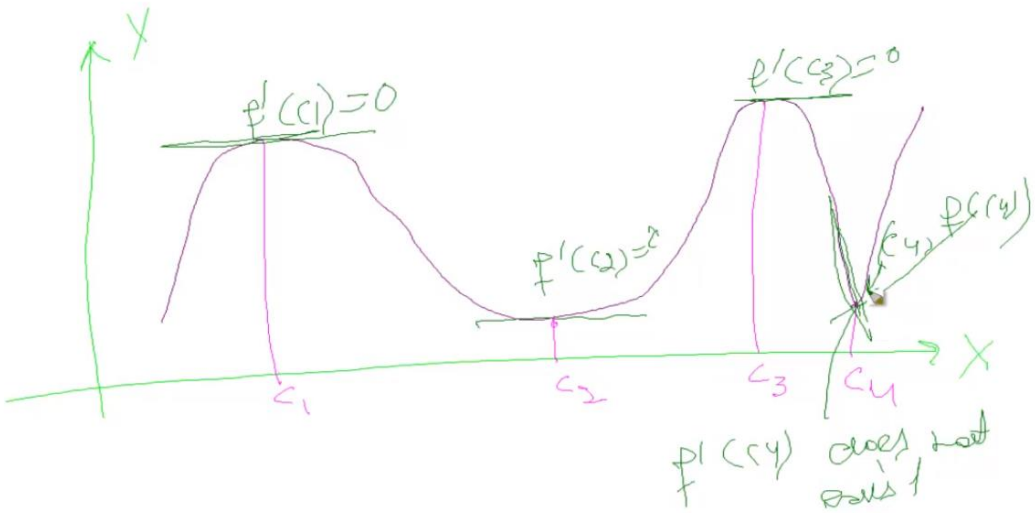
$f''(0) = 0$ $f'(0-\delta) < 0$ $f'(0+\delta) > 0$ $\forall \delta > 0$

$f'(x) = 4x^3 - 12x^2$

sign of $f'(0)$

Lecture 29:

SINGULAR POINTS



SINGULAR POINTS

$$f(x) = (x-1)^{2/3} - 3(x-1) \; ; \quad x \in [0, \infty)$$

$$\Rightarrow f'(x) = \boxed{\frac{2}{3(x-1)^{1/3}} - 3}$$

$x = 1$

$= \infty$

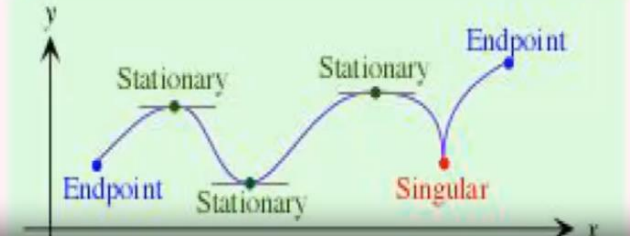
Singular Points

Example

- Discuss the singular points of $f(x) = (x-1)^{\frac{2}{3}} - 3(x-1)$ on $[0, \infty]$.
- Look for values x where $f'(x)$ is not defined, but $f(x)$ is defined.
- $f'(x) = \frac{2}{3(x-1)^{\frac{1}{3}}} - 3.$
- Notice that the denominator is zero when $x = 1$, so that $f'(x)$ is not defined when $x = 1$, even though $f(x)$ is defined when $x = 1$.

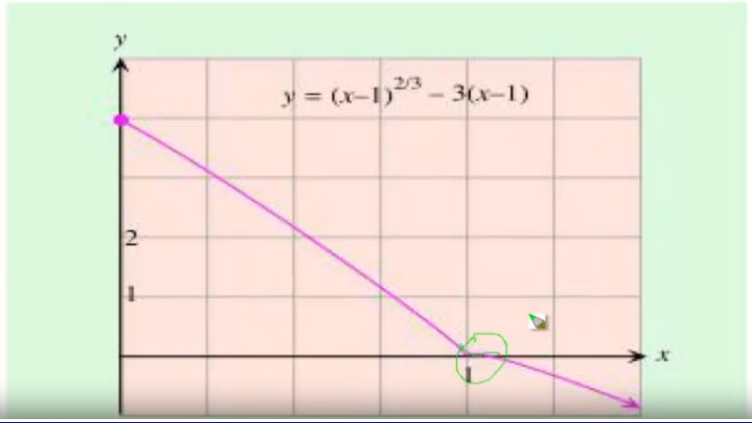
Singular Points

- A point on the curve at which the curve exhibits an extra-ordinary behavior is called a Singular Point.
- Mean to say where $f'(x)$ is not defined but $f(x)$ is defined.
- Points of inflection and multiple points are the types of singular points.



SINGULAR POINTS

EXAMPLE

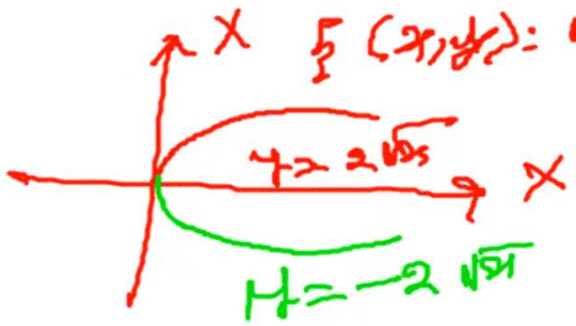


SINGULAR POINTS

DEFINITIONS

$y^2 = 4x$; $y = 2\sqrt{x}$ and $y = -2\sqrt{x}$.

$\{(x,y) : y^2 = 4x\} = \{(x,y) : y = 2\sqrt{x}\} \cup \{(x,y) : y = -2\sqrt{x}\}$



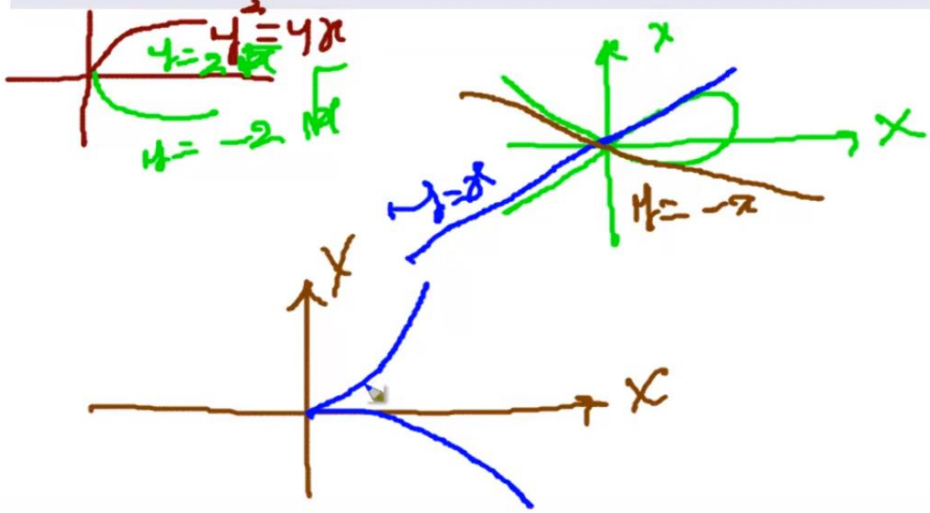
Singular Points

Definitions

- A Point on the curve through three branches of the curve pass is called Triple Point.
- A Point on the curve through which r branch of the curve pass is called Multiple Point of rth order.
- A Double Point P on a curve is called a Node if two real branches of a curve pass through P and two tangents at which are real and different. Thus the point P shown is a Node.

Singular Points

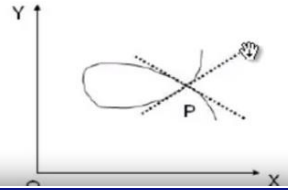
Definitions



Singular Points

Definitions

- A Point on the curve through which more than one branch of the curve pass is called Multiple Point.
- A Point on the curve through which two branches of the curve pass is called Double Point..



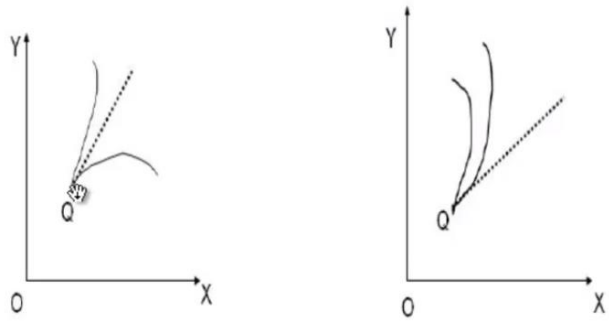
Singular Points

Definitions

- A Point on the curve through three branches of the curve pass is called Triple Point.
- A Point on the curve through which r branch of the curve pass is called Multiple Point of rth order.
- A Double Point P on a curve is called a Node if two real branches of a curve pass through P and two tangents at which are real and different. Thus the point P shown is a Node.
- A Double Point Q on a curve is called a Cusp if two real branches of a curve pass through Q and two tangents at which are real and coincident. Thus the point Q shown in the adjoining two figures is a Cusp.

Singular Points

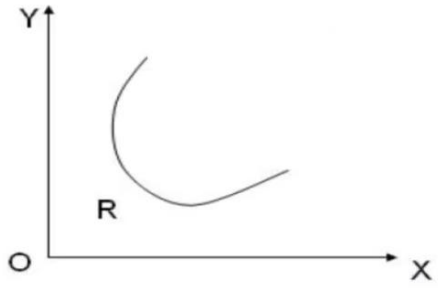
Definitions



Singular Points

Definitions

A Double Point R on a curve is called a Conjugate Point or Isolated point if there exists no real points of the curve in the neighborhood of R.



Lecture 30:

SINGULAR POINTS

WORKING RULE FOR INVESTIGATING THE NATURE OF THE DOUBLE POINT AT THE ORIGIN.

- Find the tangents at the origin by equating to zero the lowest degree terms present in the equation of the curve. If origin is a double point, then we shall get tangents real and imaginary.
- If the two tangents at the origin are imaginary, then the origin is a conjugate point.
- If the two tangents at the origin are real and different, then the tangent is a node or a conjugate point.
- If the two tangents at the origin are real and coincident, then the origin is a cusp or a conjugate point.

SINGULAR POINTS

REMARK.

To study the nature of the curve near origin

- If the tangents at the origin are $y^2 = 0$, solve the equation of the curve for y, neglecting all terms of y having powers above second. If for small non zero values of x, the values of y are real, then the branches of the curve through the origin are also real, otherwise they are imaginary.
- If the tangents at the origin are $x^2 = 0$, solve the given equation of the curve for x instead of y and proceed as mentioned in the above point.
- In other cases, solve for y or x, whichever is convenient.

Singular Points

Example

Determine the nature of the singular point $(0, 0)$ of $f(x, y) = x^3 + y^3 - 3xy$

(1) $x = 0$ and $y = 0$

(2) $y^2 = 3x$

$\Rightarrow y = \sqrt{3x}$ and $y = -\sqrt{3x}$

Singular Points

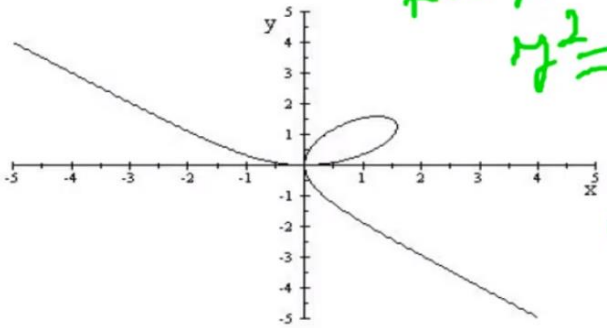
Example

Determine the nature of the singular point $(0, 0)$ of $f(x, y) = x^3 + y^3 - 3xy$

- According to the last discussions, the tangents at the origin are $x = 0$ and $y = 0$
- Hence the origin is either a node or isolated point.
- When x is so small, the equation of the curve is $y^2 = 3x$ which represent two real branches through the origin.
- Hence the origin is a node.

Singular Points

Example



$f(x,y) = x^3 + y^2 - 3xy$

$y^2 = 3x^2$

$y = \sqrt{3}x$

$y = -\sqrt{3}x$

Singular Points

Example

Determine the nature of the singular point $(0,0)$ of $f(x,y) = (x^2+y^2)(2a-x) - b^2x$.

$x = 0$

$\Rightarrow -x^3 - b^2x + 2ax^2 = 0$

$\Rightarrow x^2 - 2ax + b^2 = 0$

$x = a \pm \sqrt{a^2 - b^2}$ $\quad a^2 - b^2 \geq 0$

Singular Points

Example

Determine the nature of the singular point $(0, 0)$ of $f(x, y) = (x^2 + y^2)(2a - x) - b^2x$.

- According to the last discussions, the tangents at the origin are $x = 0$.
- Hence the origin is either a cusp or isolated point.
- To see whether the origin is a cusp or an isolated point. Solve the equation of a curve for x.
- By neglecting y^2, the equation of the curve can also be written as

Singular Points

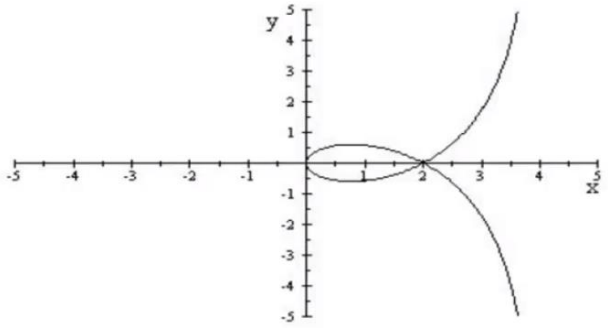
Example

$$-x^3 - b^2x + 2ax^2 = 0$$
$$x^2 - 2ax + b^2 =$$
$$x = \frac{2a \pm \sqrt{4a^2 - 4(1)(b^2)}}{2}$$
$$x = a \pm \sqrt{a^2 - b^2}.$$

Hence origin is a cusp if $a^2 - b^2 > 0$.

Singular Points

Example



Singular Points

Working rules to calculate singular points and its nature using partial derivatives

- If $f(x, y)$ be a curve then its singular points are the simultaneous solutions of $f(x, y) = 0, f_x(x, y) = 0$ and $f_y(x, y) = 0$.
- The values of tangents at singular points are the roots of $f_{yy}(y'(x))^2 + 2f_{xy}y'(x) + f_{xx} = 0$.
- If f_{xx}, f_{xy} and f_{yy} are not all zero then the point (x, y) will be a double point.
- If $(f_{xy})^2 - f_{xx}f_{yy} > 0$ the point (x, y) would be a node.
- If $(f_{xy})^2 - f_{xx}f_{yy} = 0$ the point (x, y) would be a cusp.
- If $(f_{xy})^2 - f_{xx}f_{yy} < 0$ the point (x, y) would be an isolated point.

SINGULAR POINTS

$f(x,y) = y^2 - x(x-a)^2; \quad a > 0$

$f_x = \frac{\partial f}{\partial x} = -(x-a)(3x-a)$ } $x = a$, $x = a/3$

$f_y = \frac{\partial f}{\partial y} = 2y$

$2y = 0 \Rightarrow \boxed{y = 0}$

$\{(a,0) \& (\frac{a}{3},0)\}$

SINGULAR POINTS

$[f_{xy}(a,0)]^2 - f_{xx}(a,0) f_{yy}(a,0) = 0 \quad > 0$

$f_y = 2y$

$\boxed{f_{ay} = 0}$

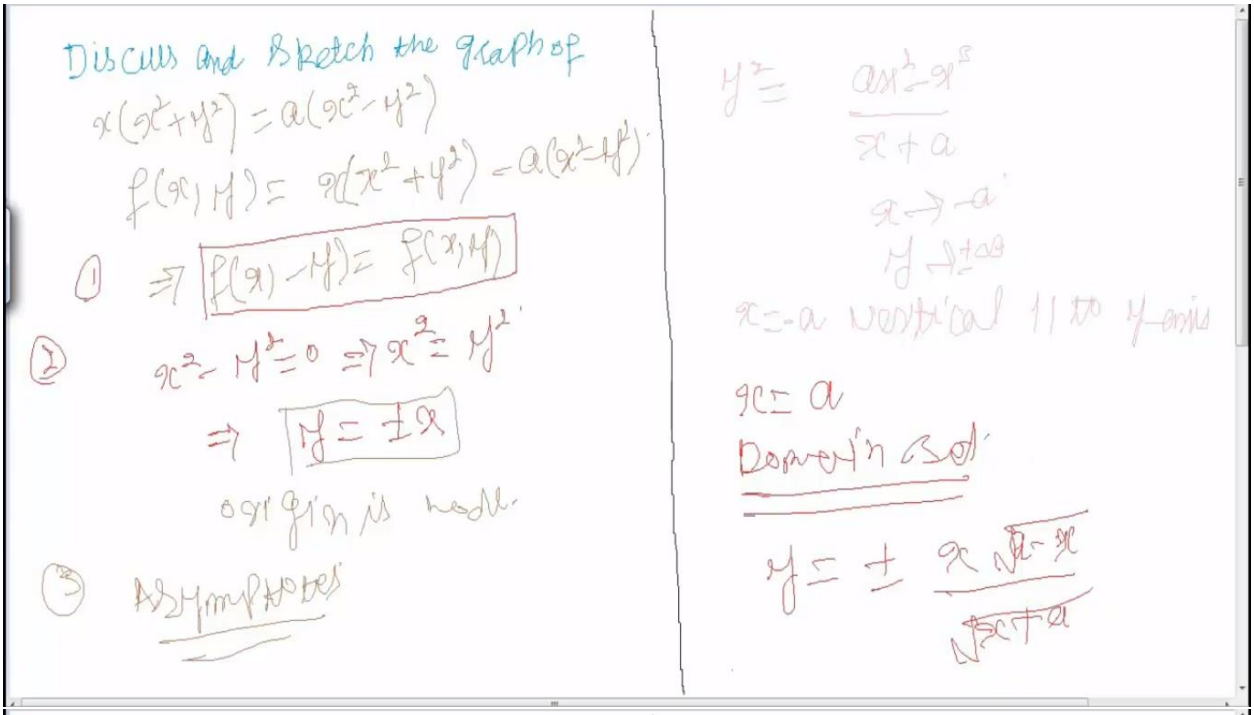
$f_x = -(x-a)(3x-a)$ < 0

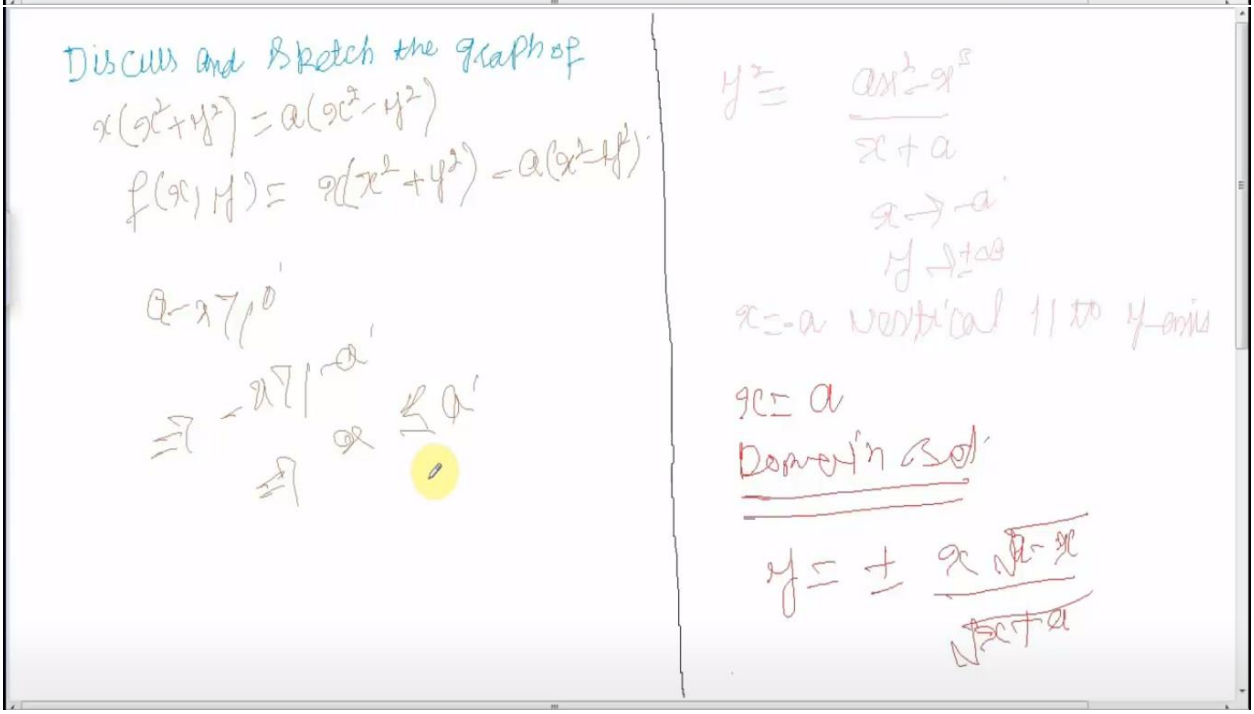
$\Rightarrow 0 - (-2a)^2$ $f_{xx} = -(3x-a) - 3(x-a)$

$= 4a^2 \Rightarrow$ $f_{yy} = 2$ $(a,0)$ Node

$4a^2 > 0 \Rightarrow$

Lecture 31:

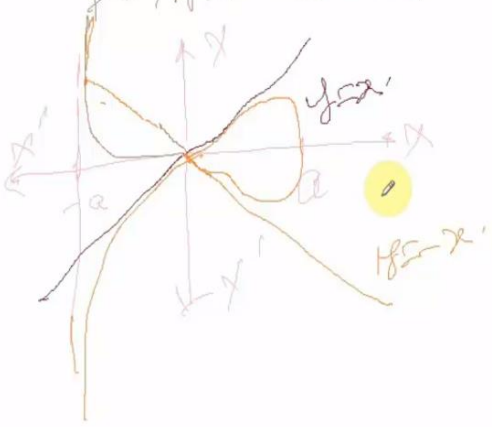

Discuss and Sketch the graph of
$x(x^2+y^2) = a(x^2-y^2)$
$f(x,y) = x(x^2+y^2) - a(x^2-y^2)$
(1) $\Rightarrow f(x,-y) = f(x,y)$
(2) $x^2 - y^2 = 0 \Rightarrow x^2 = y^2$
$\Rightarrow y = \pm x$
origin is node.
(3) Asymptotes
$y^2 = \frac{ax^2 - x^3}{x+a}$
$x \to -a$
$y \to \pm\infty$
$x = -a$ vertical || to y-axis
$x = a$
Domain's Sol.
$y = \pm \frac{x\sqrt{a-x}}{\sqrt{x+a}}$


Discuss and Sketch the graph of
$x(x^2+y^2) = a(x^2-y^2)$
$f(x,y) = x(x^2+y^2) - a(x^2-y^2)$
$a-x \neq 0$
$\Rightarrow -x \neq -a$
$\Rightarrow x \leq a$
$y^2 = \frac{ax^2 - x^3}{x+a}$
$x \to -a$
$y \to \pm\infty$
$x = -a$ vertical || to y-axis
$x = a$
Domain's Sol.
$y = \pm \frac{x\sqrt{a-x}}{\sqrt{x+a}}$

Discuss and Sketch the graph of

$x(x^2+y^2) = a(x^2-y^2)$

$f(x,y) = x(x^2+y^2) - a(x^2-y^2)$



$y^2 = \frac{ax^2-x^3}{x+a}$

$x \to -a$

$y \to \pm\infty$

$x=-a$ vertical $\parallel$ to y-axis

$x=a$

Domain :-

$y = \pm \frac{x\sqrt{a-x}}{\sqrt{x+a}}$

Discuss and Sketch the graph of

$y^2 = x^2(4-x^2)$

$f(x,y) = y^2 - x^2(4-x^2)$

Symmetrically

$f(x,-y) = f(x,y)$

$f(-x,y) = f(x,y)$

$y^2 - 4x^2 = 0$

$y = \pm 2x$

$\Rightarrow$

(3) x has not asymptote.

(4) $0 = x^2(4-x^2)$

$x=0, \; x=\pm 2$

$y=0$

(5) $y = \pm x\sqrt{(4-x^2)}$

$4-x^2 \geq 0$

$\Rightarrow -x^2 \geq -4$

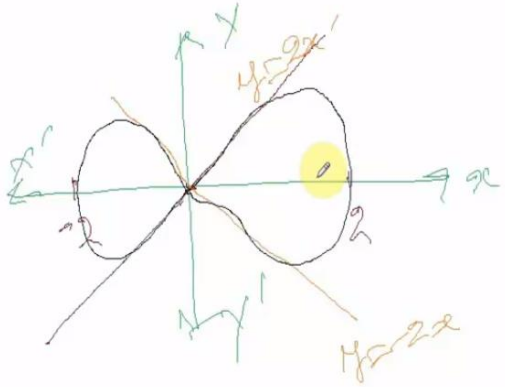
$\Rightarrow x^2 \leq 4$

$\{-2, 2\}$

Discuss and Sketch the graph of

$y^2 = x^2(4-x^2)$

$\Rightarrow f(x,y) = y^2 - x^2(4-x^2)$



(3) y has no asymptote.

(4) $0 = x^2(4-x^2)$

$x=0, \; x=\pm 2$

$y=0$

(5) $y = \pm x\sqrt{4-x^2}$

$4-x^2 \geq 0$

$\Rightarrow -x^2 \geq -4$

$\Rightarrow x^2 \leq 4$

$[-2, 2]$

Discuss and Sketch the graph of

$3ay^2 = x^2(x-a)$

$f(x,y) = 3ay^2 - x^2(x-a)$

(1) $\Rightarrow f(x,-y) = f(x,y)$

(2) $x^2 + 3y^2 = 0$

$\Rightarrow y^2 = -\frac{x^2}{3}$

Isolated point.

(3) Asymptotes: No

$0 = x^2(x-a)$

$\Rightarrow x=0$ or $x=a$

Domain:

$y^2 = \frac{x^2(x-a)}{3a}$

$\Rightarrow y = \pm x\sqrt{\frac{x-a}{3a}}$

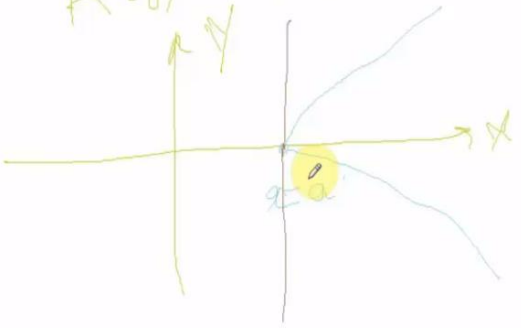
$x-a \geq 0$

$\Rightarrow x \geq a$

Discuss and Sketch the graph of

$3ay^2 = x^2(x-a)$

$f(x,y) = 3ay^2 - x^2(x-a)$



$0 = x^2(x-a)$

$\Rightarrow x = 0$ or $x = a$

Domain:

$y^2 = \frac{x^2(x-a)}{3a}$

$\Rightarrow y = \pm x\sqrt{\frac{(x-a)}{3a}}$

$x(x-a) \geq 0$

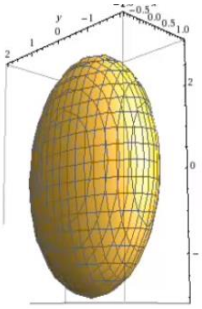
$\Rightarrow x \geq a$

Lecture 32:

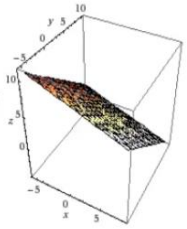
Surface: (x, y, z)

$f(x, y, z) = 0$

$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ is ellipsoid



$x + 2y + 3z = 6$ is plane →

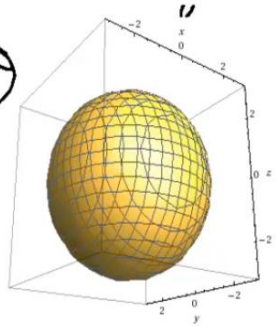


Quadratic Surface: A quadratic surface is defined as the set of all points which satisfy second degree eq.

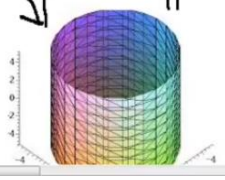
$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

where A, B, C, D, E and F are not all zero.

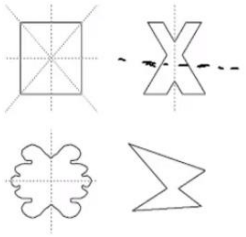
Sphere $x^2 + y^2 + z^2 = 9$ →



Cylinder $x^2 = 4y$



Properties of Surfaces



(1) Symmetry

If $f(x,y,z)=0 \Rightarrow f(x,y,-z)=0$

xy-plane

$x+y+z^2=3$ $\qquad$ $x+y+(-z)^2=3$

$\Rightarrow x+y+z^2=3$

If $f(x,y,z)=0 \Rightarrow f(-x,y,z)=0$

yz-plane

Similarly

If $f(x,y,z)=0 \Rightarrow f(x,-y,z)=0$

$\Rightarrow$ xz-plane

If $f(x,y,z)=0 \Rightarrow f(x,-y,-z)=0 \Rightarrow$ x-axis

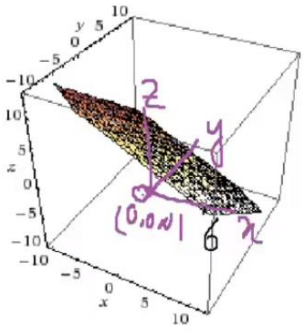
$f(x,y,z)=0 \Rightarrow f(-x,y,-z)=0 \Rightarrow$ y-axis

$f(x,y,z)=0 \Rightarrow f(-x,-y,z)=0 \Rightarrow$ z-axis

If $f(x,y,z)=0 \Rightarrow f(-x,-y,-z)=0$.

$\Rightarrow$ Symmetric about origin

Intercepts



x-intercept.
y-intercept
z-intercept.

$2x+3y+4z=12$

$y=0, z=0$

$2x+3(0)+4(0)=12 \Rightarrow 2x=12 \Rightarrow x=6$

$x=0, z=0$

$2(0)+3y+4(0)=12 \Rightarrow 3y=12 \Rightarrow y=4$

$x=0, y=0$

$2(0)+3(0)+4z=12 \Rightarrow z=4$

Traces :

$x^2 + y + z^2 = 25$

$z = 3$

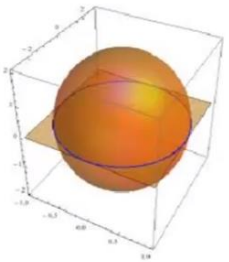
$x^2 + y^2 + 3^2 = 25$

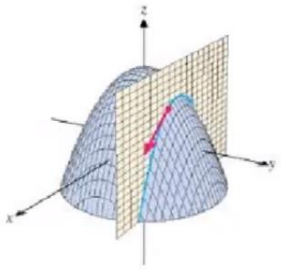
$x^2 + y^2 + 9 = 25$

$x^2 + y^2 = 25 - 9$

$x^2 + y^2 = 16$

circle



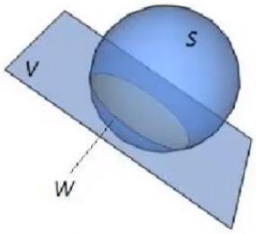


Sections :

eq. of surface

eq. of plane

(1)

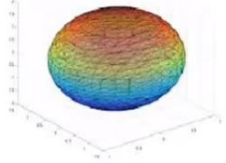


Boundedness :

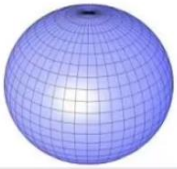
finite → bounded

infinite → unbounded

(2)



(3)

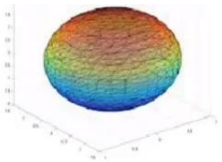


Boundaries:

finite → bounded

infinite → unbounded

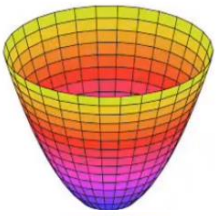
(2)



(3)



(4)

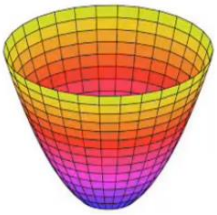


infinite → unbounded

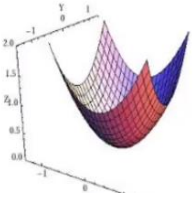
(3)



(4)



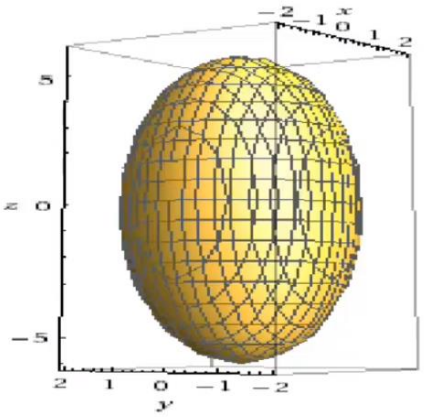
(5)



Lecture 33:

Example. Discuss the surface

$S: f(x,y,z) = x^2 + y^2 + \frac{z^2}{9} - 4 = 0$



Sol.

Symmetry.

(i) **about xy-plane**

$$\text{If } f(x,y,-z) = f(x,y,z) = 0$$

$$f(x,y,-z) = (x)^2 + (y)^2 + \frac{(-z)^2}{9} - 4 = 0$$

$$= x^2 + y^2 + \frac{z^2}{9} - 4 = 0 = f(x,y,z)$$

$\Rightarrow$ Symmetrical about xy-plane

(ii) **about yz-plane**

$$f(-x,y,z) = (-x)^2 + (y)^2 + \frac{z^2}{9} - 4 = x^2 + y^2 + \frac{z^2}{9} - 4$$

$$= f(x,y,z) = 0$$

$\Rightarrow$ Symmetrical about yz-plane.

(iii) **about xz-plane**

$$f(x,-y,z) = x^2 + (-y)^2 + \frac{z^2}{9} - 4 = x^2 + y^2 + \frac{z^2}{9} - 4 = 0$$

$$= f(x,y,z)$$

Symmetrical about xz-plane

(iv) **about x-axis**

$$f(x,-y,-z) = f(x,y,z) = 0$$

$\Rightarrow$ Symmetry

(v) about y-axis

$f(x,y,z) = f(x,y,z) = 0$

$f(-x,y,-z) = (-x)^2 + (y)^2 + \frac{(-z)^2}{9} - 4$

$= x^2 + y^2 + \frac{z^2}{9} - 4 = 0$

$= f(x,y,z)$

Symmetrical about y-axis

(vi) Symmetrical about z axis $f(-x,-y,z) = f(x,y,z) = 0$

(v) Symmetry about origin

if $f(-x,-y,-z) = 0 \Rightarrow f(x,y,z) = 0$

Traces:

i) yz-plane

put $x = 0$ in

$x^2 + y^2 + \frac{z^2}{9} - 4 = 0$

$0^2 + y^2 + \frac{z^2}{9} - 4 = 0 \Rightarrow y^2 + \frac{z^2}{9} - 4 = 0$

$\Rightarrow y^2 + \frac{z^2}{9} = 4 \Rightarrow \frac{y^2}{4} + \frac{z^2}{36} = 1 \Rightarrow$ ellipse

(ii) xy-plane

put $z=0$

$x^2+y^2-4=0 \Rightarrow x^2+y^2=4=2^2 \Rightarrow$ circle

(iii) xz-plane

put $y=0$

$x^2+\frac{z^2}{9}-4=0 \Rightarrow x^2+\frac{z^2}{9}=4$

$\Rightarrow \frac{x^2}{4}+\frac{z^2}{36}=1 \Rightarrow$ ellipse

Intercepts

x-intercept put $y=0, z=0$

$x^2-4=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

$(2,0,0), (-2,0,0)$

y-intercept $x=0, z=0$

$y^2-4=0 \Rightarrow y^2=4 \Rightarrow y=\pm 2$

$(0,2,0)$ and $(0,-2,0)$

z-intercept $y=0, x=0$

$\frac{z^2}{9}-4=0 \Rightarrow \frac{z^2}{9}=4 \Rightarrow z^2=36 \Rightarrow z=\pm 6$

$(0,0,6)$ and $(0,0,-6)$

Example Find the intercepts of the surface

$$x^2+4y^2+5xz-2x+y-3=0$$

Sol.

x-intercept $y=0, z=0$

$$x^2+4(0)^2+5x(0)-2x+0-3=0$$
$$x^2-2x-3=0$$
$$x^2-3x+x-3=0$$
$$x(x-3)+1(x-3)=0$$
$$(x-3)(x+1)=0$$
$$\Rightarrow x-3=0 \quad \text{or} \quad x+1=0$$
$$\Rightarrow x=3 \quad \text{or} \quad x=-1$$

$(3,0,0)$ and $(-1,0,0)$

y-intercept put $x=0, z=0$

$$(0)^2+4y^2+5(0)(z)-2(0)+y-3=0$$
$$4y^2+y-3=0$$
$$4y^2+4y-3y-3=0$$
$$4y(y+1)-3(y+1)=0$$
$$(y+1)(4y-3)=0$$
$$y+1=0, \quad 4y-3=0$$
$$y=-1 \qquad y=3/4$$

$(0,-1,0)$ and $(0,3/4,0)$

z-intercept

$x=0,\ y=0$

$0^2+4(0)^2+5(0)(z)-2(0)+0-3=0$

$-3=0$

which is a false eq.

Example Find the trace of the surface $x^2+y^2+2z^2-3x+5y-8z=0$ in (i) xy-plane (ii) yz-plane (iii) xz-plane

Sol. (i) xy-plane

put $z=0$

$x^2+y^2+2(0)^2-3x+5y-8(0)=0$

$x^2+y^2-3x+5y=0 \Rightarrow$ eq. of circle

(ii) yz-plane put $x=0$

$(0)^2+y^2+2z^2-3(0)+5y-8z=0$

$y^2+2z^2+5y-8z=0 \Rightarrow$ ellipse

(iii) xz-plane

put $y=0$

$x^2+(0)^2+2z^2-3x+5(0)-8z=0$

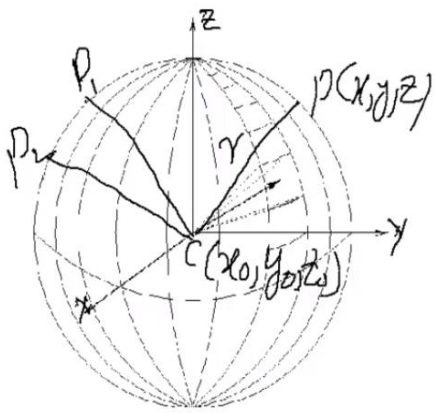
$x^2+2z^2-3x-8z=0$

$\Rightarrow$ ellipse

Lecture 34:

Sphere:

Sphere is the set of all points in the space that are equidistant from a fixed point. The fixed point is called the Centre of the sphere and the constant distance from the points of the sphere to the Centre of the circle is called as the radius of the sphere.



$$\overline{CP} = r$$

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2$$

eq. of Sphere

$C(x_0,y_0,z_0) = C(0,0,0)$, radius $= r$

$$(x-0)^2+(y-0)^2+(z-0)^2 = r^2$$

$$x^2+y^2+z^2 = r^2$$

eq. of sphere $C(0,0,0)$, radius r

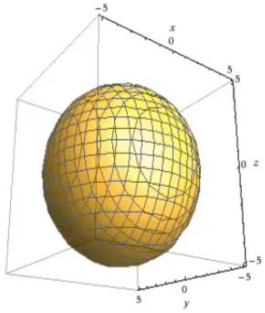
Properties of Sphere

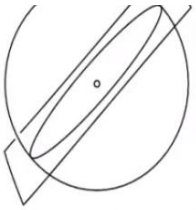
$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2$$

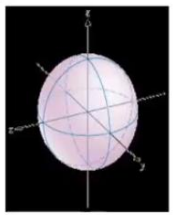
Centre (x_0,y_0,z_0), radius $= r$

centre $(0,0,0)$,

$$x^2+y^2+z^2 = r^2$$







(i) Intercepts

$$x^2 + y^2 + z^2 = 25$$

x-intercept $y=0, z=0$

$$x^2 + 0^2 + 0^2 = 25$$

$$x^2 = 25$$

$$x = \pm 5$$

$(\pm 5, 0, 0)$

y-intercept $x=0, z=0$

z-intercept $x=0, y=0$

z-intercept $x=0, y=0$

Sections.

Traces. $x^2 + y^2 + z^2 = 25$

$x=0$, or $y=0$, $z=0$

$x=0$ $y^2 + z^2 = 25 \rightarrow$ circle,

$y=0$ $x^2 + z^2 = 25 \rightarrow$ circle

$z=0$ $x^2 + y^2 = 25 \rightarrow$ Circle

$x=0$, or $y=0$, $z=0$

$x=0$ $y^2+z^2=25 \rightarrow$ circle,

$y=0$ $x^2+z^2=25 \rightarrow$ circle

$z=0$ $x^2+y^2=25 \rightarrow$ circle

Boundness

Sphere is a bounded surface

Symmetry

$x^2+y^2+z^2=25$

$(x,y,z) \rightarrow (x,y,-z)$

$x^2+y^2+(-z)^2=25 \Rightarrow x^2+y^2+z^2=25 \rightarrow$ Symmetrical about xy-plane

xz, yz origin

Example:- Find the eq. of Sphere with centre (2,1,3) and radius 4.

Sol:-

$C(x_0,y_0,z_0)=C(2,1,3)$, radius $=r=4$

$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$

$(x-2)^2+(y-1)^2+(z-3)^2=4^2$

$x^2-2(x)(2)+2^2+y^2-2y(1)+1^2+z^2-2(z)(3)+3^2=4^2$

$x^2-4x+4+y^2-2y+1+z^2-6z+9=16$

$x^2+y^2+z^2-4x-2y-6z+4+1+9-16=0$

$x^2+y^2+z^2-4x-2y-6z-2=0$

Example:- Find the centre and radius of the sphere

$$x^2+y^2+z^2+6x+4y-10z-11=0$$

Sol

$$x^2+6x+y^2+4y+z^2-10z=11$$

$$x^2+2(x)(3)+3^2+y^2+2(y)(2)+2^2+z^2-2(z)(5)+5^2=11+3^2+2^2+5^2$$

$$(x+3)^2+(y+2)^2+(z-5)^2=11+9+4+25$$

$$(x-(-3))^2+(y-(-2))^2+(z-5)^2=49=7^2$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

$$C(x_0,y_0,z_0)=C(-3,-2,5)$$

$$\text{radius}=r=7$$

Example:- Find the eq. of sphere with centre $(2,3,-1)$ and tangent to the plane $3x-2y+6z=1$

Sol:

Distance of any pt. $P(x_0,y_0,z_0)$ from a plane $ax+by+cz=d$ is given by.

$$\text{distance}=\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$$

$$\text{distance}=\frac{|3(2)-2(3)+6(-1)-1|}{\sqrt{3^2+(-2)^2+6^2}}$$

$$\text{radius} = \frac{|6-6-6-1|}{\sqrt{9+4+36}}$$

$$\text{radius} = \frac{|-7|}{\sqrt{49}} = \frac{7}{7} = 1$$

$r = 1$, $C(2,3,-1)$

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$$

$$(x-2)^2 + (y-3)^2 + (z-(-1))^2 = 1^2$$

$$x^2 - 4x + 4 + y^2 - 6y + 9 + z^2 + 2z + 1 = 1$$

$$x^2 + y^2 + z^2 - 4x - 6y + 2z + 14 - 1 = 0$$

$$x^2 + y^2 + z^2 - 4x - 6y + 2z + 13 =$$

How to recognize Sphere:

Coefficients x^2, y^2, z^2

xy, xz, yz

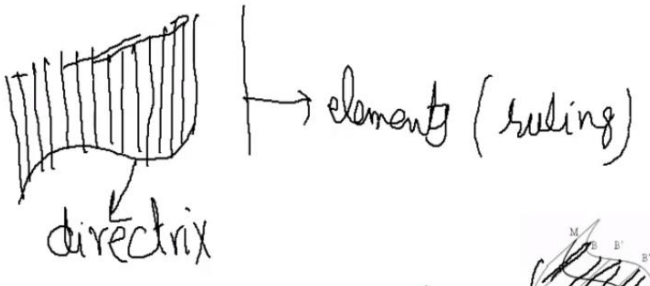
$x^2 + y^2 + z^2 = 16 \longrightarrow$ Sphere

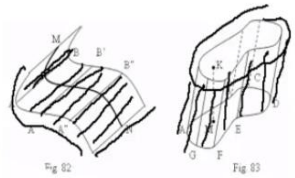
$x^2 + y^2 + z^2 - 5x - 7y = 35 \longrightarrow$ Sphere

$2x^2 + y^2 - z^2 = 42 \longrightarrow$ Not Sphere

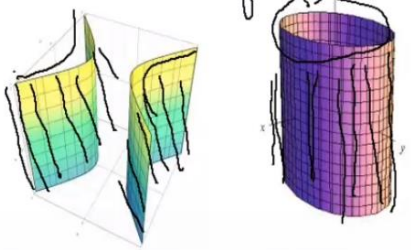
Lecture 35:

Cylinder:

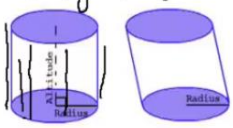




Right cylinder



Right cylinder

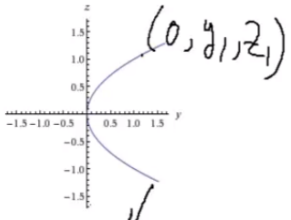


Example:

Find an equation of the cylinder with directrix $C: y - z^2 = 0$ and having elements parallel to $\vec{n} = [1 \;\; 2 \;\; 3]$.

Sol

$$\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$x = t, \quad y = y_1 + 2t, \quad z = z_1 + 3t$

$x = t, \quad y = y_1 + 2t, \quad z = z_1 + 3t$

$y = y_1 + 2x \Rightarrow y_1 = y - 2x$

$z = z_1 + 3x$

$z_1 = z - 3x$

$$y - z^2 = 0$$

$$y_1 - z_1^2 = 0$$

$$y - 2x - (z - 3x)^2 = 0$$

or $y - 2x = (z - 3x)^2$

trace of the cylinder on $x = d$.

$$y - 2d = (z - 3d)^2$$

Example:

Find an equation of the right cylinder whose directrix is the circle with centre(5,3,0) and the radius 4.

Sol.

Eq. of directrix.

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

$$C(a,b,c) = C(5,3,0), \quad r = 4$$

$$(x-5)^2 + (y-3)^2 + (z-0)^2 = 4^2$$

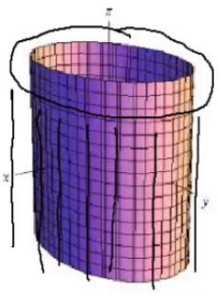
$$x^2 - 10x + 25 + y^2 - 6y + 9 + z^2 = 4^2$$

$$x^2 + y^2 + z^2 - 10x - 6y + 34 = 16$$

$$x^2 + y^2 + z^2 - 10x - 6y + 18 = 0$$

which is the eq. of cylinder

Elliptic cylinder:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{x^2}{a^2}+\frac{z^2}{b^2}=1$$

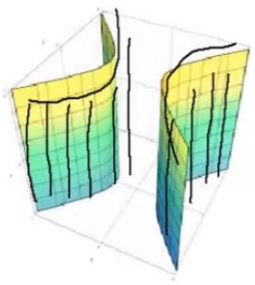
$$\frac{y^2}{a^2}+\frac{z^2}{b^2}=1$$

Example Discuss the surface $8x^2+15y^2-5=0$

Sol

$$8x^2+15y^2-5=0$$
$$8x^2+15y^2=5$$
$$\frac{8x^2}{5}+3y^2=1$$
$$\frac{x^2}{5/8}+\frac{y^2}{1/3}=1 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Hyperbolic Cylinder



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$$

$$\frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$$

Example. Discuss the surface $5x^2 - 12y^2 - 7 = 0$

Sol.

$$5x^2 - 12y^2 - 7 = 0$$

$$5x^2 - 12y^2 = 7$$

$$\frac{5x^2}{7} - \frac{12y^2}{7} = 1$$

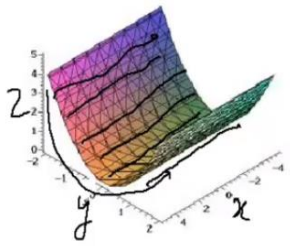
$$\frac{x^2}{7/5} - \frac{y^2}{7/12} = 1 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parabolic Cylinder

$$y = cz^2$$

$$y = cx^2$$

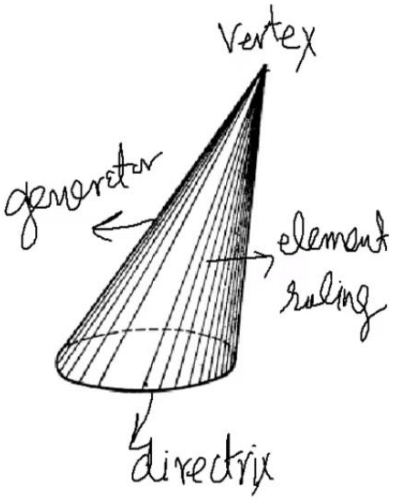
$$x = cz^2$$



Lecture 36:

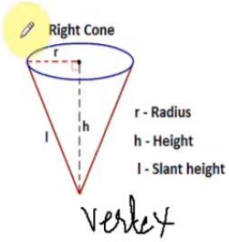
Cone

A surface generated by a straight line passing through a fixed point and intersecting a fixed plane curve not containing the fixed point is called a cone. The given curve is called the directrix and the fixed point is called called the vertex of the cone. The straight line is called the generator of and any particular position of the generator is called an element or ruling of the cone.



Right cone

A right cone is a cone with its verte above the centre of its base.



Right circular cone

A circular cone is right circular if the centres of its sections form a line perpendicular to its base.



Example: Find an eq. of cone whose directrix is

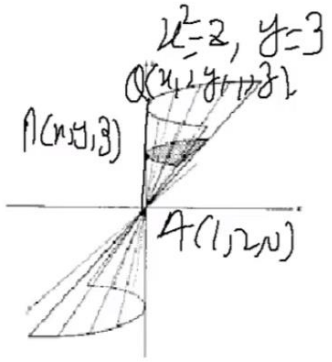
$x^2 = z, \; y = 3$ ——— (1)

and whose vertex is A(1,2,0).

Sol

$x_1^2 = z_1, \; y_1 = 3$ ——— (2)

$$\frac{x-1}{x_1-1} = \frac{y-2}{y_1-2} = \frac{z-0}{z_1-0} \quad \text{——— (3)}$$



$$\frac{x-1}{x_1-1} = \frac{y-2}{y_1-2} \Rightarrow \frac{x-1}{x_1-1} = \frac{y-2}{3-2} = y-2$$

$$x-1 = (x_1-1)(y-2)$$

$$\Rightarrow x_1 - 1 = \frac{x-1}{y-2} \Rightarrow x_1 = \frac{x-1}{y-2} + 1 = \frac{x-1+y-2}{y-2}$$

$$= \frac{x+y-3}{y-2}$$

$$\Rightarrow x_1 = \frac{x+y-3}{y-2}$$

$$\frac{z}{z_1} = \frac{y-2}{y_1-2} \Rightarrow \frac{z}{z_1} = y-2$$

$$\Rightarrow z_1 = \frac{z}{y-2}$$

$$\left(\frac{x+y-3}{y-2}\right)^2 = \frac{z}{y-2}$$

$$(x+y-3)^2 = \left(\frac{z}{y-2}\right)(y-2)^2$$

$$(x+y-3) = z(y-2)$$

$$x^2+y^2+2xy+9-6x-6y = zy-2z$$

$$x^2+y^2+2xy-6x-6y+2z-yz+9=0$$

$$\frac{z}{z_1} = \frac{y-2}{y_1-2} \Rightarrow \frac{z}{z_1} = y-2$$

$$\Rightarrow z_1 = \frac{z}{y-2}$$

$$\left(\frac{x+y-3}{y-2}\right)^2 = \frac{z}{y-2}$$

$$(x+y-3)^2 = \left(\frac{z}{y-2}\right)(y-2)^2$$

$$(x+y-3) = z(y-2)$$

$$x^2+y^2+2xy+9-6x-6y = 2yz-2z$$

$$x^2+y^2+2xy-6x-6y+2z-yz+9=0$$ which is eq. of Cone.

x-intercept $y=0, z=0$

$$x^2-6x+9=0 \Rightarrow (x-3)^2=0 \Rightarrow x=3$$

y-intercept $x=0, z=0$

$$y^2-6y+9=0 \Rightarrow (y-3)^2=0 \Rightarrow y=3$$

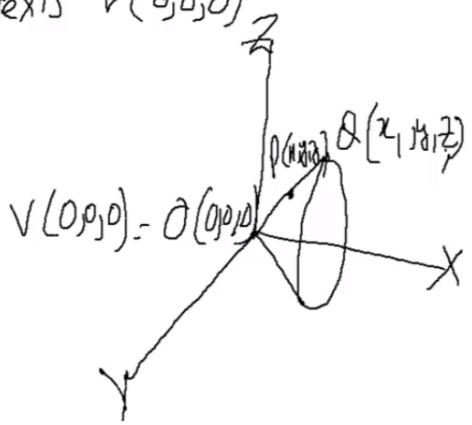
z-intercept $x=0, y=0$

$$2z+9=0 \Rightarrow z=-9/2$$

Example: Find an eq. of the cone whose directrix is $y^2+z^2=1$; $x=2$ and vertex is $V(0,0,0)$

Sol:

$$y_1^2+z_1^2=1 \; ; \; x_1=2 \quad \text{---(1)}$$



$$\frac{x-0}{x_1-0} = \frac{y-0}{y_1-0} = \frac{z-0}{z_1-0}$$

$$\Rightarrow \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$$

$$\frac{x}{x_1} = \frac{y}{y_1} \quad \text{and} \quad \frac{y}{y_1} = \frac{z}{z_1}$$

$\therefore x_1 = 2$

$\frac{x}{2} = \frac{y}{y_1}$ and $\frac{x}{2} = \frac{z}{z_1}$

$\Rightarrow y_1 = \frac{2y}{x}$ $\quad z_1 = \frac{2z}{x}$

$\left(\frac{2y}{x}\right)^2 + \left(\frac{2z}{x}\right)^2 = 1$

$\frac{4y^2}{x^2} + \frac{4z^2}{x^2} = 1$

$4y^2 + 4z^2 = x^2$ which is eq. of cone

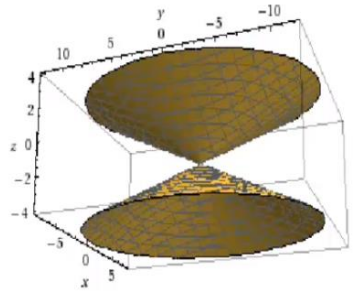
Elliptic Cone

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz^2 \quad c > 0$

→ along z-axis

if $c = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2$



$\frac{x^2}{a^2} + \frac{z^2}{b^2} = y^2 \Rightarrow$ along y-axis

$\frac{y^2}{a^2} + \frac{z^2}{b^2} = x^2 \Rightarrow$ along x-axis

Properties

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2=0 \Rightarrow f(x,y,z)=0$$

Symmetry

(i) xy-plane $f(x,y,z)=0 \Rightarrow f(x,y,-z)=0$

$$f(x,y,-z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-(-z)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+z^2=0 \Rightarrow f(x,y,z)$$

$\Rightarrow$ symmetrical about xy-plane

Intercepts $f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2=0$

x-intercept $y=0, z=0$

$$\frac{x^2}{a^2}=0 \Rightarrow x=0$$

y-intercept $x=0, z=0 \Rightarrow y=0$

z-intercept $x=0, y=0 \Rightarrow z=0$

Traces $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2 = 0$

(i) xy-plane $z=0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \Rightarrow$ represents a pt. $(0,0,0)$

(ii) xz-plane $y=0$

$\frac{x^2}{a^2} - z^2 = 0 \Rightarrow \frac{x^2}{a^2} = z^2 \Rightarrow x^2 = a^2 z^2$

$\Rightarrow x = \pm az$

(iii) yz-plane $x=0$ $\frac{y^2}{b^2} - z^2 = 0 \Rightarrow y^2 = b^2 z^2$

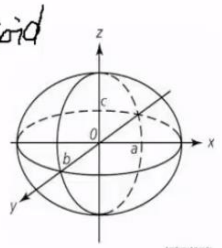
$\Rightarrow y = \pm bz$

Lecture 37:

Ellipsoid

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1 \quad \text{---(1)}$$

Ellipsoid



Centre (h,k,l)

Centre $(0,0,0)$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

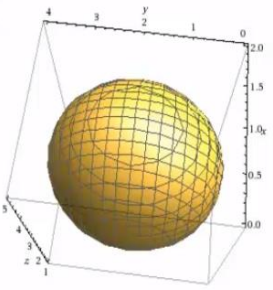
If $a=b=c$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}=1$$

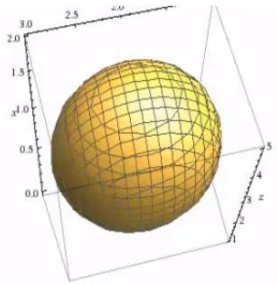
$$x^2+y^2+z^2=a^2 \rightarrow \text{Sphere}$$

$c(0,0,0)$ $r=a$

Spheroid oblate



Spheroid prolate



Spheroid

$a=b$, $c\neq a$, $c\neq b$

If $a=b$ $c<a$, $c<b$ $\rightarrow$ oblate

If $a=b$ $c>a$, $c>b$ $\rightarrow$ Prolate

Properties

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Symmetry

(i) xy-plane $f(x, y, z) = f(x, y, -z) = 0$

$f(x, y, -z) = 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(-z)^2}{c^2} = 1$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow f(x, y, z) = 0$

$\Rightarrow$ Symmetrical about xy-plane

(ii) xz-plane $f(x, -y, z) = 0 \Rightarrow f(x, y, z) = 0$

$f(x, -y, z) = \frac{x^2}{a^2} + \frac{(-y)^2}{b^2} + \frac{z^2}{c^2} = 1$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\Rightarrow f(x, y, z) = 0$

$\Rightarrow$ Symmetrical about xz-plane

(iii) similarly symmetrical about yz-plane

(iv) Symmetrical about all coordinate axes as well as about origin

Traces $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

(i) xy-plane

$z=0$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow$ ellipse

(ii) xz-plane $y=0$

$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1 \Rightarrow$ ellipse

(iii) yz-plane $x=0 \Rightarrow \frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Intercepts

(i) x-intercept Put $y=0, z=0$

$\frac{x^2}{a^2}=1 \Rightarrow x^2=a^2 \Rightarrow x=\pm a$

$(a,0,0)$ and $(-a,0,0)$

(ii) y-intercept Put $x=0, z=0$

$\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2 \Rightarrow y=\pm b$

$(0,b,0)$ and $(0,-b,0)$

(iii) z intercept $x=0, y=0$

$\frac{z^2}{c^2}=1 \Rightarrow z^2=c^2 \Rightarrow z=\pm c$

$(0,0,c)$ and $(0,0,-c)$

Example Identify the surface

$100x^2+25y^2+4z^2-400x-150y-8z+529=0$

and find its traces and intercepts

Solution

$100x^2+25y^2+4z^2-400x-150y-8z+529=0$

$100x^2-400x+25y^2-150y+4z^2-8z=-529$

$100(x^2-4x)+25(y^2-6y)+4(z^2-2z)=-529$

$100(x^2-4x+4)+25(y^2-6y+9)+4(z^2-2z+1)=-529+400+225+4$

$100(x-2)^2+25(y-3)^2+4(z-1)^2=100$

$\frac{(x-2)^2}{1}+\frac{(y-3)^2}{4}+\frac{(z-1)^2}{25}=1$

$\frac{(x-2)^2}{1^2}+\frac{(y-3)^2}{2^2}+\frac{(z-1)^2}{5^2}=1 \Rightarrow$

which is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

∴ The given eq. is the eq. of ellipsoid.

$C(2, 3, 1)$

Traces

(i) xy-plane $z = 0$

$$(x-2)^2 + \frac{(y-3)^2}{4} + \frac{(0-1)^2}{25} = 1$$

$$(x-2)^2 + \frac{(y-3)^2}{4} + \frac{1}{25} = 1$$

$$(x-2)^2 + \frac{(y-3)^2}{4} = 1 - \frac{1}{25} = \frac{24}{25}$$

$$\frac{25(x-2)^2}{24} + \frac{25(y-3)^2}{4} = 1 \Rightarrow \text{ellipse}$$

(ii) yz-plane $x = 0$

$$(0-2)^2 + \frac{(y-3)^2}{4} + \frac{(z-1)^2}{25} = 1$$

$$4 + \frac{(y-3)^2}{4} + \frac{(z-1)^2}{25} = 1$$

$$\frac{(y-3)^2}{4} + \frac{(z-1)^2}{25} = -3 \Rightarrow \text{No trace}$$

(ii) xz-plane $y=0$

$$(x-2)^2 + \frac{(0-3)^2}{4} + \frac{(z-1)^2}{25} = 1$$

$$(x-2)^2 + \frac{9}{4} + \frac{(z-1)^2}{25} = 1$$

$$(x-2)^2 + \frac{(z-1)^2}{25} = 1 - \frac{9}{4} = -\frac{5}{4}$$

$\Rightarrow$ No trace in xz-plane

Intercepts

(i) x-intercept put $y=0, z=0$

$$(x-2)^2 + \frac{(0-3)^2}{4} + \frac{(0-1)^2}{25} = 1$$

$$(x-2)^2 + \frac{9}{4} + \frac{1}{25} = 1 \Rightarrow (x-2)^2 = 1 - \frac{9}{4} - \frac{1}{25} < 0$$

$\Rightarrow$ No x-intercept

---

y-intercept $x=0, z=0$

$$\frac{(0-2)^2}{1} + \frac{(y-3)^2}{4} + \frac{(0-1)^2}{25} = 1$$

$$\frac{(y-3)^2}{4} + 4 + \frac{1}{25} = 1$$

$$\frac{(y-3)^2}{4} = 1 - 4 - \frac{1}{25} < 0 \Rightarrow$$ No y-intercept

z-intercept

$x=0,\ y=0$

$$\frac{(0-2)^2}{1}+\frac{(0-3)^2}{4}+\frac{(z-1)^2}{25}=1$$

$$4+\frac{9}{4}+\frac{(z-1)^2}{25}=1$$

$$\frac{(z-1)^2}{25}=1-4-\frac{9}{4}<0$$

$\Rightarrow$ No z-intercept.

Example Discuss the Symmetry of the ellipsoid.

$$x^2+\frac{y^2}{9}+\frac{(z-1)^2}{4}=1$$

Solution (i) xy-plane

$$f(x,y,-z)=f(x,y,z) \Rightarrow$$

$$f(x,y,-z)=x^2+\frac{y^2}{9}+\frac{(-z-1)^2}{4}=1$$

$$=x^2+\frac{y^2}{9}+\frac{(z+1)^2}{4}=1\neq f(x,y,-z)$$

$\Rightarrow$ Not Symmetrical about xy-plane.

(ii) xz-plane

$$f(x,-y,z) = x^2 + \frac{(-y)^2}{9} + \frac{(z-1)^2}{4}$$

$$= x^2 + \frac{y^2}{9} + \frac{(z-1)^2}{4}$$

$$= f(x,y,z)$$

$\Rightarrow$ Symmetrical about xz-plane.

(iii) yz-plane $f(-x,y,z) = (-x)^2 + \frac{y^2}{9} + \frac{(z-1)^2}{4}$

$$= x^2 + \frac{y^2}{9} + \frac{(z-1)^2}{4}$$

$$= f(x,y,z)$$

$\Rightarrow$ Symmetrical about yz-plane.

Lecture 38:

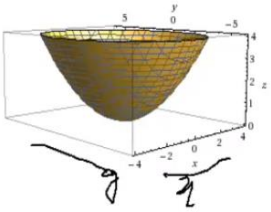
Elliptic paraboloid

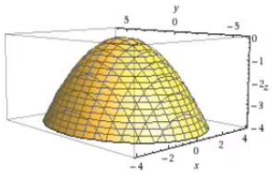
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = c(z-l)$$

$V(h,k,l)$

$V(0,0,0)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$$





if $c>0 \Rightarrow$ lies above xy-plane

if $c<0 \Rightarrow$ lies below xy-plane

e.g. $\frac{x^2}{4} + \frac{y^2}{9} = z$, $c=1$

Open along z-axis above xy-plane

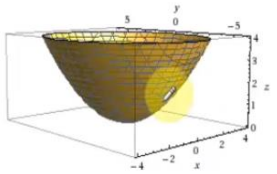
$\frac{x^2}{4} + \frac{y^2}{9} = -z$, $c=-1$

open along z-axis, lies below xy-plane

$\frac{x^2}{a^2} + \frac{z^2}{b^2} = y \Rightarrow$ open along y-axis

$\frac{y^2}{a^2} + \frac{z^2}{b^2} = x \Rightarrow$ open along x-axis

Symmetry $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$



$f(x,y,z) = 0$

$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$

(i) about xy-plane

$f(x,y,-z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + z \neq 0$

$\Rightarrow$ Not Symmetrical

(ii) xz-plane $f(x,-y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$

$\Rightarrow$ Symmetrical about xz-plane

(iii) about yz-plane

$f(-x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$

$\Rightarrow$ Symmetrical about yz-plane

(iv) about x-axis

$f(x,-y,-z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + z = 0$

$\Rightarrow$ Not Symmetrical about x-

(v) about y-axis

$$f(-x, y, -z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + z \neq 0$$

$\Rightarrow$ Not Symmetrical

(vi) about z-axis

$$f(-x, -y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$$

$\Rightarrow$ Symmetrical

(vii) about origin $f(-x, -y, -z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + z \neq 0$

$\Rightarrow$ Not Sy

Traces $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$

(i) xy-plane $z = 0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

$\Rightarrow$ which is possible only $x = 0, y = 0$

i.e. Trace is origin in xy-plane.

$z = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow$ ellipse

(ii) yz-plane $x=0$

$$\frac{y^2}{b^2} = z \Rightarrow y^2 = b^2 z \Rightarrow \text{parabola}$$

(iii) xz-plane $y=$

$$\frac{x^2}{a^2} = z \Rightarrow x^2 = a^2 z \Rightarrow \text{parabola}$$

Intercepts

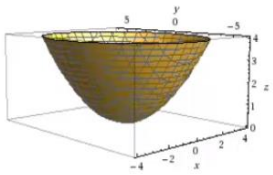
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$$

z-Intercept $x=0,\ y=0, \Rightarrow z=0$

i.e. only intercept is origin.

Boundedness

Not bounded



Example Elliptic paraboloid. Identify the surface $25x^2+y^2-2y-25z+1=0$ and find its traces in the coordinate planes and all intercepts.

Sol

$$25x^2+y^2-2y-25z+1=0$$
$$25x^2+y^2-2y+1=25z$$
$$25x^2+(y-1)^2=25z$$
$$x^2+\frac{(y-1)^2}{25}=z$$

Comparing with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$

$\Rightarrow$ elliptic

Traces

(i) xy-plane $z=0$

$$x^2+\frac{(y-1)^2}{25}=0$$

$\Rightarrow x=0,\ y-1=0$ i.e. $y=1$

i.e. $(0,1,0)$

(ii) yz-plane $x=0$

$$0^2+\frac{(y-1)^2}{25}=z \Rightarrow (y-1)^2=25z \Rightarrow \text{parabola}$$

(iii) xz-plane

$y=0$

$$x^2+\frac{(0-1)^2}{25}=z$$
$$\Rightarrow x^2+\frac{1}{25}=z$$
$$\Rightarrow x^2=z-\frac{1}{25} \Rightarrow \text{parabola}$$

Intercepts

(i) x-intercept $y=0, z=0$

$x^2 + \frac{(0-1)^2}{25} = 0 \Rightarrow x^2 + \frac{1}{25} = 0$

$\Rightarrow x^2 = -\frac{1}{25} \Rightarrow$ impossible

$\Rightarrow$ No-Intercept with x-axis

(ii) y-intercept $x=0, z=0$

$0 + \frac{(y-1)^2}{25} = 0 \Rightarrow (y-1)^2 = 0 \Rightarrow y-1=0 \Rightarrow y=1$

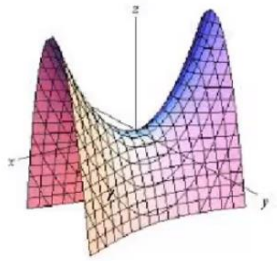
$(0, 1, 0)$

(iii) z-intercept $x=0, y=0$ $\quad 0 + \frac{1}{25} = z \Rightarrow z = \frac{1}{25}$

$(0, 0, \frac{1}{25})$

Hyperbolic Paraboloid

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = z$



Traces xy-plane put $z=0$

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 0 \Rightarrow \frac{y^2}{b^2} = \frac{x^2}{a^2} \Rightarrow y = \pm\frac{b}{a}x \Rightarrow$ Pair of intersecting lines

xz-plane put $y=0$

$-\frac{x^2}{a^2} = z \Rightarrow x^2 = -a^2 z \Rightarrow$ Parabola.

yz-plane put $x=0$

$y^2/b^2 = z \Rightarrow y^2 = b^2 z \Rightarrow$ Parabola

Put $z=1$

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \Rightarrow$ Hyperbola

Symmetry $\frac{y^2}{b^2} - \frac{x^2}{a^2} = z$

$f(x,y,z) = \frac{y^2}{b^2} - \frac{x^2}{a^2} - z = 0 \Rightarrow (2)$

(i) xy-plane $f(x,y,-z) = \frac{y^2}{b^2} - \frac{x^2}{a^2} + z \neq 0$ by (2).

$\Rightarrow$ Not Symmetrical

(ii) yz-plane

$f(-x,y,z) = \frac{y^2}{b^2} - \frac{(-x)^2}{a^2} - z = \frac{y^2}{b^2} - \frac{x^2}{a^2} - z = 0$ by (2)

$\Rightarrow$ Symmetrical

(iii) xz-plane $f(x,-y,z) = \frac{y^2}{b^2} - \frac{x^2}{a^2} - z = 0$ by (2)

$\Rightarrow$ Symmetrical about xz-plane

(iv) x-axis $f(x,-y,-z) = \frac{y^2}{b^2} - \frac{x^2}{a^2} + z \neq 0 \Rightarrow$ Not Symmetrical

(v) y-axis $f(-x,y,-z) = \frac{y^2}{b^2} - \frac{x^2}{a^2} + z \neq 0 \Rightarrow$ Not Symmetrical

(vi) z-axis $f(-x,-y,z) = \frac{y^2}{b^2} - \frac{x^2}{a^2} - z = 0 \Rightarrow$ Symmetrical

(vi) origin $f(-x,-y,-z) = \frac{y^2}{b^2} - \frac{x^2}{a^2} + z \neq 0$ Not

Intercepts

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = z.$$

$y=0, z=0$

$$\frac{0^2}{b^2} - \frac{x^2}{a^2} = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$$

$x=0, y=0, z=0$

$O(0,0,0)$

Example Identify the surface

$-x^2+4y^2-4x-16y-4z+12=0$

and find its traces in the coordinate planes and intercepts.

Sol.

$-x+4y^2-4x-16y-4z+12=0$

$4y^2-16y-x^2-4x+12=4z$

$4(y^2-4y+4)-(x^2+4x+4)+12=4z+16-4$

$4(y-2)^2-(x+2)^2+12=4z+12$

$(y-2)^2-\frac{(x+2)^2}{4}=z$

Hyperbolic paraboloid

Traces $(y-2)^2 - \frac{(x+2)^2}{4} = z$

xy-plane put $z=0 \Rightarrow (y-2)^2 = \frac{(x+2)^2}{4}$

$\Rightarrow y-2 = \pm\frac{(x+2)}{2}$

$y-2 = \frac{x+2}{2}$ or $y-2 = -\frac{(x+2)}{2}$

$2y-4 = x+2$ or $2(y-2) = -x-2$

$x-2y+6=0$ $\quad 2y-4 = -x-2$

$x+2y-2=0$

pair of

yz-plane $x=0$

$(y-2)^2 = z \Rightarrow$ parabola

xz-plane $y=0$

$(-2)^2 - \frac{(x+2)^2}{4} = z$

$4 - \frac{(x+2)^2}{4} = z$

$\frac{(x+2)^2}{4} = 4-z \Rightarrow (x+2)^2 = 4(4-z)$

Intercepts

x-intercept $y=0, z=0$

$$(0-2)^2 - \frac{(x+2)^2}{4} = 0 \Rightarrow 4 - \frac{(x+2)^2}{4} = 0$$

$$\frac{(x+2)^2}{4} = 4 \Rightarrow (x+2)^2 = 16 \Rightarrow x+2 = \pm 4$$

$x+2=4$ or $x+2=-4$

$x=2$ or $x=-6$

$(2,0,0)$ and $(-6,0,0)$

y-intercept $x=0, z=0$

$$(y-2)^2 - \frac{(0+2)^2}{4} = 0$$

$$(y-2)^2 - \frac{4}{4} = 0$$

$$(y-2)^2 = 1 \Rightarrow y-2 = \pm 1$$

$y-2=1$ or $y-2=-1$

$y=3$ $y=1$

$(0,3,0)$ and $(0,1,0)$

z-intercept $x=0,\ y=0$

$$(0-2)^2-\frac{(0+2)^2}{4}=z$$

$$4-\frac{4}{4}=z$$

$$\Rightarrow z=4-1$$

$$z=3$$

$$(0,0,3)$$

Lecture 39:

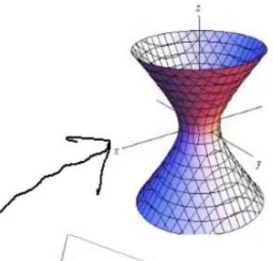
Hyperboloid of one sheet

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = 1$$

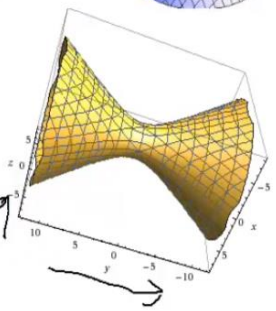
Centre (h,k,l)

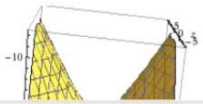
if Centre $(0,0,0)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \Rightarrow \text{along } z\text{-axis}$$

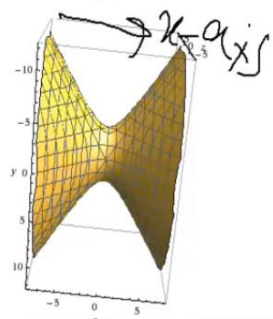


$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \text{along } y\text{-axis}$$



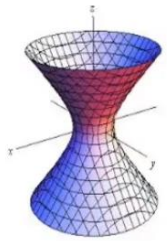


$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \text{along } x\text{-axis}$$



Symmetry

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$$

(i) xy-plane

$$f(x,y,-z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=f(x,y,z)=0$$

$\Rightarrow$ Symmetrical

(ii) yz-plane

$$f(-x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=f(x,y,z)=0$$

$\Rightarrow$ Symmetrical

(iii) xz-plane

$$f(x,-y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=f(x,y,z)$$

$\Rightarrow$ Symmetrical

(iv) x-axis

$$f(x,-y,-z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=f(x,y,z)=0$$

$\Rightarrow$ Symmetrical

(v) y-axis

$$f(-x,y,-z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=f(x,y,z)=0$$

$\Rightarrow$ Symmetrical

(vi) z-axis

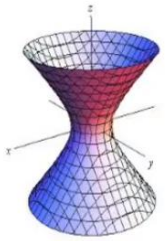
$$f(-x,-y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = f(x,y,z) = 0$$

$\Rightarrow$ Symmetrical

(vii) origin $f(-x,-y,-z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = f(x,y,z) = 0$

$\Rightarrow$ Symmetrical

Traces $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



(i) xy-plane put $z = 0$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow$ ellipse

(ii) xz-plane put $y = 0$

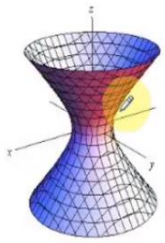
$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \Rightarrow$ Hyperbola

(iii) yz-plane put $x = 0$

$\frac{y^2}{a^2} - \frac{z^2}{c^2} = 1 \Rightarrow$ Hyperbola

Intercepts

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



(i) x-intercept put $y=0,\ z=0$

$$\frac{x^2}{a^2}=1 \Rightarrow x^2=a^2 \Rightarrow x=\pm a$$

$(a,0,0)$ and $(-a,0,0)$

(ii) y-intercept Put $x=0,\ z=0$

$$\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2 \Rightarrow y=\pm b$$

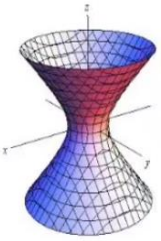
$(0,b,0)$ and $(0,-b,0)$

(iii) z-intercept

$x=0,\ y=0$

$$-\frac{z^2}{c^2}=1 \Rightarrow z^2=-c^2 \Rightarrow \text{No real sol.}$$

$\Rightarrow$ No z-intercept.



Boundness

Not bounded

Example Hyperboloid of One Sheet

Identify the surface

$4x^2+y^2-4z^2-16x-4y+8z+12=0$

and find its Centre and intercepts

Sol

$4x^2+y^2-4z^2-16x-4y+8z+12=0$

$4x^2-16x+y^2-4y-4z^2+8z=-12$

$4(x^2-4x+4)+(y^2-4y+4)-4(z^2-2z+1)=-12+16+4-4$

$4(x-2)^2+(y-1)^2-4(z-1)^2=4$

$(x-2)^2+\frac{(y-1)^2}{4}-(z-1)^2=1$

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}-\frac{(z-l)^2}{c^2}=1$

$\Rightarrow$ Hyperboloid of one sheet

Centre $(2,1,1)$

Traces

$4x^2+y^2-4z^2-16x-4y+8z+12=0$

(i) x-intercept: Put $y=0, z=0$

$4x^2-16x+12=0 \Rightarrow x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0$

$x-1=0$ or $x-3=0$

$x=1$ or $x=3$

$(1,0,0)$ and $(3,0,0)$

(ii) y-intercept: Put $x=0, z=0$

$y^2-4y+12=0 \Rightarrow$ No Solution

No y-intercept

(iii) z-intercept $x=0, y=0$

$$-4z^2+8z+12=0$$
$$z^2-2z-3=0$$
$$(z-3)(z+1)=0$$
$$z-3=0 \qquad z+1=0$$
$$z=3 \qquad z=-1$$

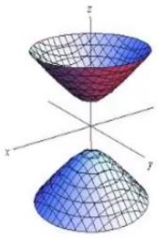
$(0,0,3)$ and $(0,0,-1)$

Hyperboloid of Two sheets

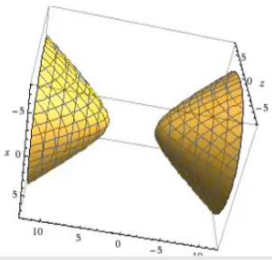
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}-\frac{(z-l)^2}{c^2}=-1$$

centre $=(h,k,l)$

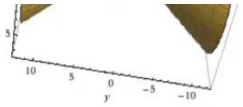
Centre $(0,0,0)$ $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ ⟶

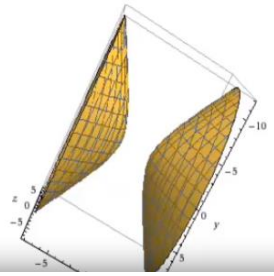


$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$$ ⟶



$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$$





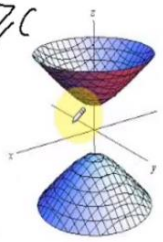
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}-1$$

i.e. $\frac{z^2}{c^2}-1\geq 0$ or $\frac{z^2}{c^2}\geq 1$

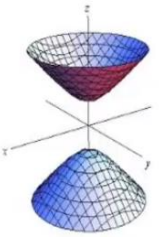
or $z^2\geq c^2$ or $z\leq -c$ or $z\geq c$

$z\geq c$



$z\leq -c$

Symmetry $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$



$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0 \longrightarrow (1)$$

(i) xy-plane $f(x,y,-z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=f(x,y,z)=0$

$\Rightarrow$ Symmetrical

(ii) xz-plane $f(x,-y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=f(x,y,z)=0$

$\Rightarrow$ Symmetrical xz-plane

(iii) yz-plane $f(-x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=f(x,y,z)=0$

(iv) x-axis

$$f(x,-y,-z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0=f(x,y,z)$$

Symmetrical

(v) y-axis $f(-x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=f(x,y,z)=0$

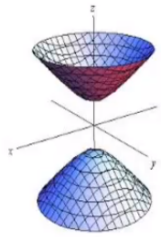
Symmetrical

(vi) z-axis $f(-x,-y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=f(x,y,z)=0$

$\Rightarrow$ Symmetrical

(vii) origin $f(-x,-y,-z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=f(x,y,z)=0$

Traces $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$



(i) xy-plane put $z=0$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1 \Rightarrow$ No solution

$\Rightarrow$ No trace

(ii) yz-plane put $x=0$

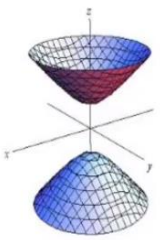
$\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1 \Rightarrow \frac{z^2}{c^2}-\frac{y^2}{b^2}=1 \Rightarrow$ Hyperbola

(iii) xz-plane $y=0$

$\frac{x^2}{a^2}-\frac{z^2}{c^2}=-1 \Rightarrow \frac{z^2}{c^2}-\frac{x^2}{a^2}=1 \Rightarrow$ Hyperbola

Intercepts

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$



(i) x-intercept Put $y=0, z=0$

$$\frac{x^2}{a^2}=-1 \Rightarrow x^2=-a^2 \Rightarrow \text{No Solution}$$

$\Rightarrow$ No x-intercept

(ii) y-intercept $x=0, z=0$

$$\frac{y^2}{b^2}=-1 \Rightarrow y^2=-b^2 \Rightarrow \text{No y-intercept}$$

(iii) z-intercept $x=0, y=0$ $-z^2/c^2=-1 \Rightarrow z^2=c^2 \Rightarrow z=\pm c$

$(0,0,c)$ and $(0,0,-c)$

Example Hyperboloid of Two sheets

Identify the Surface

$$9x^2+9y^2-z^2-18x-18y+2z+26=0$$

find its centre and traces in coordinate planes

Solution

$$9x^2+9y^2-z^2-18x-18y+2z+26=0$$

$$9x^2-18x+9y^2-18y-z^2+2z=-26$$

$$9(x^2-2x+1)+9(y^2-2y+1)-(z^2-2z+1)=-26+9+9-1$$

$$9(x-1)^2+9(y-1)^2-(z-1)^2=-9$$

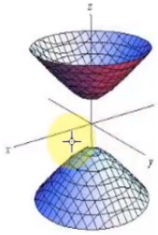
$$(x-1)^2 + (y-1)^2 - \frac{(z-1)^2}{9} = -1$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \frac{(z-l)^2}{c^2} = -1$$

Hyperboloid of two sheets

$C(1,1,1)$

Traces $(x-1)^2 + (y-1)^2 - \frac{(z-1)^2}{9} = -1$



(i) xy-plane $z=0$

$$(x-1)^2 + (y-1)^2 - \frac{1}{9} = -1$$

$$(x-1)^2 + (y-1)^2 = -1 + \frac{1}{9} = -\frac{8}{9}$$

$\Rightarrow$ No solution

$\Rightarrow$ No trace

$\Rightarrow$ No trace

(ii) yz-plane $x=0$ $\quad 1 + (y-1)^2 - \frac{(z-1)^2}{9} = -1$

$$(y-1)^2 - \frac{(z-1)^2}{9} = -2$$

$$\frac{(z-1)^2}{9} - (y-1)^2 = 2 \Rightarrow \text{Hyperbola}$$

(iii) xz-plane $y=0$

$$(x-1)^2 + 1 - \frac{(z-1)^2}{9} = -1 \Rightarrow (x-1)^2 - \frac{(z-1)^2}{9} = -2$$

$$\frac{(z-1)^2}{9} - (x-1)^2 = 2 \Rightarrow \text{Hyperbola}$$

Lecture 40:

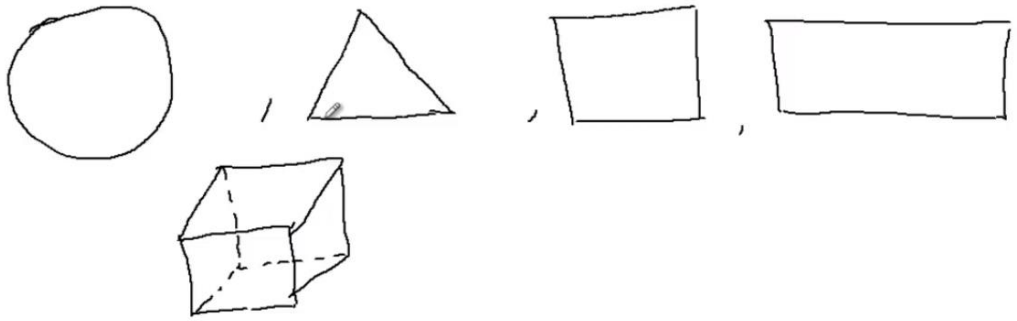
Spherical Trigonometry

Main Branches of Mathematics

- **Arithmetic**
- **Algebra**
- **Geometry**

Geometry

- The mathematical study of the properties, measurement, and relationships of points, circles,lines, planes, surfaces, angles etc.



Types of Geometry

- Plane geometry
- Solid geometry
- Spherical geometry

Plane geometry :





Example



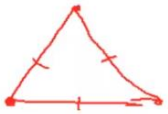
$C = 2\pi r$

$A = \pi r^2$ $A = \frac{1}{2} r^2 \theta$

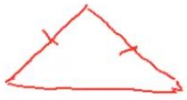
$l = r\theta$

Trigonometry

- Trigonometry is a branch of geometry that studies relationships involving lengths and angles of triangles.



Equilateral Δ



Iss. Δ





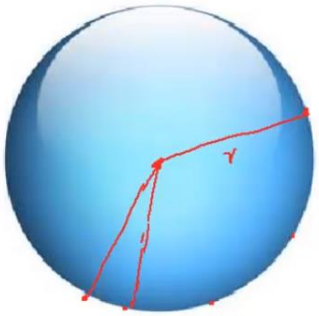
Scalene Δ.

Solid geometry

- It is the study about three dimensional objects like spheres, cubes, prisms, cylinders etc.

Example





$A = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

Spherical geometry

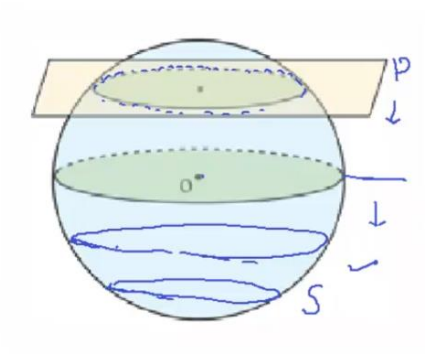
- The study of figures on the surface of a sphere.

Spherical Trigonometry

- Spherical Trigonometry deals with triangles drawn on the surface of sphere.

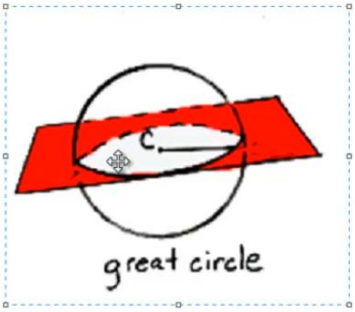
Lecture 41:

Intersection of a plane and a sphere:



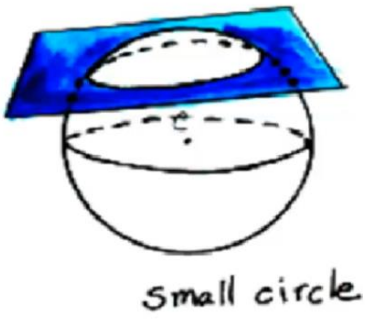
Great Circle:

A great circle of a sphere is the intersection of the sphere and a plane such that the plane passes through the center of the sphere.



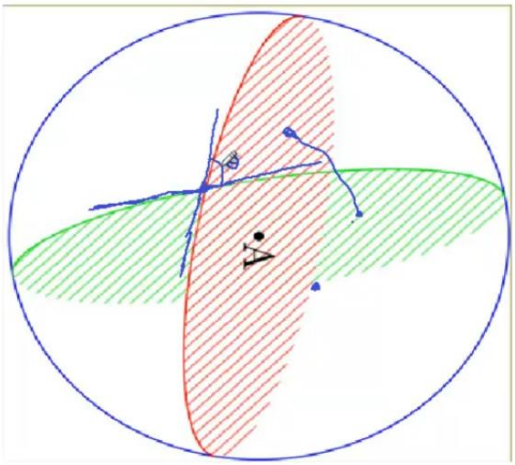
Small Circle:

A small circle of a sphere is the intersection of the sphere and a plane such that the plane does not pass through the center of the sphere.



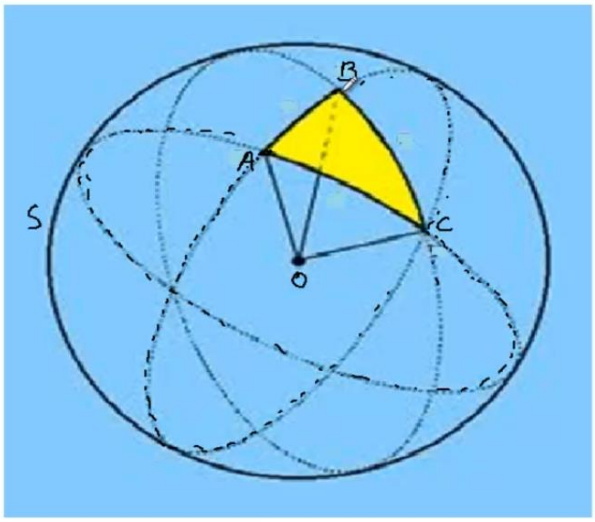
Spherical Angle

It is the angle between two intersecting arcs on a sphere, and is measured by the angle between the planes of great circles containing the arcs.



Spherical Triangle:

A spherical triangle is a part of surface of sphere formed by three great circular arcs intersecting pairwise in three vertices.



Spherical triangle:

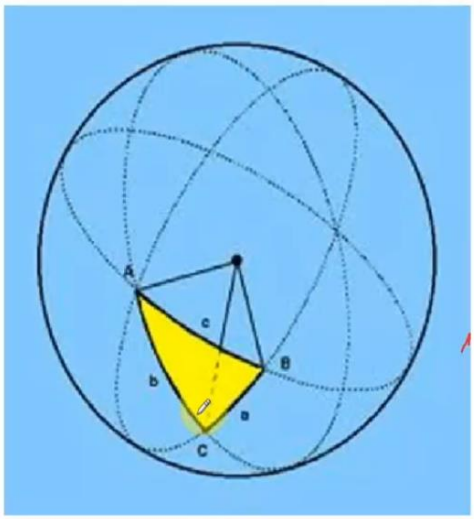
1. The three sides are all the arcs of great circles.
2. Any two sides are together longer than the third side.
3. The sum of the three angles is greater than 180^0.
4. Each individual spherical angle is less than 180^0.

Plane triangle:

1. The three sides are straight lines on the surface of plane.
2. Any two sides are together longer than the third side.
3. The sum of the three angles is always equal to 180^0.
4. Each individual angle is less than 180^0.

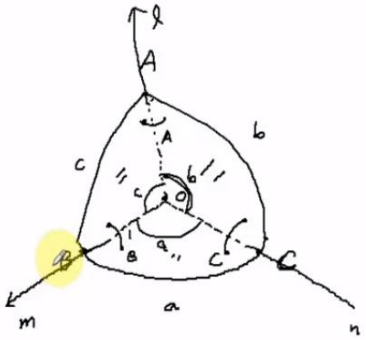
Lecture 42:

The Cosine Formula



The Lagrange's identity

$$(l \times m) \cdot (l \times n) = \begin{vmatrix} l \cdot l & l \cdot n \\ m \cdot l & m \cdot n \end{vmatrix} \longrightarrow (1)$$



Take $\begin{vmatrix} l \cdot l & l \cdot n \\ m \cdot l & m \cdot n \end{vmatrix} = \begin{vmatrix} |l||l|\cos 0 & |l||n|\cos b \\ |m||l|\cos c & |m||n|\cos a \end{vmatrix}$

$$R.H.S = \begin{vmatrix} 1 & \cos b \\ \cos c & \cos a \end{vmatrix} \quad \because |l| = |m| = |n| = 1$$



$l = r\theta$

$b = (1)\theta$

$a = b$

$$R.H.S = \cos a - \cos b \cos c \longrightarrow (2)$$

$$L.H.S = (l \times m) \cdot (l \times n)$$

$$= |l \times m|\,|l \times n| \cos A$$

$$= |l||m| \sin c \cdot |l||n| \sin b \cdot \cos A$$

$$= \sin c \sin b \cos A \longrightarrow (3)$$

From eqns 2 & 3

$$\cos a - \cos b \cos c = \sin c \sin b \cos A$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

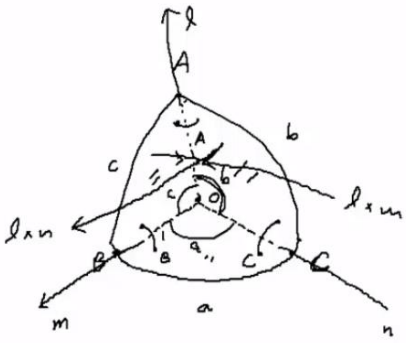
} Cosine formulas

The Sine Formula

$$(l \times m) \times (l \times n) = [l\,m\,n]\,l - [l\,m\,l]\,n$$

$$(l \times m) \times (l \times n) = [l\,m\,n]\,l \qquad \because [l\,m\,l] = 0$$

$\longrightarrow$ (1)



$$(l \times m) \times (l \times n) = |l \times m|\,|l \times n| \sin A\, l$$

$$= |l||m| \sin c \cdot |l||n| \sin b \cdot \sin A\, l$$

$$(l \times m) \times (l \times n) = \sin b \sin c \sin A\, l \qquad \because |l| = |m| = |n| = 1$$

$l = r\theta$

$b = 1 \cdot \theta$

$\theta = b$

$\theta = b$

Eq (1) $\Rightarrow$ $\sin b \sin c \sin A\, l = [l\,m\,n]\,l$

$\Rightarrow$ $\sin b \sin c \sin A = [l\,m\,n]$ $\longrightarrow$ (2)

$\sin a \sin b \sin C = [l\,m\,n]$ $\longrightarrow$ (3)

$\sin a \sin c \sin B = [l\,m\,n]$ $\longrightarrow$ (4)

$\Rightarrow \sin b \sin c \sin A = \sin a \sin b \sin C = \sin a \sin c \sin B$

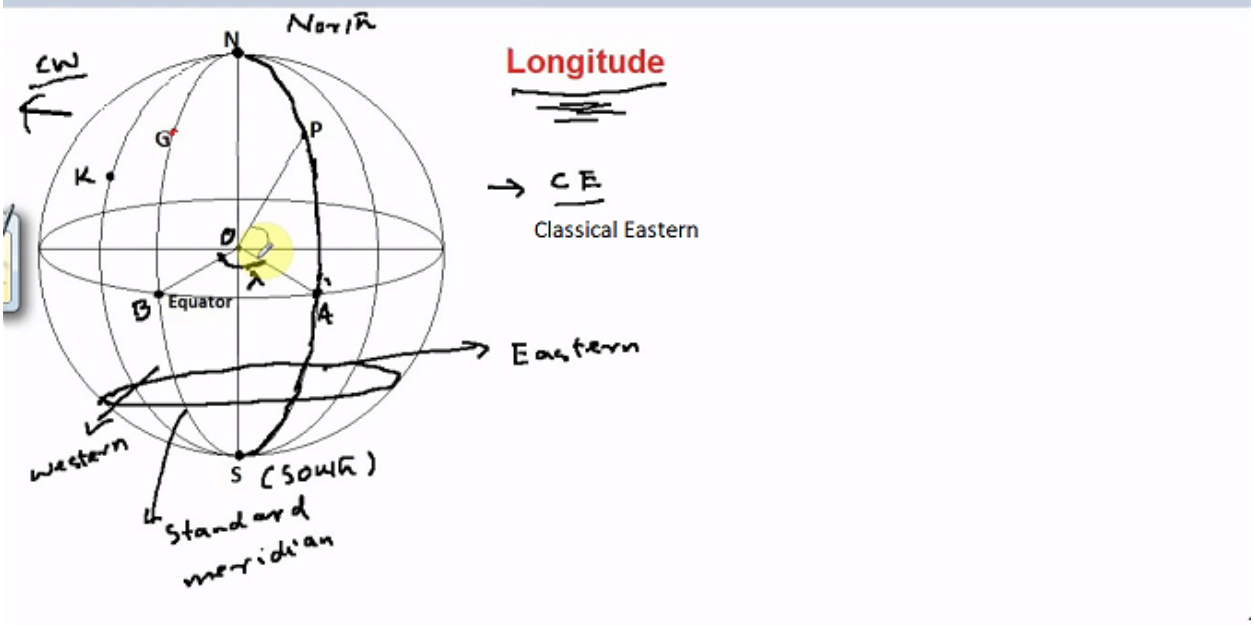
Dividing by $\sin a \sin b \sin c$

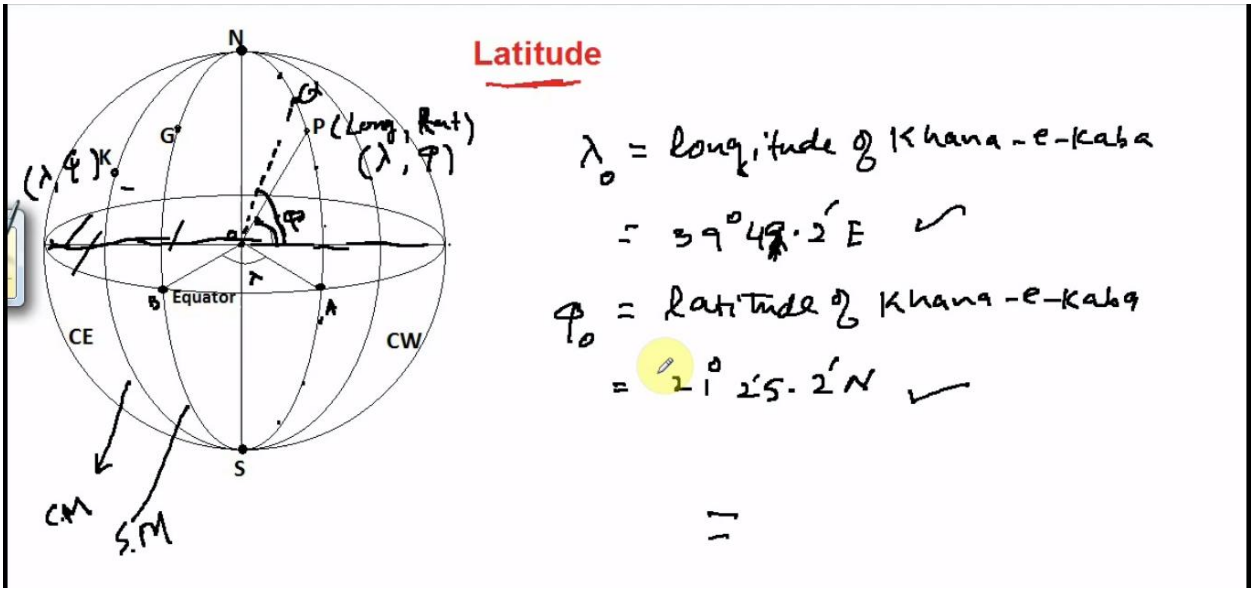
$\Rightarrow$

$$\boxed{\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c} = \frac{\sin B}{\sin b}}$$

The Sine Formula.

Lecture 43:


North
N
CW
G
P
K
O
λ
B
Equator
A
S (South)
Western
Standard meridian
Longitude
CE
Classical Eastern
Eastern


N
G
P (Long, Lat)
(λ, φ)
(λ, φ)K
φ
λ
B
Equator
A
CE
CW
S
CM
SM
Latitude
λ0 = longitude of Khana-e-Kaba
= 39°49.2' E
φ0 = latitude of Khana-e-Kaba
= 21° 25.2' N

Lecture 44:

Method to find the Direction of Qibla

λ = longitude of any place on the surface of earth

φ = latitude " " " " " " " "

$\lambda_0 = 39^\circ 49.2' E$ = longitude of Khan-e-kaba

$\varphi_0 = 21^\circ 25.2' N$ = latitude " " " "

$$P = \frac{\sin\varphi}{\tan l}, \quad Q = \frac{\cos\varphi \tan\varphi_0}{\sin l}$$

$$\tan i = P - Q$$

$$i = \tan^{-1}(P - Q)$$

The classical longitude l of a place can be found by the following way.

1. If $\lambda_0 < \lambda^0 E \leq 180^0$ then $l = (\lambda - \lambda_0)^0 CE$
2. If $0^0 \leq \lambda^0 E < \lambda_0$ then $l = (\lambda_0 - \lambda)^0 CW$
3. If $0 < \lambda^0 W < 180^0 - \lambda_0$ then $l = (\lambda_0 + \lambda)^0 CW$

Q Find the direction of Qibla of Islamabad

$\lambda = 73^\circ 8' E$, $\varphi = 33^\circ 40' N$

Solution

Longitude of Islamabad = $\lambda = 73^\circ 8' E$

Latitude " " " = $\varphi = 33^\circ 40' N$

Longitude of Khana-e-kaba = $\lambda_0 = 39^\circ 49.2' E$

Latitude " " " " = $\varphi_0 = 21^\circ 25.2' N$

$$p = \frac{\sin\varphi}{\tan l}, \qquad l = \lambda - \lambda_0 = 73^\circ 8' - 39^\circ 49.2'$$

$$l = 33^\circ 18.8' \ (CE)$$

$$p = \frac{\sin 33^\circ 40'}{\tan 33^\circ 18.8'} = 0.8435$$

$$q = \frac{\cos\varphi \tan\varphi_0}{\sin l} = \frac{\cos 33^\circ 40' \tan 21^\circ 25.2'}{\sin 33^\circ 18.8'}$$

$$q = 0.1793$$

As $\tan i = p - q$

$= 0.8435 - 0.1793$

$= 0.6642$

$i = \tan^{-1}(0.6642) = 33^\circ 35.5'$ South of west