

CHAP # 1

TRIGONOMETRY

Complex No. complex number is an element (x, y) of the set $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$

OR Number which can be written in form $a+ib$.

Polar Rectangular Components:

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Exercise # 1.1

Express each of the following complex no. in polar form. (Problem 1-6)

1. $-\sqrt{3} + i$

$x = -\sqrt{3}$, $y = 1$

$r = |z| = \sqrt{3+1} = \sqrt{4}$

$\boxed{r=2}$

$\cos \theta = -\frac{\sqrt{3}}{2}$, $\sin \theta = \frac{1}{2}$

θ is in 2nd Quadrant

Reference angle $= \alpha = \frac{\pi}{6}$

$\theta = \pi - \alpha = \pi - \pi/6$

$\theta = 5\pi/6$

$z = r [\cos \theta + i \sin \theta]$

$= 2 \left[\cos \left(\frac{5\pi}{6} \right) + i \sin \left(\frac{5\pi}{6} \right) \right]$

2. $-i$

$x = 0$, $y = -1$

$r = |z| = \sqrt{0+1} = 1 \Rightarrow \boxed{r=1}$

$\cos \theta = 0$, $\sin \theta = -1$

Here θ is quadrantal angle

$\theta = -\pi/2$

$z = r [\cos \theta + i \sin \theta]$

$z = 1 \left[\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right]$

$z = \cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right)$

$$\textcircled{2} \quad r = \sqrt{x^2 + y^2}$$

3. $-1 - \sqrt{3}i$

$$x = -1, \quad y = -\sqrt{3}$$

$$|z| = r = \sqrt{1+3} = \sqrt{4}$$

$$\Rightarrow \boxed{r=2}$$

$$\cos \theta = -1/2, \quad \sin \theta = -\sqrt{3}/2$$

θ is in 3rd quadrant

$$\text{Reference Angle} = \alpha = \pi/3$$

$$\theta = \alpha - \pi = \frac{\pi}{3} - \pi$$

$$\theta = -\frac{2\pi}{3}$$

$$z = 2 \left[\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right]$$

5. $(-2+2i)(1-i)$

$$= -2 + 2i + 2i - 2i^2$$

$$= -2 + 4i + 2$$

$$z = 4i$$

$$x=0, \quad y=4 \quad \boxed{|z|=4}$$

$$\cos \theta = 0, \quad \sin \theta = 1$$

Here θ is Quadrantal Angle

$$\therefore \theta = \pi/2$$

$$z = r [\cos \theta + i \sin \theta]$$

$$= 4 [\cos(\pi/2) + i \sin(\pi/2)]$$

Quadrantal Angle:

Angle which lies on boundaries b/w two quadrants

like $(0, 90, 180, 270, 360)$

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4. $-1 + i$

$$x = -1, \quad y = 1$$

$$|z| = r = \sqrt{1+1} \Rightarrow \boxed{r=\sqrt{2}}$$

$$\cos \theta = -\frac{1}{\sqrt{2}}, \quad \sin \theta = \frac{1}{\sqrt{2}}$$

θ is in 2nd quadrant

$$\text{Reference angle} = \alpha = \frac{\pi}{4}$$

$$\theta = \pi - \alpha = \pi - \frac{\pi}{4}$$

$$\boxed{\theta = 3\pi/4}$$

$$z = \sqrt{2} \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right]$$

6. $\frac{-34i}{5-3i}$

$$= \frac{-34i}{5-3i} \times \frac{5+3i}{5+3i}$$

$$= \frac{-34(5i+3i^2)}{(5)^2 - (3i)^2}$$

$$= \frac{-34(5i-3)}{25+9}$$

$$= \frac{34(3-5i)}{34}$$

$$z = 3-5i$$

$$x=3, \quad y=-5$$

$$z = \sqrt{9+25} = \sqrt{34}$$

$$\cos \theta = 3/\sqrt{34}, \quad \sin \theta = -5/\sqrt{34}$$

$$\frac{\sin \theta}{\cos \theta} = \frac{-5/\sqrt{34}}{3/\sqrt{34}}$$

$$\tan \theta = -\frac{5}{3}$$

$$\theta = \tan^{-1}\left(-\frac{5}{3}\right)$$

$$z = r [\cos \theta + i \sin \theta]$$

$$= \sqrt{34} \left[\cos\left(\tan^{-1}\left(-\frac{5}{3}\right)\right) + i \sin\left(\tan^{-1}\left(-\frac{5}{3}\right)\right) \right]$$

(3)

Express the given complex numbers in cartesian form and plot on an argand diagram (Problem 7-10)

7. $2 \text{Cis}(\pi/6)$

$$\begin{aligned}
 &= 2 [\cos(\pi/6) + i \sin(\pi/6)] \\
 &= 2 \left[\frac{\sqrt{3}}{2} + i \sin(\pi/6) \right] \\
 &= 2 \left(\frac{\sqrt{3}}{2} + i \frac{1}{2} \right) \\
 &= \frac{2}{2} (\sqrt{3} + i) \\
 &= \sqrt{3} + i
 \end{aligned}$$

8. $5 \text{Cis}(\frac{3\pi}{4})$

$$\begin{aligned}
 &= 5 \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] \\
 &= 5 \left[-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right] \\
 &= -5/\sqrt{2} (1 - i)
 \end{aligned}$$

$4 \times 5 \times 3 = 135$
 $\downarrow$
 2nd Quad
 $\downarrow$
 Cos -ve
 Sin +ve

9. $\sqrt{3} \text{Cis}\left(\frac{7\pi}{6}\right)$

$$\begin{aligned}
 &= \sqrt{3} \left[\cos\left(\frac{7\pi}{6}\right) + i \sin\left(\frac{7\pi}{6}\right) \right] \\
 &= \sqrt{3} \left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right] \\
 &= -\frac{\sqrt{3}}{2} (1 + i)
 \end{aligned}$$

$30 \times 7 = 210$
 3rd quad.
 Sin -ve, Cos -ve

10. $5 \text{Cis}(\pi/3)$
 $2 \text{Cis}(\pi/2)$

$$\begin{aligned}
 &= \frac{5}{2} \text{Cis}\left(\frac{\pi}{3} - \frac{\pi}{2}\right) \\
 &= \frac{5}{2} \text{Cis}\left(-\frac{\pi}{6}\right) \\
 &= \frac{5}{2} \left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right] \\
 &= \frac{5}{2} \left[\frac{\sqrt{3}}{2} - i \frac{1}{2} \right] \\
 &= \frac{5\sqrt{3}}{4} - i \frac{5}{4}
 \end{aligned}$$

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Formula

if

$$z = z_1 \cdot z_2 \cdot z_3$$

$$|z| = |z_1| |z_2| |z_3|$$

$$|z| = \sqrt{x^2 + y^2}$$

11. Find $|z|$.

(i)

$$z = -2i(1+i)(2+4i)(3+i)$$

$$z_1 = -2i \Rightarrow |z_1| = \sqrt{4} = 2$$

$$z_2 = 1+i \Rightarrow |z_2| = \sqrt{1+1} = \sqrt{2}$$

$$z_3 = 2+4i \Rightarrow |z_3| = \sqrt{4+16} = \sqrt{20}$$

$$z_4 = 3+i \Rightarrow |z_4| = \sqrt{9+1} = \sqrt{10}$$

$$|z| = |z_1| \cdot |z_2| \cdot |z_3| \cdot |z_4|$$

$$= 2 \cdot \sqrt{2} \cdot \sqrt{20} \cdot \sqrt{10}$$

$$= 2 \sqrt{2 \times 20 \times 10}$$

$$= 2 \sqrt{20 \times 20}$$

$$= 2(20)$$

$$|z| = 40$$

Formula: if $z = \frac{z_1 \cdot z_2}{z_3 \cdot z_4}$

$$\text{then } |z| = \frac{|z_1| |z_2|}{|z_3| |z_4|}$$

$$(ii) z = \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$$

$$|z_1| = 3+4i \Rightarrow |z_1| = \sqrt{9+16} = \sqrt{25}$$

$$|z_1| = 5$$

$$z_2 = -1+2i \Rightarrow |z_2| = \sqrt{1+4} = \sqrt{5}$$

$$z_3 = -1-i \Rightarrow |z_3| = \sqrt{1+1} = \sqrt{2}$$

$$z_4 = 3-i \Rightarrow |z_4| = \sqrt{9+1} = \sqrt{10}$$

$$|z| = \frac{|z_1| |z_2|}{|z_3| |z_4|}$$

$$= \frac{5 \cdot \sqrt{5}}{\sqrt{2} \cdot \sqrt{10}} = 5 \sqrt{\frac{5}{2 \times 10}}$$

$$= 5 \sqrt{\frac{1}{4}} = \frac{5}{2}$$

$$|z| = \frac{5}{2}$$

12. Show that $z = a+ib$ is

(i) real iff $z = \bar{z}$

Suppose $z = \bar{z} \rightarrow (1)$

$$z = a+ib \Rightarrow \bar{z} = a-ib$$

$$a+ib = a-ib \quad \text{from (1)}$$

$$ib+ib=0$$

$$2ib=0$$

$$\Rightarrow 2 \neq 0, i \neq 0 \Rightarrow b=0$$

put in z

$$z = a+0i = a$$

$\Rightarrow z$ is real

Conversely;

Suppose z is real

$$\Rightarrow z = a$$

$$\Rightarrow \bar{z} = a$$

here

$$z = \bar{z} = a$$

proved

(ii) pure imaginary iff

$$z = -\bar{z}$$

Suppose

$$z = -\bar{z}$$

$$z = a+ib \Rightarrow \bar{z} = a-ib$$

$$\therefore z = -\bar{z}$$

$$a+ib = -(a-ib)$$

$$a+ib = -a+ib$$

$$a+a=0$$

$$2a=0$$

$$\therefore 2 \neq 0 \Rightarrow a=0$$

put in z

$$z = 0+ib = ib$$

$\Rightarrow z$ is pure imaginary

Conversely, suppose

z is imaginary,

$$\Rightarrow z = ib$$

$$\Rightarrow \bar{z} = -ib$$

here

$$z = -\bar{z}$$

proved

13. Prove analytically that ⁽⁵⁾ for Complex numbers z_1, z_2 .

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

Sol. $|z_1 + z_2| \leq |z_1| + |z_2|$

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + \bar{z}_1z_2 + z_2\bar{z}_2 \\ &= |z_1|^2 + z_1\bar{z}_2 + \bar{z}_1z_2 + |z_2|^2 \\ &= |z_1|^2 + 2\operatorname{Re}(z_1\bar{z}_2) + |z_2|^2 \\ &\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \end{aligned}$$

$$\because z\bar{z} = |z|^2$$

$$\because \bar{\bar{z}} = z$$

$$\begin{aligned} \text{if } z &= a+ib \\ \Rightarrow \bar{z} &= a-ib \end{aligned}$$

$$\begin{aligned} z + \bar{z} &= 2a \\ &= 2\operatorname{Re}(z) \end{aligned}$$

$$\because \operatorname{Re}(z) < |z|$$

$$\because a < \sqrt{a^2 + b^2}$$

$$\because |z_1z_2| = |z_1||z_2|$$

$$\because |z_1| = |\bar{z}_1|$$

$$\begin{aligned} |z_1 + z_2|^2 &\leq (|z_1| + |z_2|)^2 \\ |z_1 + z_2| &\leq |z_1| + |z_2| \rightarrow \text{A} \end{aligned}$$

Now for $||z_1| - |z_2|| \leq |z_1 - z_2|$

$$\begin{aligned} |z_1| &= |z_1 - z_2 + z_2| \\ |z_1| &\leq |z_1 - z_2| + |z_2| \\ |z_1| - |z_2| &\leq |z_1 - z_2| \\ \text{from (i) \& (ii)} \quad \downarrow \text{ii)} \\ -|z_1 - z_2| &\leq |z_1| - |z_2| \leq |z_1 - z_2| \\ \Rightarrow ||z_1| - |z_2|| &\leq |z_1 - z_2| \rightarrow \text{B} \end{aligned}$$

Obviously:

$$|z_1 - z_2| \leq |z_1 + z_2| \rightarrow \text{C}$$

from A, B & C.

$$||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\Rightarrow ||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

Proved.

14. Let $z_1 = 24 + 7i$ and $|z_2| = 6$. Find the greatest and least value of $|z_1 + z_2|$.

Sol. $\because$ greatest value of $|z_1 + z_2| = |z_1| + |z_2| =$

$$|z_1| = \sqrt{(24)^2 + (7)^2} = \sqrt{576 + 49} = \sqrt{625} = 25$$

$$|z_2| = \sqrt{0 + 6} = 6$$

greatest value = $25 + 6 = 31$

(6)

Least value of $|z_1 + z_2| = ||z_1| - |z_2||$
 $= |25 - 6| = 19$

15. if z_1, z_2 are complex numbers. show that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

Sol: $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$ $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2})$
 $= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$ $= (z_1 - z_2)(\overline{z_1} - \overline{z_2})$
 $= (z_1\overline{z_1} + z_2\overline{z_2} + \overline{z_1}z_2 + z_1\overline{z_2})$ $= (z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - z_2\overline{z_1})$
 $|z_1 + z_2|^2 = (|z_1|^2 + z_1\overline{z_2} + \overline{z_1}z_2 + |z_2|^2)$ $|z_1 - z_2|^2 = (|z_1|^2 + |z_2|^2 - z_1\overline{z_2} - \overline{z_1}z_2)$
① ②

adding ① and ②

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = |z_1|^2 + z_1\overline{z_2} + \overline{z_1}z_2 + |z_2|^2 + |z_1|^2 + |z_2|^2 - z_1\overline{z_2} - \overline{z_1}z_2$$

$$= 2|z_1|^2 + 2|z_2|^2$$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

16. Prove that $\left| \frac{az+b}{b\overline{z}+a} \right| = 1$ for $|z|=1$

$$\left| \frac{az+b}{b\overline{z}+a} \right| = \frac{|az+b|}{|b\overline{z}+a|} = \frac{|az+b|}{|\overline{b\overline{z}+a}|} \quad \because |z|=|\overline{z}|$$

$$= \frac{|az+b|}{1|\overline{b\overline{z}+a}|}$$

$$= \frac{|az+b|}{|az+b|}$$

$$\overline{\overline{a+b}} = \overline{a+b}$$

$$\overline{ab} = \overline{a}\overline{b}$$

$$\overline{\overline{a}} = a$$

$$= \frac{|az+b|}{|z(b\overline{z}+a)|}$$

$$= \frac{|az+b|}{|az+b|}$$

$$= \frac{|az+b|}{|b|z|^2+az|}$$

$$= \frac{|az+b|}{|az+b|}$$

$$z\overline{z} = |z|^2$$

$$|z|=1$$

$$|z|^2=1$$

$$\left| \frac{az+b}{b\overline{z}+a} \right| = 1$$

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(7)

17. Find locus of points in the plane satisfying each of the given conditions.

(i) $|z-5|=6$

$$= |x+iy-5|=6$$

$$= |(x-5)+iy|=6$$

$$\sqrt{(x-5)^2+y^2}=6$$

$$(x-5)^2+(y-0)^2=36=(6)^2$$

locus of circle with centre (5,0) and radius 6

(iii) $\operatorname{Re}(z+2)=-1$

$$\operatorname{Re}(x+iy+2)=-1$$

$$\operatorname{Re}(x+2+iy)=-1$$

$$x+2=-1$$

$$x=-3$$

locus is line parallel to y-axis

(v) $|z+i|=|z-i|$

$$|x+iy+i|=|x+iy-i|$$

$$|x+i(y+1)|=|x+i(y-1)|$$

$$\sqrt{x^2+(y+1)^2}=\sqrt{x^2+(y-1)^2}$$

Squaring both sides

$$x^2+(y+1)^2=x^2+(y-1)^2$$

$$(y+1)^2=(y-1)^2$$

$$y^2+1+2y=y^2+1-2y$$

$$4y=0$$

$$y=0$$

x-axis.

(ii) $|z-2i| \geq 1$

$$|x+iy-2i| \geq 1$$

$$|x+(y-2)i| \geq 1$$

$$\sqrt{x^2+(y-2)^2} \geq 1$$

$$(x-0)^2+(y-2)^2 \geq 1$$

Locus is set of points on and outside the circle with center (0,2) and radius 1

(iv) $\operatorname{Re}(i\bar{z})=3$

$$\operatorname{Re}(i(x+iy))=3$$

$$\operatorname{Re}(i(x-yi))=3$$

$$\operatorname{Re}(ix+y^2)=3$$

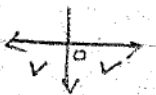
$$y=3$$

locus is line parallel to x-axis.

(vii) $\operatorname{Im}(z) < 0$

$$\operatorname{Im}(x+iy) < 0$$

$$y < 0$$



3rd & 4th quadrant.

(vii) $-1 < \operatorname{Re}(z) < 1$

$$-1 < \operatorname{Re}(x+iy) < 1$$

$$-1 < x < 1$$

$$x \in]-1, 1[$$

Locus is interval on real line

8

(vi) $|z+3| + |z+1| = 4$

$$|x+iy+3| + |x+iy+1| = 4 \quad = |x+3+iy| + |x+1+iy| = 4$$

$$\sqrt{(x+3)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4$$

$$\sqrt{(x+3)^2 + y^2} = 4 - \sqrt{(x+1)^2 + y^2}$$

$$\sqrt{x^2 + 9x + 6x + y^2} = 4 - \sqrt{x^2 + 2x + 1 + y^2}$$

Squaring both sides

using formula
 $(x+y)^2 = x^2 + y^2 + 2xy$

$$x^2 + 9 + 6x + y^2 = 16 + x^2 + 2x + 1 + y^2 - 2(4)\sqrt{x^2 + 2x + 1 + y^2}$$

$$6x - 2x + 9 - 16 - 1 = -8\sqrt{x^2 + 2x + 1 + y^2}$$

$$4x - 8 = -8\sqrt{x^2 + 2x + 1 + y^2}$$

$$4(x-2) = -8\sqrt{x^2 + 2x + 1 + y^2}$$

$$x-2 = -2\sqrt{x^2 + 2x + 1 + y^2}$$

$$x^2 - 4x + 4 = 4(x^2 + 2x + 1 + y^2)$$

Squaring both sides

$$x^2 - 4x + 4 = 4x^2 + 8x + 4 + 4y^2$$

$$4x^2 - x^2 + 8x + 4x + 4y^2 = 0$$

$$3x^2 + 12x + 4y^2 = 0$$

$$3(x^2 + 4x) + 4(y^2) = 0$$

∴ Completing Square

$$x^2 + 4x = \frac{x^2 + 4x + 4 - 4}{1} = (x+2)^2 - 4$$

$$3[(x+2)^2 - 4] + 4(y^2) = 0$$

$$3(x+2)^2 - 12 + 4y^2 = 0$$

$$3(x+2)^2 + 4y^2 = 12$$

$$\frac{(x+2)^2}{4} + \frac{y^2}{3} = 1$$

divide both sides by 12.

locus is ellipse.

(ix) $\text{Arg}(z) = \pi/3$

$$\text{Arg}(x+iy) = \pi/3$$

$$\tan^{-1}(y/x) = \pi/3$$

$$\tan(\pi/3) = y/x$$

$$y/x = \sqrt{3}$$

$$y = \sqrt{3}x$$

locus is straight line

(x) $\text{Arg}(z-1) = -\frac{3\pi}{4}$

$$\text{Arg}(x+iy-1) = -3\pi/4$$

$$\text{Arg}((x-1)+iy) = -3\pi/4$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right) = -\frac{3\pi}{4}$$

$$\frac{y}{x-1} = \tan\left(-\frac{3\pi}{4}\right)$$

$$\frac{y}{x-1} = -\tan(3\pi/4)$$

$$\frac{y}{x-1} = -(-1) = 1$$

$$\frac{y}{x-1} = 1$$

$$y = x-1$$

$$x-y-1=0$$

locus is straight line

(9)

De Moivre's Theorem:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad n \in \mathbb{Z}$$

Proof:

For +ve integers, we use Mathematical Induction.

CASE I: Put $n=1$

$$\begin{aligned} (\cos \theta + i \sin \theta)^1 &= \cos 1\theta + i \sin 1\theta \\ &= \cos \theta + i \sin \theta. \end{aligned}$$

This is true for $n=1$.

CASE 2: Suppose this is true for $n=k$ where $k \in \mathbb{Z}^+$

$$(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$$

Multiply both sides by $(\cos \theta + i \sin \theta)$

$$(\cos \theta + i \sin \theta)^k (\cos \theta + i \sin \theta) = (\cos k\theta + i \sin k\theta) (\cos \theta + i \sin \theta)$$

$$\begin{aligned} (\cos \theta + i \sin \theta)^{k+1} &= \cos k\theta \cos \theta + i \cos k\theta \sin \theta + i \sin k\theta \cos \theta \\ &\quad + i^2 \sin k\theta \sin \theta \end{aligned}$$

$$= (\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i (\cos k\theta \sin \theta + \sin k\theta \cos \theta)$$

$$= \cos(k\theta + \theta) + i \sin(k\theta + \theta)$$

$$(\cos \theta + i \sin \theta)^{k+1} = \cos(k+1)\theta + i \sin(k+1)\theta$$

$\Rightarrow$ This is true for $n=k+1$ where $k \in \mathbb{Z}^+$

$\Rightarrow$ Theorem is valid for +ve integers.

for $n=0$

$$(\cos \theta + i \sin \theta)^0 = \cos(0)\theta + i \sin(0)\theta$$

$$1 = 1 + i(0)$$

$$1 = 1$$

True.

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for $n \in \mathbb{Z}^-$

Let $n = -m$ where $m \in \mathbb{Z}^+$

$$(\cos \theta + i \sin \theta)^n = (\cos \theta + i \sin \theta)^{-m} = [(\cos \theta + i \sin \theta)^m]^{-1}$$

$$= [\cos m\theta + i \sin m\theta]^{-1} \quad \text{valid for } m \in \mathbb{Z}^+$$

$$= \frac{1}{\cos m\theta + i \sin m\theta} \times \frac{\cos m\theta - i \sin m\theta}{\cos m\theta - i \sin m\theta}$$

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$\cos^2 m\theta + \sin^2 m\theta = 1$$

$$= \cos(-m\theta) + i \sin(-m\theta)$$

$$\because \sin(-\theta)$$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

$$= -\sin \theta$$

Hence Theorem is True for All integers

EXERCISE 1.2 ⁽¹⁰⁾

Write each of the following expression in the form $a+ib$.

1. $(-\sqrt{3}+i)^2$

$$z = -\sqrt{3} + i$$

$$x = -\sqrt{3}, y = 1$$

$$\therefore r = \sqrt{3+1} = 2$$

$$\cos \theta = \frac{-\sqrt{3}}{2}, \sin \theta = \frac{1}{2}$$

θ is in IInd quadrant

reference angle = $\pi/6$

$$\theta = \pi - \pi/6 = 5\pi/6$$

$$z = r(\cos \theta + i \sin \theta)$$

$$z = 2 \left[\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) \right]$$

$$(z)^2 = z^2 \left[\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) \right]^2$$

by de-Moivre's Theorem

$$(-\sqrt{3}+i)^2 = 4 \left[\cos\left(2 \times \frac{5\pi}{6}\right) + i \sin\left(2 \times \frac{5\pi}{6}\right) \right]$$

$$= 4 \left[\cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) \right]$$

$$= 4 \left[\frac{1}{2} + i \left(-\frac{\sqrt{3}}{2}\right) \right]$$

$$= \frac{4}{2} (1 - \sqrt{3}i)$$

$$= 2(1 - \sqrt{3}i)$$

$$(-\sqrt{3}+i)^2 = 2 - 2\sqrt{3}i$$

(ii) $(-3i)^4 = (-3)^4 (i)^4$

$$= 81 (i^2)^2$$

$$= 81 (-1)^2$$

$$(-3i)^4 = 81$$

(iii) $\left(\frac{1 - \sqrt{3}i}{1 + \sqrt{3}i} \right)^6$

here $z = \frac{1 - \sqrt{3}i}{1 + \sqrt{3}i} \times \frac{1 - \sqrt{3}i}{1 - \sqrt{3}i}$

$$z = \frac{(1 - \sqrt{3}i)^2}{1 + 3} = \frac{(1 - \sqrt{3}i)^2}{4}$$

$$z^6 = \left[\frac{(1 - \sqrt{3}i)^2}{4} \right]^6$$

$$\left(\frac{1 - \sqrt{3}i}{1 + \sqrt{3}i} \right)^6 = \frac{(1 - \sqrt{3}i)^{12}}{4096} = \frac{z_1}{4096}$$

Let $z_1 = 1 - \sqrt{3}i$

$$|z_1| = \sqrt{1+3} = 2$$

$$\cos \theta = 1/2, \sin \theta = -\sqrt{3}/2$$

IVth quadrant.

reference Angle = $\alpha = \pi/3$

$$\theta = -\frac{\pi}{3} \therefore \theta = -\alpha$$

$$z_1 = r [\cos \theta + i \sin \theta]$$

$$z_1 = 2 \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right]$$

$$(z_1)^{12} = (2)^{12} \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right]^{12}$$

$$(z_1)^{12} = 4096 \left[\cos\left(-\frac{12\pi}{3}\right) + i \sin\left(-\frac{12\pi}{3}\right) \right]$$

$$\frac{(z_1)^{12}}{4096} = \cos(-4\pi) + i \sin(-4\pi)$$

$$\left(\frac{1 - \sqrt{3}i}{1 + \sqrt{3}i} \right)^6 = 1 + 0 = 1$$

(11)

2. Simplify.

$$\begin{aligned}
 \text{(i)} \quad & \frac{(\cos 2\theta + i \sin 2\theta)^5 (\cos 3\theta - i \sin 3\theta)^6}{(\cos 4\theta + i \sin 4\theta)^7 (\cos 5\theta + i \sin 5\theta)^8} \\
 &= \frac{(\cos 2\theta + i \sin 2\theta)^5 (\cos(-3\theta) + i \sin(-3\theta))^6}{(\cos(-4\theta) + i \sin(-4\theta))^7 (\cos 5\theta + i \sin 5\theta)^8} \\
 &= \frac{(\cos 10\theta + i \sin 10\theta)(\cos(-18\theta) + i \sin(-18\theta))}{(\cos(-28\theta) + i \sin(-28\theta))(\cos 40\theta + i \sin 40\theta)} \\
 &= \frac{\text{cis}(10\theta) \text{cis}(-18\theta)}{\text{cis}(-28\theta) \text{cis}(40\theta)} \\
 &= \text{cis}(10 - 18 + 28 - 40)\theta \\
 &= \text{cis}(-20)\theta \\
 &= \cos(-20)\theta + i \sin(-20)\theta \\
 &= \cos 20\theta - i \sin 20\theta
 \end{aligned}$$

$$\begin{aligned}
 \therefore \cos(-\theta) &= \cos \theta \\
 \sin(-\theta) &= -\sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \therefore \frac{\text{cis}(\alpha)}{\text{cis}(\beta)} &= \text{cis}(\alpha - \beta) \\
 \& \text{cis}(\alpha) \cdot \text{cis}(\beta) &= \text{cis}(\alpha + \beta)
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \frac{(\cos \alpha - i \sin \alpha)^{11}}{(\cos \beta + i \sin \beta)^9} = \frac{(\cos(-\alpha) + i \sin(-\alpha))^{11}}{(\cos \beta + i \sin \beta)^9} \\
 &= \frac{\cos(-11\alpha) + i \sin(-11\alpha)}{\cos(9\beta) + i \sin(9\beta)} \\
 &= \frac{\text{cis}(-11\alpha)}{\text{cis}(9\beta)} \\
 &= \text{cis}(-11\alpha - 9\beta) \\
 &= \cos(-(11\alpha + 9\beta)) + i \sin(-(11\alpha + 9\beta)) \\
 &= \cos(11\alpha + 9\beta) - i \sin(11\alpha + 9\beta)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & \frac{(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)}{(\cos \gamma + i \sin \gamma)(\cos \delta + i \sin \delta)} = \frac{\text{cis} \alpha \cdot \text{cis} \beta}{\text{cis} \gamma \cdot \text{cis} \delta} \\
 &= \text{cis}(\alpha + \beta - \gamma - \delta)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \frac{(3 \text{cis} \pi/6)^7}{(4 \text{cis} \pi/3)^5} = \frac{3^7 \text{cis}(7\pi/6)}{4^5 \text{cis}(5\pi/3)} \\
 &= \frac{3^7}{4^5} \text{cis}\left(\frac{7\pi}{6} - \frac{5\pi}{3}\right) = \frac{3^7}{4^5} \text{cis}\left(\frac{7\pi - 10\pi}{6}\right) \\
 &= \frac{3^7}{4^5} \text{cis}\left(-\frac{\pi}{2}\right) = \frac{3^7}{4^5} \left(-\frac{\pi}{2}\right)
 \end{aligned}$$

(12)

3. Prove that

$$(i) \left[(\cos \theta - \cos \phi) + i(\sin \theta - \sin \phi) \right]^n + \left[(\cos \theta - \cos \phi) - i(\sin \theta - \sin \phi) \right]^n = 2^{n+1} \sin^n \left(\frac{\theta - \phi}{2} \right) \cos n \left(\frac{\theta + \phi}{2} \right)$$

Proof:

Let $\cos \theta - \cos \phi = r \cos \alpha + i$, $\sin \theta - \sin \phi = r \sin \alpha \rightarrow (ii)$
 Squaring & adding (i) & (ii)

$$(\cos \theta - \cos \phi)^2 + (\sin \theta - \sin \phi)^2 = r^2 \cos^2 \alpha + r^2 \sin^2 \alpha$$

$$\cos^2 \theta - \cos^2 \phi - 2 \cos \theta \cos \phi + \sin^2 \theta + \sin^2 \phi - 2 \sin \theta \sin \phi = r^2 (\sin^2 \alpha + \cos^2 \alpha)$$

$$(\cos^2 \theta + \sin^2 \theta) + (\cos^2 \phi + \sin^2 \phi) - 2(\cos \theta \cos \phi + \sin \theta \sin \phi) = r^2 (1)$$

$$1 + 1 - 2 \cos(\theta - \phi) = r^2$$

$$2 - 2 \cos(\theta - \phi) = r^2$$

$$2(1 - \cos(\theta - \phi)) = r^2$$

$$2 \left(2 \sin^2 \left(\frac{\theta - \phi}{2} \right) \right) = r^2$$

$$r^2 = 4 \sin^2 \left(\frac{\theta - \phi}{2} \right)$$

$$\Rightarrow r = 2 \sin \left(\frac{\theta - \phi}{2} \right)$$

Square Root

 $\rightarrow (iii)$

Dividing (ii) by (i)

$$\frac{r \sin \alpha}{r \cos \alpha} = \frac{\sin \theta - \sin \phi}{\cos \theta - \cos \phi}$$

$$\tan \alpha = \frac{2 \cos \left(\frac{\theta + \phi}{2} \right) \sin \left(\frac{\theta - \phi}{2} \right)}{-2 \sin \left(\frac{\theta + \phi}{2} \right) \sin \left(\frac{\theta - \phi}{2} \right)}$$

$$\tan \alpha = -\cot \left(\frac{\theta + \phi}{2} \right)$$

$$\tan \alpha = \tan \left(\frac{\pi}{2} + \frac{\theta + \phi}{2} \right)$$

$$\tan \alpha = \tan \left(\frac{\pi}{2} + \frac{\theta + \phi}{2} \right)$$

$$\therefore \tan \left(\frac{\pi}{2} + \alpha \right) = -\cot \alpha$$

$$\alpha = \frac{\pi}{2} + \frac{\theta + \phi}{2}$$

$$\alpha = \frac{\pi + \theta + \phi}{2} \rightarrow (iv)$$

from (A)

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$$\text{L.H.S} = (r \cos \alpha + i r \sin \alpha)^n + (r \cos \alpha - i r \sin \alpha)^n$$

$$= r^n (\cos \alpha + i \sin \alpha)^n + r^n (\cos \alpha - i \sin \alpha)^n$$

$$= r^n [(\cos n \alpha + i \sin n \alpha) + (\cos(-\alpha) + i \sin(-\alpha))^n] \quad \text{De-Moivre's Theorem}$$

$$= r^n [\cos n \alpha + i \sin n \alpha + (\cos(-n \alpha) + i \sin(-n \alpha))] \quad \text{De-Moivre's Theorem}$$

$$= r^n [\cos n \alpha + i \sin n \alpha + \cos n \alpha - i \sin n \alpha]$$

$$= r^n [2 \cos n \alpha]$$

$$= \left[2 \sin \left(\frac{\theta - \phi}{2} \right) \right]^n \cdot 2 \cos n \left(\frac{\pi + \theta + \phi}{2} \right)$$

from (iii) & (iv)

(13)

$$= 2^n \sin^n \left(\frac{\theta - \phi}{2} \right) 2 \cos n \left(\frac{\theta + \phi + \pi}{2} \right)$$

$$= 2^{n+1} \sin^n \left(\frac{\theta - \phi}{2} \right) \cos n \left(\frac{\theta + \phi + \pi}{2} \right) = \text{R.H.S proved}$$

$$(ii) \left(\frac{1 + \sin x + i \cos x}{1 + \sin x - i \cos x} \right)^n = \cos n \left(\frac{\pi}{2} - x \right) + i \sin n \left(\frac{\pi}{2} - x \right)$$

$$\frac{1 + \sin x + i \cos x}{1 + \sin x - i \cos x} = \frac{(1 + \sin x) + i(\cos x)}{(1 + \sin x) - i(\cos x)} \times \frac{(1 + \sin x) + i(\cos x)}{(1 + \sin x) + i(\cos x)}$$

$$= \frac{((1 + \sin x) + i \cos x)^2}{(1 + \sin x)^2 - (i \cos x)^2}$$

$$= \frac{(1 + \sin x)^2 + (i \cos x)^2 + 2(1 + \sin x)(i \cos x)}{1 + \sin^2 x + 2 \sin x - i^2 \cos^2 x}$$

 $\therefore i = -1$

$$= \frac{(1 + \sin x)^2 - \cos^2 x + 2(1 + \sin x)(i \cos x)}{1 + \sin^2 x + 2 \sin x + \cos^2 x}$$

 $\therefore \cos^2 x + \sin^2 x = 1$ $\therefore \cos^2 x = 1 - \sin^2 x$

$$= \frac{(1 + \sin x)^2 - (1 - \sin^2 x) + 2(1 + \sin x)(i \cos x)}{(1 + \sin x)^2 + (1 - \sin^2 x)}$$

 $\therefore a^2 - b^2 = (a+b)(a-b)$

$$= \frac{(1 + \sin x)^2 - (1 - \sin x)(1 + \sin x) + 2(1 + \sin x)(i \cos x)}{(1 + \sin x)^2 + (1 - \sin x)(1 + \sin x)}$$

$$= \frac{(1 + \sin x) [1 + \sin x - (1 - \sin x) + 2i \cos x]}{(1 + \sin x) [(1 + \sin x) + (1 - \sin x)]}$$

$$= \frac{1 + \sin x - 1 + \sin x + 2i \cos x}{2}$$

$$= \frac{2 \sin x + 2i \cos x}{2}$$

$$= \frac{2(\sin x + i \cos x)}{2}$$

$$\frac{1 + \sin x + i \cos x}{1 + \sin x - i \cos x} = i \cos x + \sin x$$

$$\left(\frac{1 + \sin x + i \cos x}{1 + \sin x - i \cos x} \right)^n = (\sin x + i \cos x)^n$$

$$= \left[\cos \left(\frac{\pi}{2} - x \right) + i \sin \left(\frac{\pi}{2} - x \right) \right]^n$$

De-Moivre's Theorem

$$= \cos n \left(\frac{\pi}{2} - x \right) + i \sin n \left(\frac{\pi}{2} - x \right) = \text{R.H.S}$$

PROVED

4. if $2 \cos \theta = x + \frac{1}{x}$, $2 \cos \phi = y + \frac{1}{y}$, $2 \cos \psi = z + \frac{1}{z}$
then prove that



Allied Angles

$$\therefore \cos \left(\frac{\pi}{2} - x \right) = \sin x$$

$$\& \sin \left(\frac{\pi}{2} - x \right) = \cos x$$

(14)

$$(i) 2\cos(\theta + \phi + \psi) = xyz + \frac{1}{xyz}$$

Sol.

$$\text{Let } x = \cos\theta + i\sin\theta = \text{Cis}(\theta)$$

$$\frac{1}{x} = \cos\theta - i\sin\theta = \text{Cis}(-\theta)$$

$$\Rightarrow x + \frac{1}{x} = 2\cos\theta$$

Similarly,

$$y = \cos\phi + i\sin\phi = \text{Cis}(\phi)$$

$$\frac{1}{y} = \cos\phi - i\sin\phi = \text{Cis}(-\phi)$$

&

$$\text{also } z = \cos\psi + i\sin\psi = \text{Cis}(\psi)$$

$$\frac{1}{z} = \cos\psi - i\sin\psi = \text{Cis}(-\psi)$$

$$xyz = \text{Cis}(\theta) \text{Cis}(\phi) \text{Cis}(\psi)$$

$$xyz = \text{Cis}(\theta + \phi + \psi) \rightarrow (1)$$

$$\text{and } \frac{1}{x} \cdot \frac{1}{y} \cdot \frac{1}{z} = \text{Cis}(-\theta) \text{Cis}(-\phi) \text{Cis}(-\psi)$$

$$\frac{1}{xyz} = \text{Cis}(-(\theta + \phi + \psi)) \rightarrow (2)$$

adding (1) & (2)

$$xyz + \frac{1}{xyz} = \text{Cis}(\theta + \phi + \psi) + \text{Cis}(-(\theta + \phi + \psi))$$

$$= \cos(\theta + \phi + \psi) + i\sin(\theta + \phi + \psi) + \cos(-(\theta + \phi + \psi)) + i\sin(-(\theta + \phi + \psi))$$

$$= \cos(\theta + \phi + \psi) + i\sin(\theta + \phi + \psi) + \cos(\theta + \phi + \psi) - i\sin(\theta + \phi + \psi)$$

$$xyz + \frac{1}{xyz} = 2\cos(\theta + \phi + \psi)$$

Proved.

$$(ii) x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$$

$$x = \cos\theta + i\sin\theta$$

$$x^m = (\cos\theta + i\sin\theta)^m$$

$$x^m = \cos m\theta + i\sin m\theta$$

$$x^m = \text{Cis}(m\theta)$$

&

$$y = \cos\phi + i\sin\phi$$

$$y^n = (\cos\phi + i\sin\phi)^n$$

$$= \cos n\phi + i\sin n\phi$$

$$y^n = \text{Cis}(n\phi)$$

$$\frac{1}{x} = \cos\theta - i\sin\theta = \text{Cis}(-\theta) + i\sin(-\theta)$$

$$\frac{1}{x^m} = (\text{Cis}(-\theta) + i\sin(-\theta))^m$$

$$\frac{1}{x^m} = \cos(-m\theta) + i\sin(-m\theta)$$

$$\frac{1}{x^m} = \text{Cis}(-m\theta)$$

$$\frac{1}{y} = \cos\phi - i\sin\phi = \text{Cis}(-\phi) + i\sin(-\phi)$$

$$\frac{1}{y^n} = (\text{Cis}(-\phi) + i\sin(-\phi))^n$$

$$= \cos(-n\phi) + i\sin(-n\phi)$$

$$\frac{1}{y^n} = \text{Cis}(-n\phi)$$

(15)

$$x^m y^n = \text{Cis}(m\theta) \text{Cis}(n\phi)$$

$$x^m y^n = \text{Cis}(m\theta + n\phi) \rightarrow \textcircled{1}$$

$$\begin{aligned} \frac{1}{x^m} \cdot \frac{1}{y^n} &= \text{Cis}(-m\theta) \text{Cis}(-n\phi) \\ &= \text{Cis}(-(m\theta + n\phi)) \rightarrow \textcircled{2} \end{aligned}$$

adding $\textcircled{1}$ & $\textcircled{2}$

$$x^m y^n + \frac{1}{x^m y^n} = \text{Cis}(m\theta + n\phi) + \text{Cis}(-(m\theta + n\phi))$$

$$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \cos(-(m\theta + n\phi)) + i\sin(-(m\theta + n\phi))$$

$$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi) + \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$$

$$\frac{x^m y^n + \frac{1}{x^m y^n}}{x^m y^n} = 2\cos(m\theta + n\phi)$$

5. Find the three cube roots of $8i$.

$$-iz = 8i \quad \text{here } x=0, y=8, |z|=r = \sqrt{0+8^2} = 8$$

$$\cos\theta = 0, \sin\theta = 1$$

$$\cos\theta = x/r, \sin\theta = y/r$$

Here θ is quadrantal Angle
& $\theta = 90^\circ = \pi/2$

formula for finding n-Roots

$$z_k = |z|^{1/n} \left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right) \right], k = 0, 1, 2, \dots, n-1$$

for 3-roots.

$$z_k = |z|^{1/3} \left[\cos\left(\frac{\theta + 2k\pi}{3}\right) + i\sin\left(\frac{\theta + 2k\pi}{3}\right) \right], k = 0, 1, 2$$

for $8i$, $\theta = \pi/2$ & $|z| = 8$

$$z_k = [8]^{1/3} \left[\cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right) \right], k = 0, 1, 2$$

$$z_k = 2 \left[\cos\left(\frac{\pi + 4k\pi}{6}\right) + i\sin\left(\frac{\pi + 4k\pi}{6}\right) \right], k = 0, 1, 2$$

For $k=0$

$$z_0 = 2 \left[\cos\left(\frac{\pi+0}{6}\right) + i\sin\left(\frac{\pi+0}{6}\right) \right]$$

$$= 2 \left(\frac{\sqrt{3}}{2} + i\frac{1}{2} \right)$$

$$z_0 = \sqrt{3} + i$$

For $k=1$

$$z_1 = 2 \left[\cos\left(\frac{\pi+4\pi}{6}\right) + i\sin\left(\frac{\pi+4\pi}{6}\right) \right]$$

$$z_1 = 2 \left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) \right]$$

$$z_1 = 2 \left[-\frac{\sqrt{3}}{2} + i\frac{1}{2} \right]$$

$$z_1 = -\sqrt{3} + i$$

(16)

$$\begin{aligned}
 \text{For } k=2 \\
 z_2 &= 2 \left[\cos\left(\frac{\pi+8\pi}{6}\right) + i \sin\left(\frac{\pi+8\pi}{6}\right) \right] \\
 &= 2 \left[\cos\left(\frac{9\pi}{6}\right) + i \sin\left(\frac{9\pi}{6}\right) \right] \\
 &= 2 \left[\cos\left(3 \cdot \frac{\pi}{2}\right) + i \sin\left(3 \cdot \frac{\pi}{2}\right) \right] \\
 &= 2(0 + i(-1)) \\
 &= -2i
 \end{aligned}$$

Three roots of 8i are $\sqrt{3}+i$, $-\sqrt{3}+i$, $-2i$

Find four fourth roots of each of the complex numbers.

$-16i$ here $x=0$, $y=-16$, $r=|z|=16$
 $\cos\theta=0$, $\sin\theta=-1$
 θ is quadrant angle & $\theta = -\pi/2$

$$z_k = |z|^{1/4} \left[\cos\left(\frac{\theta+2k\pi}{4}\right) + i \sin\left(\frac{\theta+2k\pi}{4}\right) \right]; k=0,1,2,3$$

$$z_k = (16)^{1/4} \left[\cos\left(\frac{-\pi/2+2k\pi}{4}\right) + i \sin\left(\frac{-\pi/2+2k\pi}{4}\right) \right]; k=0,1,2,3$$

$$z_k = 2 \left[\cos\left(\frac{-\pi+4k\pi}{8}\right) + i \sin\left(\frac{-\pi+4k\pi}{8}\right) \right]; k=0,1,2,3$$

for $k=0$

$$z_0 = 2 \left[\cos\left(-\frac{\pi}{8}\right) + i \sin\left(-\frac{\pi}{8}\right) \right]$$

for $k=1$

$$z_1 = 2 \left[\cos\left(\frac{-\pi+4\pi}{8}\right) + i \sin\left(\frac{-\pi+4\pi}{8}\right) \right]$$

$$z_1 = 2 \left[\cos\left(\frac{3\pi}{8}\right) + i \sin\left(\frac{3\pi}{8}\right) \right]$$

for $k=2$

$$z_2 = 2 \left[\cos\left(\frac{-\pi+8\pi}{8}\right) + i \sin\left(\frac{-\pi+8\pi}{8}\right) \right]$$

$$z_2 = 2 \left[\cos\left(\frac{7\pi}{8}\right) + i \sin\left(\frac{7\pi}{8}\right) \right]$$

for $k=3$

$$z_3 = 2 \left[\cos\left(\frac{-\pi+12\pi}{8}\right) + i \sin\left(\frac{-\pi+12\pi}{8}\right) \right]$$

$$z_3 = 2 \left[\cos\left(\frac{11\pi}{8}\right) + i \sin\left(\frac{11\pi}{8}\right) \right]$$

(ii) 64

here $x=64$, $y=0$, $|z|=64$

$$\cos\theta = \frac{x}{r} = 1, \sin\theta = \frac{y}{r} = 0$$

θ is quadrant angle
 & $\theta = 0^\circ$

$$z_k = (64)^{1/4} \left[\cos\left(\frac{0+2k\pi}{4}\right) + i \sin\left(\frac{0+2k\pi}{4}\right) \right]; k=0,1,2,3$$

(17)

$$\begin{aligned}
 & (4^2)^{1/4} \cdot 4^{1/4} \\
 &= (2^4)^{1/4} (2^2)^{1/4} \\
 &= 2 \cdot \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 z_k &= (4^2 \cdot 4)^{1/4} \left[\cos\left(\frac{k\pi}{2}\right) + i \sin\left(\frac{k\pi}{2}\right) \right] ; k=0,1,2,3 \\
 &= 2\sqrt{2} \left[\cos\left(\frac{k\pi}{2}\right) + i \sin\left(\frac{k\pi}{2}\right) \right] ; k=0,1,2,3
 \end{aligned}$$

for $k=0$

$$z_0 = 2\sqrt{2} \left[\cos(0) + i \sin(0) \right]$$

$$z_0 = 2\sqrt{2}$$

for $k=2$

$$z_2 = 2\sqrt{2} \left[\cos\pi + i \sin\pi \right]$$

$$z_2 = -2\sqrt{2}$$

for $k=1$

$$z_1 = 2\sqrt{2} \left[\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right]$$

$$z_1 = 2\sqrt{2}(i)$$

for $k=3$

$$z_3 = 2\sqrt{2} \left[\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2} \right]$$

$$z_3 = -2\sqrt{2}i$$

Forth roots of 64 are $+2\sqrt{2}, \pm 2\sqrt{2}i$

$$(iii) -2\sqrt{3} + 2i$$

$$\text{here } x = -2\sqrt{3}, y = 2$$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{4(3) + 4} = \sqrt{12 + 4} = \sqrt{16} = 4 \Rightarrow |z| = 4$$

$$\cos\theta = \frac{x}{r} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$$

$$\sin\theta = \frac{y}{r} = \frac{2}{4} = \frac{1}{2}$$

θ lies in IInd quadrant.
reference Angle $= \alpha = \pi/6$

$$\theta = \pi - \alpha = \pi - \frac{\pi}{6} \Rightarrow \theta = \frac{5\pi}{6}$$

Formula

$$\begin{aligned}
 z_k &= (4)^{1/4} \left[\cos\left(\frac{5\pi/6 + 2k\pi}{4}\right) + i \sin\left(\frac{5\pi/6 + 2k\pi}{4}\right) \right] ; k=0,1,2,3 \\
 &= (2)^{2/4} \left[\cos\left(\frac{5\pi + 12k\pi}{24}\right) + i \sin\left(\frac{5\pi + 12k\pi}{24}\right) \right] ; k=0,1,2,3
 \end{aligned}$$

For $k=0$

$$z_0 = \sqrt{2} \left[\cos\left(\frac{5\pi}{24}\right) + i \sin\left(\frac{5\pi}{24}\right) \right]$$

$$z_1 = \sqrt{2} \left[\cos\left(\frac{5\pi + 12\pi}{24}\right) + i \sin\left(\frac{5\pi + 12\pi}{24}\right) \right] = \sqrt{2} \left[\cos\left(\frac{17\pi}{24}\right) + i \sin\left(\frac{17\pi}{24}\right) \right]$$

$$z_2 = \sqrt{2} \left[\cos\left(\frac{5\pi + 24\pi}{24}\right) + i \sin\left(\frac{5\pi + 24\pi}{24}\right) \right] = \sqrt{2} \left[\cos\left(\frac{29\pi}{24}\right) + i \sin\left(\frac{29\pi}{24}\right) \right]$$

$$z_3 = \sqrt{2} \left[\cos\left(\frac{5\pi + 36\pi}{24}\right) + i \sin\left(\frac{5\pi + 36\pi}{24}\right) \right] = \sqrt{2} \left[\cos\left(\frac{41\pi}{24}\right) + i \sin\left(\frac{41\pi}{24}\right) \right]$$

6. Find six 6th roots of

(i) -1

$z^6 = -1$

$z = -1 + 0i$

$r = \sqrt{1+0} = 1$

$x = -1$

$y = 0$

$\cos \theta = -1$

$\sin \theta = 0$

quadrantal angle $\theta = \pi$

$$z_k = (1)^{1/6} \left[\cos \left(\frac{\pi + 2k\pi}{6} \right) + i \sin \left(\frac{\pi + 2k\pi}{6} \right) \right], k = 0, 1, 2, 3, 4, 5$$

For $k=0$

$$z_0 = \cos \left(\frac{\pi}{6} \right) + i \sin \left(\frac{\pi}{6} \right) = \frac{\sqrt{3}}{2} + i \frac{1}{2}$$

$$z_1 = \cos \left(\frac{\pi + 2\pi}{6} \right) + i \sin \left(\frac{\pi + 2\pi}{6} \right) = \cos \left(\frac{3\pi}{6} \right) + i \sin \left(\frac{3\pi}{6} \right) = 0 + i$$

$$z_2 = \cos \left(\frac{\pi + 4\pi}{6} \right) + i \sin \left(\frac{\pi + 4\pi}{6} \right) = \cos \left(\frac{5\pi}{6} \right) + i \sin \left(\frac{5\pi}{6} \right) = -\frac{\sqrt{3}}{2} + i \frac{1}{2}$$

$$z_3 = \cos \left(\frac{\pi + 6\pi}{6} \right) + i \sin \left(\frac{\pi + 6\pi}{6} \right) = \cos \left(\frac{7\pi}{6} \right) + i \sin \left(\frac{7\pi}{6} \right) = -\frac{\sqrt{3}}{2} - i \frac{1}{2}$$

$$z_4 = \cos \left(\frac{\pi + 8\pi}{6} \right) + i \sin \left(\frac{\pi + 8\pi}{6} \right) = \cos \left(\frac{9\pi}{6} \right) + i \sin \left(\frac{9\pi}{6} \right) = 0 - i$$

$$z_5 = \cos \left(\frac{\pi + 10\pi}{6} \right) + i \sin \left(\frac{\pi + 10\pi}{6} \right) = \cos \left(\frac{11\pi}{6} \right) + i \sin \left(\frac{11\pi}{6} \right) = \frac{\sqrt{3}}{2} - i \frac{1}{2}$$

(ii) $1 + i = z$

Here $x = 1, y = 1, |z| = \sqrt{1+1} = \sqrt{2}$

$\cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = \frac{1}{\sqrt{2}}$

 θ lies in 1st quadrant

$\alpha = \theta = \pi/4$

$$z_k = (\sqrt{2})^{1/6} \left[\cos \left(\frac{\pi/4 + 2k\pi}{6} \right) + i \sin \left(\frac{\pi/4 + 2k\pi}{6} \right) \right], k = 0, 1, 2, 3, 4, 5$$

$$= (2)^{1/12} \left[\cos \left(\frac{\pi + 8k\pi}{24} \right) + i \sin \left(\frac{\pi + 8k\pi}{24} \right) \right], k = 0, 1, 2, 3, 4, 5$$

$$z_0 = (2)^{1/12} \left[\cos \left(\frac{\pi}{24} \right) + i \sin \left(\frac{\pi}{24} \right) \right] = (2)^{1/12} \text{Cis} \left(\frac{\pi}{24} \right)$$

$$z_1 = (2)^{1/12} \left[\cos \left(\frac{\pi + 8\pi}{24} \right) + i \sin \left(\frac{\pi + 8\pi}{24} \right) \right] = (2)^{1/12} \text{Cis} \left(\frac{9\pi}{24} \right)$$

$$z_2 = (2)^{1/12} \left[\cos \left(\frac{\pi + 16\pi}{24} \right) + i \sin \left(\frac{\pi + 16\pi}{24} \right) \right] = (2)^{1/12} \text{Cis} \left(\frac{17\pi}{24} \right)$$

$$z_3 = (2)^{1/12} \left[\cos \left(\frac{\pi + 24\pi}{24} \right) + i \sin \left(\frac{\pi + 24\pi}{24} \right) \right] = (2)^{1/12} \text{Cis} \left(\frac{25\pi}{24} \right)$$

$$z_4 = (2)^{1/12} \left[\cos \left(\frac{\pi + 32\pi}{24} \right) + i \sin \left(\frac{\pi + 32\pi}{24} \right) \right] = (2)^{1/12} \text{Cis} \left(\frac{33\pi}{24} \right)$$

$$z_5 = (2)^{1/12} \left[\cos \left(\frac{\pi + 40\pi}{24} \right) + i \sin \left(\frac{\pi + 40\pi}{24} \right) \right] = (2)^{1/12} \text{Cis} \left(\frac{41\pi}{24} \right)$$

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7. Find the squares of all 5th roots of

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Sol. $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} \Rightarrow |z| = 1$

$$\cos \theta = \frac{1}{2}, \quad \sin \theta = \frac{\sqrt{3}}{2}$$

θ is in 1st quadrant, $\theta = \alpha = \pi/3$
for five roots.

$$z_k = (1)^{1/5} \left[\cos\left(\frac{\pi/3 + 2k\pi}{5}\right) + i \sin\left(\frac{\pi/3 + 2k\pi}{5}\right) \right], k=0,1,2,3,4$$

$$z_k = \left[\cos\left(\frac{\pi + 6k\pi}{15}\right) + i \sin\left(\frac{\pi + 6k\pi}{15}\right) \right], k=0,1,2,3,4$$

Taking square on both side $\therefore$ (we find squares of 5th roots)

$$z_k^2 = \left[\cos\left(\frac{\pi + 6k\pi}{15}\right) + i \sin\left(\frac{\pi + 6k\pi}{15}\right) \right]^2$$

$$z_k^2 = \cos 2\left(\frac{\pi + 6k\pi}{15}\right) + i \sin 2\left(\frac{\pi + 6k\pi}{15}\right) \quad \text{by De-Moivre's Theorem}$$

$$z_k^2 = \cos\left(\frac{2\pi + 12k\pi}{15}\right) + i \sin\left(\frac{2\pi + 12k\pi}{15}\right)$$

for $k=0, 2$

$$z_0^2 = \cos\left(\frac{2\pi}{15}\right) + i \sin\left(\frac{2\pi}{15}\right) = \text{Cis}\left(\frac{2\pi}{15}\right)$$

for $k=1$

$$z_1^2 = \cos\left(\frac{2\pi + 12\pi}{15}\right) + i \sin\left(\frac{2\pi + 12\pi}{15}\right) = \text{Cis}\left(\frac{14\pi}{15}\right)$$

for $k=2$

$$z_2^2 = \cos\left(\frac{2\pi + 24\pi}{15}\right) + i \sin\left(\frac{2\pi + 24\pi}{15}\right) = \text{Cis}\left(\frac{26\pi}{15}\right)$$

for $k=3$

$$z_3^2 = \cos\left(\frac{2\pi + 36\pi}{15}\right) + i \sin\left(\frac{2\pi + 36\pi}{15}\right) = \text{Cis}\left(\frac{38\pi}{15}\right)$$

for $k=4$

$$z_4^2 = \cos\left(\frac{2\pi + 48\pi}{15}\right) + i \sin\left(\frac{2\pi + 48\pi}{15}\right) = \text{Cis}\left(\frac{50\pi}{15}\right)$$

8. Solve the following equations.

(i) $x^7 + 1 = 0$

$$x^7 = (-1)$$

$$x = (-1)^{1/7}$$

Let

$$z = 1 + 0i$$

$$x = -1$$

$$y = 0$$

$$|z| = 1$$

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$$\left. \begin{aligned} \cos \theta &= \frac{x}{r} = -1 \\ \sin \theta &= \frac{y}{r} = 0 \end{aligned} \right\} \text{quadrantal Angle.}$$

$$\theta = \pi$$

$$Z_k = (1)^{1/7} \left[\cos \left(\frac{\pi + 2k\pi}{7} \right) + i \sin \left(\frac{\pi + 2k\pi}{7} \right) \right], k=0, 1, 2, 3, 4, 5, 6$$

for $k=0$

$$Z_0 = \left[\cos \left(\frac{\pi}{7} \right) + i \sin \left(\frac{\pi}{7} \right) \right] = \text{Cis} \left(\frac{\pi}{7} \right)$$

for $k=1$

$$Z_1 = \left[\cos \left(\frac{\pi + 2\pi}{7} \right) + i \sin \left(\frac{\pi + 2\pi}{7} \right) \right] = \text{Cis} \left(\frac{3\pi}{7} \right)$$

for $k=2$

$$Z_2 = \left[\cos \left(\frac{\pi + 4\pi}{7} \right) + i \sin \left(\frac{\pi + 4\pi}{7} \right) \right] = \text{Cis} \left(\frac{5\pi}{7} \right)$$

for $k=3$

$$Z_3 = \left[\cos \left(\frac{\pi + 6\pi}{7} \right) + i \sin \left(\frac{\pi + 6\pi}{7} \right) \right] = \cos(\pi) + i \sin(\pi)$$

$$Z_3 = -1 + 0i = -1$$

for $k=4$

$$Z_4 = \left[\cos \left(\frac{\pi + 8\pi}{7} \right) + i \sin \left(\frac{\pi + 8\pi}{7} \right) \right] = \text{Cis} \left(\frac{9\pi}{7} \right)$$

for $k=5$

$$Z_5 = \left[\cos \left(\frac{\pi + 10\pi}{7} \right) + i \sin \left(\frac{\pi + 10\pi}{7} \right) \right] = \text{Cis} \left(\frac{11\pi}{7} \right)$$

for $k=6$

$$Z_6 = \left[\cos \left(\frac{\pi + 12\pi}{7} \right) + i \sin \left(\frac{\pi + 12\pi}{7} \right) \right] = \text{Cis} \left(\frac{13\pi}{7} \right)$$

$$(ii) \quad x^7 + x^4 + x^3 + 1 = 0$$

$$x^4(x^3 + 1) + 1(x^3 + 1) = 0$$

$$(x^3 + 1)(x^4 + 1) = 0$$

$$\text{Either } x^3 + 1 = 0$$

$$x^3 = -1$$

$$x = (-1)^{1/3}$$

$$; \quad x^4 + 1 = 0$$

$$; \quad x^4 = -1$$

$$; \quad x = (-1)^{1/4}$$

$$\text{Let } Z = (-1)$$

$$x = -1, \quad y = 0, \quad |Z| = 1$$

$$\left. \begin{aligned} \cos \theta &= -1 \\ \sin \theta &= 0 \end{aligned} \right\} \quad \theta = \pi$$

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$$z_k = (1)^{1/3} \left[\cos\left(\frac{\pi+2k\pi}{3}\right) + i \sin\left(\frac{\pi+2k\pi}{3}\right) \right], k=0,1,2$$

$$z_k = \cos\left(\frac{\pi+2k\pi}{3}\right) + i \sin\left(\frac{\pi+2k\pi}{3}\right)$$

for $k=0$

$$z_0 = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i \frac{\sqrt{3}}{2}$$

for $k=1$

$$z_1 = \cos\left(\frac{\pi+2\pi}{3}\right) + i \sin\left(\frac{\pi+2\pi}{3}\right) = \cos(\pi) + i \sin(\pi) = -1$$

for $k=2$

$$z_2 = \cos\left(\frac{\pi+4\pi}{3}\right) + i \sin\left(\frac{\pi+4\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right)$$

$$z_2 = \frac{1}{2} - i \frac{\sqrt{3}}{2}$$

When:

$$x = (-1)^{1/4}, \text{ Let } z = -1, |z| = 1$$

$$\left. \begin{array}{l} \cos \theta = -1 \\ \sin \theta = 0 \end{array} \right\} \theta = \pi$$

$$z_k = (1)^{1/4} \left[\cos\left(\frac{\pi+2k\pi}{4}\right) + i \sin\left(\frac{\pi+2k\pi}{4}\right) \right]; k=0,1,2,3$$

for $k=0$

$$z_0 = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$$

for $k=1$

$$z_1 = \cos\left(\frac{\pi+2\pi}{4}\right) + i \sin\left(\frac{\pi+2\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)$$

$$z_1 = -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$$

for $k=2$

$$z_2 = \cos\left(\frac{\pi+4\pi}{4}\right) + i \sin\left(\frac{\pi+4\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)$$

$$z_2 = -\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}$$

for $k=3$

$$z_3 = \cos\left(\frac{\pi+6\pi}{4}\right) + i \sin\left(\frac{\pi+6\pi}{4}\right) = \cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right)$$

$$z_3 = \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}$$

(22)

(iii)

$$x^6 + 1 = \sqrt{3}i$$

$$x^6 = -1 + \sqrt{3}i$$

$$x = (-1 + \sqrt{3}i)^{1/6}$$

Let

$$z = -1 + \sqrt{3}i$$

$$x = -1, y = \sqrt{3}, |z| = \sqrt{1+3} = \sqrt{4} \Rightarrow |z| = r$$

$$\cos \theta = -1/2, \theta \in \text{II quadrant}$$

$$\sin \theta = \sqrt{3}/2 \text{ reference angle} = \alpha = \pi/3$$

$$\theta = \pi - \alpha = \pi - \pi/3 = \frac{2\pi}{3}$$

$$x_k = (2)^{1/6} \left[\cos \left(\frac{2\pi/3 + 2k\pi}{6} \right) + i \sin \left(\frac{2\pi/3 + 2k\pi}{6} \right) \right]; k=0,1,2,3,4,5$$

$$x_k = (2)^{1/6} \left[\cos \left(\frac{2\pi + 6k\pi}{18} \right) + i \sin \left(\frac{2\pi + 6k\pi}{18} \right) \right]$$

$$x_k = (2)^{1/6} \left[\cos \left(\frac{\pi + 3k\pi}{9} \right) + i \sin \left(\frac{\pi + 3k\pi}{9} \right) \right]; k=0,1,2,3,4,5$$

for $k=0$

$$x_0 = (2)^{1/6} \left[\cos \left(\frac{\pi}{9} \right) + i \sin \left(\frac{\pi}{9} \right) \right] = (2)^{1/6} \text{Cis} \left(\frac{\pi}{9} \right)$$

for $k=1$

$$x_1 = (2)^{1/6} \left[\cos \left(\frac{\pi + 3\pi}{9} \right) + i \sin \left(\frac{\pi + 3\pi}{9} \right) \right] = (2)^{1/6} \text{Cis} \left(\frac{4\pi}{9} \right)$$

for $k=2$

$$x_2 = (2)^{1/6} \left[\cos \left(\frac{\pi + 6\pi}{9} \right) + i \sin \left(\frac{\pi + 6\pi}{9} \right) \right] = (2)^{1/6} \text{Cis} \left(\frac{7\pi}{9} \right)$$

for $k=3$

$$x_3 = (2)^{1/6} \left[\cos \left(\frac{\pi + 9\pi}{9} \right) + i \sin \left(\frac{\pi + 9\pi}{9} \right) \right] = (2)^{1/6} \text{Cis} \left(\frac{10\pi}{9} \right)$$

for $k=4$

$$x_4 = (2)^{1/6} \left[\cos \left(\frac{\pi + 12\pi}{9} \right) + i \sin \left(\frac{\pi + 12\pi}{9} \right) \right] = (2)^{1/6} \text{Cis} \left(\frac{13\pi}{9} \right)$$

for $k=5$

$$x_5 = (2)^{1/6} \left[\cos \left(\frac{\pi + 15\pi}{9} \right) + i \sin \left(\frac{\pi + 15\pi}{9} \right) \right] = (2)^{1/6} \text{Cis} \left(\frac{16\pi}{9} \right)$$

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9. Solve the equation $x^{12} - 1$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$

Sol. $x^{12} - 1 = 0 \Rightarrow (x^6)^2 - (1)^2 = 0$
 $(x^6 - 1)(x^6 + 1) = 0$

Either $x^6 - 1 = 0$ or $x^6 + 1 = 0$

$\Rightarrow x^6 = 1$
 $\Rightarrow x = (1)^{1/6}$

here $z = 1$

$x = 1, y = 0, |z| = 1$
 $\cos \theta = 1$
 $\sin \theta = 0$ } quadrantal angle,
 $\theta = 0^\circ$

$x_k = (1)^{1/6} \left[\cos\left(\frac{2k\pi}{6}\right) + i \sin\left(\frac{2k\pi}{6}\right) \right]$

$x_k = \cos\left(\frac{k\pi}{3}\right) + i \sin\left(\frac{k\pi}{3}\right)$

$k = 0, 1, 2, 3, 4, 5$

for $k = 0$

$x_0 = \cos(0) + i \sin(0) = 1$

for $k = 1$

$x_1 = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i \frac{\sqrt{3}}{2}$

for $k = 2$

$x_2 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$

for $k = 3$

$x_3 = \cos\left(\frac{3\pi}{3}\right) + i \sin\left(\frac{3\pi}{3}\right) = -1$

for $k = 4$

$x_4 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$

for $k = 5$

$x_5 = \cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i \frac{\sqrt{3}}{2}$

$x^6 = -1$
 $x = (-1)^{1/6}$

Same Solution as in Q.6 part (i)

$z_0 = \frac{\sqrt{3}}{2} + \frac{i}{2}$

$z_1 = i$

$z_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$z_3 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$

$z_4 = -i$

$z_5 = \frac{\sqrt{3}}{2} - \frac{i}{2}$

So roots of $x^{12} - 1 = 0$ are

$\pm 1, \pm i, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$\frac{\sqrt{3}}{2} \pm \frac{i}{2}$ and $-\frac{\sqrt{3}}{2} \pm \frac{i}{2}$

Now $x^6 - 1 = 0$

$\Rightarrow (x^2 - 1)(x^4 + x^2 - 1) = 0$

$x^2 - 1 = 0, x^4 + x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

Hence roots of

$x^4 + x^2 - 1 = 0$ are

$\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

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10. Express the following in series of sines or cosines of multiples of θ .

(i) $\cos^4 \theta$

$$x = \cos \theta + i \sin \theta$$

$$2 \cos \theta = x + \frac{1}{x}$$

$$(2 \cos \theta)^4 = \left(x + \frac{1}{x}\right)^4$$

$$2^4 \cos^4 \theta = \binom{4}{0} x^4 \left(\frac{1}{x}\right)^0 + \binom{4}{1} x^3 \frac{1}{x} + \binom{4}{2} x^2 \frac{1}{x^2} + \binom{4}{3} x \frac{1}{x^3} + \binom{4}{0} x^0 \frac{1}{x^4}$$

$$2^4 \cos^4 \theta = x^4 + 4x^2 + 6 + 4\left(\frac{1}{x^2}\right) + \frac{1}{x^4}$$

$$= \left(x^4 + \frac{1}{x^4}\right) + 4\left(x^2 + \frac{1}{x^2}\right) + 6$$

$$2^4 \cos^4 \theta = 2 \cos 4\theta + 4(2 \cos 2\theta) + 6$$

$$2^4 \cos^4 \theta = 2(\cos 4\theta + 4 \cos 2\theta + 3)$$

$$2^3 \cos^4 \theta = \cos 4\theta + 4 \cos 2\theta + 3$$

$$\cos^4 \theta = \frac{1}{8} (\cos 4\theta + 4 \cos 2\theta + 3)$$

(ii) $\sin^4 \theta$

$$x = \cos \theta + i \sin \theta$$

$$x - \frac{1}{x} = 2i \sin \theta$$

$$\therefore i^4 = (i^2)^2 = (-1)^2 = 1$$

$$(2i \sin \theta)^4 = \left(x - \frac{1}{x}\right)^4$$

$$2^4 i^4 \sin^4 \theta = \binom{4}{0} x^4 \left(\frac{1}{x}\right)^0 - \binom{4}{1} x^3 \frac{1}{x} + \binom{4}{2} x^2 \frac{1}{x^2} - \binom{4}{3} x \frac{1}{x^3} + \binom{4}{4} x^0 \frac{1}{x^4}$$

$$\sin^4 \theta = \frac{1}{2^4} \left[x^4 - 4x^2 + 6 - 4\left(\frac{1}{x^2}\right) + \frac{1}{x^4} \right]$$

$$= \frac{1}{2^4} \left[\left(x^4 + \frac{1}{x^4}\right) - 4\left(x^2 + \frac{1}{x^2}\right) + 6 \right]$$

$$= \frac{1}{2^4} [2 \cos 4\theta - 4(2 \cos 2\theta) + 6]$$

$$= \frac{2}{2^4} [\cos 4\theta - 4 \cos 2\theta + 3] = \frac{1}{2^3} [\cos 4\theta - 4 \cos 2\theta + 3]$$

$$\sin^4 \theta = \frac{1}{8} [\cos 4\theta - 4 \cos 2\theta + 3]$$

(iii) $\sin^6 \theta$

$$x = \cos \theta + i \sin \theta$$

$$2i \sin \theta = x - \frac{1}{x}$$

$$(2i \sin \theta)^6 = \left(x - \frac{1}{x}\right)^6$$

$$2^6 i^6 \sin^6 \theta = \binom{6}{0} x^6 \left(\frac{1}{x}\right)^0 - \binom{6}{1} x^5 \frac{1}{x} + \binom{6}{2} x^4 \frac{1}{x^2} - \binom{6}{3} x^3 \frac{1}{x^3} + \binom{6}{4} x^2 \frac{1}{x^4} - \binom{6}{5} x \frac{1}{x^5} + \binom{6}{6} x^0 \frac{1}{x^6}$$

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$$i^6 = (i^2)^3 = (-1)^3 = -1$$

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$$\begin{aligned} -\sin^6 \theta &= \frac{1}{2^6} \left[x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6} \right] \\ &= \frac{1}{2^6} \left[\left(x^6 + \frac{1}{x^6} \right) - 6 \left(x^4 + \frac{1}{x^4} \right) + 15 \left(x^2 + \frac{1}{x^2} \right) - 20 \right] \\ &= \frac{1}{64} \left[2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20 \right] \\ &= \frac{2}{64} \left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10 \right] \\ \sin^6 \theta &= \frac{1}{32} \left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10 \right] \quad \text{Ans.} \end{aligned}$$

(iv) $\cos^7 \theta$

$$\begin{aligned} x &= \cos \theta + i \sin \theta \\ x + \frac{1}{x} &= 2\cos \theta \end{aligned}$$

$$\begin{aligned} (2\cos \theta)^7 &= \left(x + \frac{1}{x} \right)^7 \\ 2^7 \cos^7 \theta &= \binom{7}{0} x^7 \left(\frac{1}{x} \right)^0 + \binom{7}{1} x^6 \frac{1}{x} + \binom{7}{2} x^5 \left(\frac{1}{x^2} \right) + \binom{7}{3} x^4 \frac{1}{x^3} \\ &\quad + \binom{7}{4} x^3 \frac{1}{x^4} + \binom{7}{5} x^2 \frac{1}{x^5} + \binom{7}{6} x \frac{1}{x^6} + \binom{7}{7} x^0 \frac{1}{x^7} \\ 2^7 \cos^7 \theta &= x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7} \\ &= \left(x^7 + \frac{1}{x^7} \right) + 7 \left(x^5 + \frac{1}{x^5} \right) + 21 \left(x^3 + \frac{1}{x^3} \right) + 35 \left(x + \frac{1}{x} \right) \\ &= 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos \theta) \\ \cos^7 \theta &= \frac{1}{2^7} (2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos \theta)) \\ \cos^7 \theta &= \frac{1}{2^6} (\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos \theta) \quad \text{Ans.} \end{aligned}$$

(v) $\sin^9 \theta$

$$x - \frac{1}{x} = 2i \sin \theta$$

$$(2i \sin \theta)^9 = \left(x - \frac{1}{x} \right)^9$$

$$\begin{aligned} i^9 &= (i^2)^4 i \\ &= (-1)^4 i = i \end{aligned}$$

$$\begin{aligned} 2^9 i^9 \sin^9 \theta &= \binom{9}{0} x^9 \left(\frac{1}{x} \right)^0 - \binom{9}{1} x^8 \frac{1}{x} + \binom{9}{2} x^7 \frac{1}{x^2} - \binom{9}{3} x^6 \frac{1}{x^3} + \binom{9}{4} x^5 \frac{1}{x^4} \\ &\quad - \binom{9}{5} x^4 \frac{1}{x^5} + \binom{9}{6} x^3 \frac{1}{x^6} - \binom{9}{7} x^2 \frac{1}{x^7} + \binom{9}{8} x \frac{1}{x^8} - \binom{9}{9} x^0 \frac{1}{x^9} \\ 2^9 i \sin^9 \theta &= x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x} + \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9} \\ &= \left(x^9 - \frac{1}{x^9} \right) - 9 \left(x^7 - \frac{1}{x^7} \right) + 36 \left(x^5 - \frac{1}{x^5} \right) - 84 \left(x^3 - \frac{1}{x^3} \right) + 126 \left(x - \frac{1}{x} \right) \\ &= 2i \sin 9\theta - 9(2i \sin 7\theta) + 36(2i \sin 5\theta) - 84(2i \sin 3\theta) \\ &\quad + 126(2i \sin \theta) \\ 2^9 i \sin^9 \theta &= 2i (\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin \theta) \\ \sin^9 \theta &= \frac{1}{2^8} [\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin \theta] \end{aligned}$$

(26)

$$i^6 = (i^2)^3 = (-1)^3 = -1$$

(vi) $\sin^6 \theta \cos^2 \theta$

$$x + \frac{1}{x} = 2\cos \theta$$

$$x - \frac{1}{x} = 2i\sin \theta$$

$$(2\cos \theta)^2 (2i\sin \theta)^6 = \left(x + \frac{1}{x}\right)^2 \left(x - \frac{1}{x}\right)^6$$

$$2^4 \cos^2 \theta \cdot 2^6 i^6 \sin^6 \theta = \left(x + \frac{1}{x}\right)^2 \left(x - \frac{1}{x}\right)^2 \left(x - \frac{1}{x}\right)^4$$

$$-2^8 \sin^6 \theta \cos^2 \theta = \left(\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)\right)^2 \left(x - \frac{1}{x}\right)^4$$

$$= \left(x^2 - \frac{1}{x^2}\right)^2 \left(x - \frac{1}{x}\right)^4$$

$$-2^8 \sin^6 \theta \cos^2 \theta = \left(x^4 + \frac{1}{x^4} - 2\right) \left[\binom{4}{0} x^4 \cdot \frac{1}{x^0} - \binom{4}{1} x^3 \cdot \frac{1}{x} + \binom{4}{2} x^2 \cdot \frac{1}{x^2} - \binom{4}{3} x \cdot \frac{1}{x^3} + \binom{4}{4} x^0 \cdot \frac{1}{x^4} \right]$$

$$= \left(x^4 + \frac{1}{x^4} - 2\right) \left(x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}\right)$$

$$= x^8 - 4x^6 + 6x^4 - 4x^2 + 1 + 1 - \frac{4}{x^2} + \frac{6}{x^4} - \frac{4}{x^6} + \frac{1}{x^8}$$

$$- 2x^4 + 8x^2 - 12 + \frac{8}{x^2} - \frac{2}{x^4}$$

$$-2^8 \sin^6 \theta \cos^2 \theta = \left(x^8 + \frac{1}{x^8}\right) - 4\left(x^6 + \frac{1}{x^6}\right) + 4\left(x^4 + \frac{1}{x^4}\right) + 4\left(x^2 + \frac{1}{x^2}\right) - 10$$

$$\sin^6 \theta \cos^2 \theta = -\frac{1}{2^8} \left[2\cos 8\theta - 4(2\cos 6\theta) + 4(2\cos 4\theta) + 4(2\cos 2\theta) - 10 \right]$$

$$= -\frac{2}{2^8} [\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5]$$

$$\sin^6 \theta \cos^2 \theta = \frac{1}{2^7} [\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5]$$

(vii) $\cos^4 \theta \sin^3 \theta$

$$i^3 = i i^2 = -i$$

$$(2\cos \theta)^4 (2i\sin \theta)^3 = \left(x + \frac{1}{x}\right)^4 \left(x - \frac{1}{x}\right)^3$$

$$2^4 \cos^4 \theta \cdot 2^3 i^3 \sin^3 \theta = \left(x + \frac{1}{x}\right)^3 \left(x - \frac{1}{x}\right)^3 \left(x + \frac{1}{x}\right)$$

$$-2^7 i \cos^4 \theta \sin^3 \theta = \left(x^2 - \frac{1}{x^2}\right)^3 \left(x + \frac{1}{x}\right)$$

$$= \left[\binom{3}{0} (x^2)^3 \cdot \frac{1}{(x^2)^0} - \binom{3}{1} (x^2)^2 \cdot \frac{1}{x^2} + \binom{3}{2} (x^2) \cdot \frac{1}{(x^2)^2} - \binom{3}{3} (x^2)^0 \cdot \frac{1}{(x^2)^3} \right] \left(x + \frac{1}{x}\right)$$

$$= \left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6} \right] \left(x + \frac{1}{x}\right)$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} + x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}$$

$$-2^7 i \cos^4 \theta \sin^3 \theta = \left(x^7 - \frac{1}{x^7}\right) - 3\left(x^3 - \frac{1}{x^3}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x - \frac{1}{x}\right)$$

$$\cos^4 \theta \sin^3 \theta = -\frac{1}{2^7 i} \left[2i\sin 7\theta - 3(2i\sin 3\theta) + 2i\sin 5\theta - 3(2i\sin \theta) \right]$$

(27)

1

$$= -\frac{2\sqrt{2}}{2^{2\sqrt{2}}} [\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin \theta]$$

$$\cos^4 \theta \sin^3 \theta = -\frac{1}{2^6} [\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin \theta]$$

(viii) $\cos^5 \theta \sin^7 \theta$

$$i^7 = (i^2)^3 \cdot i = -i$$

$$\begin{aligned} (2\cos\theta)^5 (2i\sin\theta)^7 &= \left(x + \frac{1}{x}\right)^5 \left(x - \frac{1}{x}\right)^7 \\ 2^5 \cos^5 \theta \cdot 2^7 i^7 \sin^7 \theta &= \left(x + \frac{1}{x}\right)^5 \left(x - \frac{1}{x}\right)^5 \left(x - \frac{1}{x}\right)^2 \\ -2^{12} i \cos^5 \theta \sin^7 \theta &= \left(x^2 - \frac{1}{x^2}\right)^5 \left(x^2 + \frac{1}{x^2} - 2\right) \\ &= \left[\binom{5}{0} (x^2)^5 \left(\frac{1}{x^2}\right)^0 - \binom{5}{1} (x^2)^4 \left(\frac{1}{x^2}\right) + \binom{5}{2} (x^2)^3 \left(\frac{1}{x^2}\right)^2 - \binom{5}{3} (x^2)^2 \left(\frac{1}{x^2}\right)^3 \right. \\ &\quad \left. + \binom{5}{4} (x^2) \left(\frac{1}{x^2}\right)^4 - \binom{5}{5} (x^2)^0 \left(\frac{1}{x^2}\right)^5 \right] \left(x^2 + \frac{1}{x^2} - 2\right) \end{aligned}$$

$$= \left(x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right) \left(x^2 + \frac{1}{x^2} - 2\right)$$

$$= x^{12} - 5x^8 + 10x^4 - \frac{5}{x^4} - \frac{1}{x^8} - 5x^4 + 10 - \frac{10}{x^4} + \frac{5}{x^8} - \frac{1}{x^{12}} - 2x^{10} + 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} - \frac{2}{x^{10}}$$

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

$$-2^{12} i \cos^5 \theta \sin^7 \theta = 2i \sin 12\theta - 2(2i \sin 10\theta) - 4(2i \sin 8\theta) + 10(2i \sin 6\theta) + 5(2i \sin 4\theta) - 20(2i \sin 2\theta)$$

$$-2^{12} i \cos^5 \theta \sin^7 \theta = 2i (\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)$$

$$\cos^5 \theta \sin^7 \theta = -\frac{1}{2^{11}} (\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)$$

11. Show that $\cos^4 \theta + \sin^4 \theta = \frac{1}{4} (3 + \cos 4\theta)$

From question 10 part (i) & (ii)

$$\cos^4 \theta = \frac{1}{8} (\cos 4\theta + 4\cos 2\theta + 3) \rightarrow \text{A}$$

$$\sin^4 \theta = \frac{1}{8} (\cos 4\theta - 4\cos 2\theta + 3) \rightarrow \text{B}$$

Adding A and B

$$\begin{aligned} \cos^4 \theta + \sin^4 \theta &= \frac{1}{8} (\cos 4\theta + 4\cos 2\theta + 3 + \cos 4\theta - 4\cos 2\theta + 3) \\ &= \frac{1}{8} (2\cos 4\theta + 6) \end{aligned}$$

$$= \frac{x}{84} (\cos 4\theta + 3)$$

$$\cos^4 \theta + \sin^4 \theta = \frac{1}{4} (3 + \cos 4\theta) \quad \text{Proved}$$

12. Prove that.

$$64 (\cos^8 \theta + \sin^8 \theta) = \cos^8 \theta + 28 \cos 4\theta + 35$$

Proof:

$$(2\cos \theta)^8 = \left(x + \frac{1}{x}\right)^8$$

$$2^8 \cos^8 \theta = \binom{8}{0} x^8 \cdot \frac{1}{x^0} + \binom{8}{1} x^7 \cdot \frac{1}{x} + \binom{8}{2} x^6 \cdot \frac{1}{x^2} + \binom{8}{3} x^5 \cdot \frac{1}{x^3} + \binom{8}{4} x^4 \cdot \frac{1}{x^4}$$

$$+ \binom{8}{5} x^3 \cdot \frac{1}{x^5} + \binom{8}{6} x^2 \cdot \frac{1}{x^6} + \binom{8}{7} x \cdot \frac{1}{x^7} + \binom{8}{8} x^0 \cdot \frac{1}{x^8}$$

$$2^8 \cos^8 \theta = x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + \frac{56}{x^2} + \frac{28}{x^4} + \frac{8}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$$2^2 \cdot 2^6 \cos^8 \theta = 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70 \rightarrow (1)$$

$$(2i\sin \theta)^8 = \left(x - \frac{1}{x}\right)^8$$

$$i^8 = (i^2)^4 = 1$$

$$2^8 i^8 \sin^8 \theta = \binom{8}{0} x^8 \cdot \frac{1}{x^0} - \binom{8}{1} x^7 \cdot \frac{1}{x} + \binom{8}{2} x^6 \cdot \frac{1}{x^2} - \binom{8}{3} x^5 \cdot \frac{1}{x^3} + \binom{8}{4} x^4 \cdot \frac{1}{x^4}$$

$$- \binom{8}{5} x^3 \cdot \frac{1}{x^5} + \binom{8}{6} x^2 \cdot \frac{1}{x^6} - \binom{8}{7} x \cdot \frac{1}{x^7} + \binom{8}{8} x^0 \cdot \frac{1}{x^8}$$

$$= x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - \frac{56}{x^2} + \frac{28}{x^4} - \frac{8}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) - 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) - 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$$2^2 \cdot 2^6 \sin^8 \theta = 2\cos 8\theta - 8(2\cos 6\theta) + 28(2\cos 4\theta) - 56(2\cos 2\theta) + 70 \rightarrow (2)$$

adding (1) & (2)

$$2^2 \cdot 2^6 [\cos^8 \theta + \sin^8 \theta] = 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta)$$

$$+ 70 + 2\cos 8\theta - 8(2\cos 6\theta) + 28(2\cos 4\theta) - 56(2\cos 2\theta) + 70$$

$$64 [\cos^8 \theta + \sin^8 \theta] = 1 [4\cos 8\theta + 4(28)\cos 4\theta + 2(70)]$$

$$= \frac{1}{2^2} (2^2) [\cos 8\theta + 28\cos 4\theta + 35]$$

$$64 [\cos^8 \theta + \sin^8 \theta] = \cos 8\theta + 28\cos 4\theta + 35$$

PROVED

13. Prove.

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Sol. $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= \cos^3\theta + i^3\sin^3\theta + 3(\cos\theta)(i\sin\theta)(\cos\theta + i\sin\theta) \\ &= \cos^3\theta - i\sin^3\theta + 3\cos^2\theta\sin\theta i - 3\cos\theta\sin^2\theta\end{aligned}$$

$$\cos 3\theta + i\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Equating real and imaginary part.

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$

$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(iii) $\sin 4\theta = 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

Sol. $(i\sin\theta + \cos\theta)^4 = (\cos\theta + i\sin\theta)^4$

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= \binom{4}{0}(\cos\theta)^4(i\sin\theta)^0 + \binom{4}{1}(\cos\theta)^3(i\sin\theta) + \binom{4}{2}(\cos\theta)^2(i\sin\theta)^2 \\ &\quad + \binom{4}{3}(\cos\theta)(i\sin\theta)^3 + \binom{4}{4}(\cos\theta)^0(i\sin\theta)^4\end{aligned}$$

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6\cos^2\theta\sin^2\theta i^2 + 4\cos\theta\sin^3\theta i^3 + \sin^4\theta$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + (4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)i + \sin^4\theta$$

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)\end{aligned}$$

Equating real and imaginary part

$\cos 4\theta = \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta$

$= \cos^4\theta + (\sin^2\theta)^2 - 6\cos^2\theta(1 - \cos^2\theta)$

$= \cos^4\theta + (1 - \cos^2\theta)^2 - 6\cos^2\theta + 6\cos^4\theta$

$= 7\cos^4\theta + 1 + \cos^4\theta - 2\cos^2\theta - 6\cos^2\theta$

$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

proved

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$= 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$

Proved.

(30)

$$(v) \frac{\sin 5\theta}{\sin \theta} = 16\cos^4 \theta - 12\cos^2 \theta + 1$$

$$\begin{aligned} (\cos 5\theta + i \sin 5\theta) &= (\cos \theta + i \sin \theta)^5 \\ &= \binom{5}{0}(\cos \theta)^5(i \sin \theta)^0 + \binom{5}{1}(\cos \theta)^4(i \sin \theta)^1 \\ &\quad + \binom{5}{2}(\cos \theta)^3(i \sin \theta)^2 + \binom{5}{3}(\cos \theta)^2(i \sin \theta)^3 + \binom{5}{4}(\cos \theta)(i \sin \theta)^4 + \binom{5}{5}(\cos \theta)^0(i \sin \theta)^5 \\ &= \cos^5 \theta + 5\cos^4 \theta \sin \theta i + 10\cos^3 \theta \sin^2 \theta i^2 + 10\cos^2 \theta \sin^3 \theta i^3 \\ &\quad + 5\cos \theta \sin^4 \theta i^4 + \sin^5 \theta i^5 \\ &= (\cos^5 \theta + 5\cos \theta \sin^4 \theta - 10\cos^3 \theta \sin^2 \theta) + i(5\cos^4 \theta \sin \theta - 10\cos^2 \theta \sin^3 \theta + \sin^5 \theta) \end{aligned}$$

Equating real and imaginary part

Imaginary Part:

$$\begin{aligned} \sin 5\theta &= 5\cos^4 \theta \sin \theta - 10\cos^2 \theta \sin^3 \theta + \sin^5 \theta \rightarrow (\sin^2 \theta)^2 \\ \sin 5\theta &= \sin \theta (5\cos^4 \theta - 10\cos^2 \theta \sin^2 \theta + \sin^4 \theta) \\ \sin 5\theta &= \sin \theta (5\cos^4 \theta - 10\cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2) \\ &= \sin \theta [5\cos^4 \theta - 10\cos^2 \theta + 10\cos^4 \theta + 1 + \cos^4 \theta - 2\cos^2 \theta] \\ \sin 5\theta &= \sin \theta [16\cos^4 \theta - 12\cos^2 \theta + 1] \\ \frac{\sin 5\theta}{\sin \theta} &= 16\cos^4 \theta - 12\cos^2 \theta + 1 \quad \text{PROVED} \end{aligned}$$

14. Prove that $\tan 6\theta = 2t \left(\frac{3-10t^2+3t^4}{1-15t^2+15t^4-t^6} \right)$, where $t = \tan \theta$

Proof:

$$\begin{aligned} (\cos 6\theta + i \sin 6\theta) &= (\cos \theta + i \sin \theta)^6 \\ &= \binom{6}{0}(\cos \theta)^6(i \sin \theta)^0 + \binom{6}{1}(\cos \theta)^5(i \sin \theta)^1 + \binom{6}{2}(\cos \theta)^4(i \sin \theta)^2 \\ &\quad + \binom{6}{3}(\cos \theta)^3(i \sin \theta)^3 + \binom{6}{4}(\cos \theta)^2(i \sin \theta)^4 + \binom{6}{5}(\cos \theta)(i \sin \theta)^5 + \binom{6}{6}(i \sin \theta)^6 \\ &= \cos^6 \theta + 6\cos^5 \theta \sin \theta i + 15\cos^4 \theta \sin^2 \theta i^2 + 20\cos^3 \theta \sin^3 \theta i^3 \\ &\quad + 15\cos^2 \theta \sin^4 \theta i^4 + 6\cos \theta \sin^5 \theta i^5 + i^6 \sin^6 \theta \\ &= (\cos^6 \theta - 15\cos^4 \theta \sin^2 \theta + 15\cos^2 \theta \sin^4 \theta - \sin^6 \theta) \\ &\quad + i(6\cos^5 \theta \sin \theta - 20\cos^3 \theta \sin^3 \theta + 6\cos \theta \sin^5 \theta) \end{aligned}$$

Equating Real and imaginary part

$$\cos 6\theta = \cos^6 \theta - 15\cos^4 \theta \sin^2 \theta + 15\cos^2 \theta \sin^4 \theta - \sin^6 \theta \rightarrow \text{①}$$

$$\sin 6\theta = 6\cos^5 \theta \sin \theta - 20\cos^3 \theta \sin^3 \theta + 6\cos \theta \sin^5 \theta \rightarrow \text{②}$$

(31)

Dividing (iii) by (i)

$$\frac{\sin 6\theta}{\cos 6\theta} = \frac{6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta}{\cos^6\theta - 15\cos^4\theta \sin^2\theta + 15\cos^2\theta \sin^4\theta - \sin^6\theta}$$

$$= \frac{2\sin\theta}{\cos\theta} \left[\frac{3\cos^5\theta - 10\cos^3\theta \sin^2\theta + 3\cos\theta \sin^4\theta}{\cos^5\theta - 15\cos^3\theta \sin^2\theta + 15\cos\theta \sin^4\theta - \frac{\sin^6\theta}{\cos\theta}} \right]$$

∴ dividing Numerator & denominator by $\cos^5\theta$

$$\tan 6\theta = \frac{2\sin\theta}{\cos\theta} \left[\frac{\frac{3\cos^5\theta}{\cos^5\theta} - \frac{10\cos^3\theta \sin^2\theta}{\cos^3\theta \cos^2\theta} + \frac{3\cos\theta \sin^4\theta}{\cos\theta \cos^4\theta}}{\frac{\cos^5\theta}{\cos^5\theta} - \frac{15\cos^3\theta \sin^2\theta}{\cos^3\theta \cos^2\theta} + \frac{15\cos\theta \sin^4\theta}{\cos\theta \cos^4\theta} - \frac{\sin^6\theta}{\cos^5\theta \cos\theta}} \right]$$

$$= \frac{2\sin\theta}{\cos\theta} \left[\frac{3 - 10\tan^2\theta + 3\tan^4\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta} \right]$$

$$\therefore \tan\theta = t$$

$$\tan 6\theta = 2t \left[\frac{3 - 10t^2 + 3t^4}{1 - 15t^2 + 15t^4 - t^6} \right]$$

PROVED

15.

Prove that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ and hence solve the equation $1 - 3t^2 = 3t - t^3$ where $t = \tan\theta$.

Proof

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + i^3\sin^3\theta + 3\cos\theta i^2\sin^2\theta + 3i\sin\theta \cos^2\theta$$

$$= (\cos^3\theta - 3\cos\theta \sin^2\theta) + i(3\sin\theta \cos^2\theta - \sin^3\theta)$$

Equating Real & Imaginary part.

$$\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta \quad \rightarrow (1)$$

$$\sin 3\theta = 3\sin\theta \cos^2\theta - \sin^3\theta \quad \rightarrow (2)$$

dividing (1) & (2)

$$\frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin\theta \cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta \sin^2\theta}$$

$$= \frac{\frac{3\sin\theta \cos^2\theta}{\cos^2\theta \cos\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta \sin^2\theta}{\cos\theta \cos^2\theta}}$$

Dividing Numerator and denominator by $\cos^3\theta$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\therefore \tan\theta = t$$

$$\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$$

$$\text{Put } \tan 3\theta = 1$$

$$1 = \frac{3t - t^3}{1 - 3t^2}$$

$$\therefore 3t - t^3 = 1 - 3t^2$$

when

$$t = \tan 15$$

$$t = \tan (45 - 30)$$

$$t = \frac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$t = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{(\sqrt{3})^2 + (1)^2 - 2\sqrt{3}}{(\sqrt{3})^2 - (1)^2}$$

$$= \frac{3 + 1 - 2\sqrt{3}}{3 - 1}$$

$$= \frac{2(2 - \sqrt{3})}{2}$$

$$\boxed{t = 2 - \sqrt{3}}$$

$$\Rightarrow \tan 3\theta = 1$$

$$3\theta = 45, 225$$

$$\theta = \frac{45}{3}, \frac{225}{3}$$

$$\theta = 15, 75$$

when

$$t = \tan 35$$

$$t = \tan (45 + 30)$$

$$t = \frac{\tan 45 + \tan 30}{1 + \tan 45 \tan 30}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$t = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{(\sqrt{3})^2 + (1)^2 + 2\sqrt{3}}{(\sqrt{3})^2 - (1)^2}$$

$$= \frac{3 + 1 + 2\sqrt{3}}{3 - 1}$$

$$= \frac{2(2 + \sqrt{3})}{2}$$

$$\boxed{t = 2 + \sqrt{3}}$$

16. Prove that $\cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2}$

Proof: Let $x^7 - 1 = 0$
 $x^7 = 1$
 $x = (1)^{1/7}$

Here $z = 1 + 0i$ $|z| = 1$

$\cos \theta = 1$
 $\sin \theta = 0$ } quadrantal angle
 $\theta = 0^\circ$

$$x_k = (1)^{1/7} \left[\cos \left(\frac{0 + 2k\pi}{7} \right) + i \sin \left(\frac{0 + 2k\pi}{7} \right) \right] \quad k = -3, -2, -1, 0, 1, 2, 3$$

$$x_k = 1 \left[\cos \left(\frac{2k\pi}{7} \right) + i \sin \left(\frac{2k\pi}{7} \right) \right] \quad k = -3, -2, -1, 0, 1, 2, 3$$

$$x_k = \sum_{k=-3}^3 \left[\cos \left(\frac{2k\pi}{7} \right) + i \sin \left(\frac{2k\pi}{7} \right) \right] = 0 + 0i$$

Equating real part

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$$\therefore \sum_{k=-3}^3 \cos\left(\frac{2k\pi}{7}\right) = 0$$

$$\cos\left(-\frac{6\pi}{7}\right) + \cos\left(-\frac{4\pi}{7}\right) + \cos\left(-\frac{2\pi}{7}\right) + \cos 0 + \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = 0$$

$$\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) + 1 + \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = 0$$

$$2\cos\left(\frac{6\pi}{7}\right) + 2\cos\left(\frac{4\pi}{7}\right) + 2\cos\left(\frac{2\pi}{7}\right) = -1$$

$$\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

$$\cos\left(\pi - \frac{\pi}{7}\right) + \cos\left(\pi - \frac{3\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

$$-\cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

$$\cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) = \frac{1}{2} \quad \text{Proved}$$

17. Prove the following Relations

(i) $z^m z^n = z^{m+n}$

$$z = r \operatorname{Cis} \theta$$

$$z^m = r^m \operatorname{Cis}(m\theta)$$

$$z^n = r^n \operatorname{Cis} n\theta$$

$$z^m \cdot z^n = r^m \operatorname{Cis}(m\theta) \cdot r^n \operatorname{Cis}(n\theta)$$

$$= r^{m+n} \operatorname{Cis}(m\theta + n\theta)$$

$$= r^{m+n} \operatorname{Cis}(m+n)\theta$$

$$= z^{m+n}$$



(ii) $(z^m)^n = z^{mn}$

$$z = r \operatorname{Cis} \theta$$

$$z^m = (r \operatorname{Cis} \theta)^m$$

$$z^m = r^m \operatorname{Cis}(m\theta)$$

By De-Moivre's Theorem

$$(z^m)^n = (r^m \operatorname{Cis}(m\theta))^n$$

$$= (r^m)^n \operatorname{Cis}(m(m\theta))$$

$$= r^{mn} \operatorname{Cis}(mn)\theta$$

$$(z^m)^n = z^{mn}$$

proved

(iii) $(z_1 z_2)^n = z_1^n z_2^n$

$$z_1 = r_1 \operatorname{Cis} \theta_1$$

$$z_2 = r_2 \operatorname{Cis} \theta_2$$

$$z_1^n = r_1^n \operatorname{Cis} n\theta_1$$

$$z_2^n = r_2^n \operatorname{Cis} n\theta_2$$

$$z_1 z_2 = r_1 \text{Cis } \theta_1 \cdot r_2 \text{Cis } \theta_2$$

$$z_1 z_2 = r_1 r_2 \text{Cis } (\theta_1 + \theta_2)$$

$$(z_1 z_2)^n = [r_1 r_2 \text{Cis } (\theta_1 + \theta_2)]^n$$

$$(z_1 z_2)^n = r_1^n r_2^n \text{Cis } n(\theta_1 + \theta_2) \rightarrow \textcircled{1}$$

$$\begin{aligned} z_1^n z_2^n &= r_1^n \text{Cis } n\theta_1 \cdot r_2^n \text{Cis } n\theta_2 \\ &= r_1^n r_2^n \text{Cis } (n\theta_1 + n\theta_2) \end{aligned}$$

$$z_1^n z_2^n = r_1^n r_2^n \text{Cis } (\theta_1 + \theta_2)n \rightarrow \textcircled{2}$$

from $\textcircled{1}$ & $\textcircled{2}$

$$(z_1 z_2)^n = (z_1)^n (z_2)^n$$

$$17. (iv) \frac{z^m}{z^n} = z^{m-n}$$

$$z = r \text{Cis } \theta$$

$$z^m = r^m \text{Cis } m\theta$$

$$z^n = r^n \text{Cis } n\theta$$

$$\frac{z^m}{z^n} = \frac{r^m \text{Cis } m\theta}{r^n \text{Cis } n\theta}$$

$$\begin{aligned} &= r^{m-n} \text{Cis } (m\theta - n\theta) \\ &= r^{m-n} \text{Cis } (m-n)\theta \end{aligned}$$

$$\frac{z^m}{z^n} = z^{m-n}$$

$$(v) \left(\frac{z_1}{z_2} \right)^n = \frac{z_1^n}{z_2^n} \quad \text{Provided } z_2 \neq 0$$

$$z_1 = r_1 \text{Cis } \theta_1, \quad z_2 = r_2 \text{Cis } \theta_2$$

$$\frac{z_1}{z_2} = \frac{r_1 \text{Cis } \theta_1}{r_2 \text{Cis } \theta_2}$$

$$= \frac{r_1}{r_2} \text{Cis } (\theta_1 - \theta_2) \rightarrow \textcircled{1}$$

$$z_1^n = r_1^n \text{Cis } n\theta_1, \quad z_2^n = r_2^n \text{Cis } n\theta_2$$

$$\frac{z_1^n}{z_2^n} = \frac{r_1^n \text{Cis } n\theta_1}{r_2^n \text{Cis } n\theta_2}$$

$$= \frac{r_1^n}{r_2^n} \text{Cis } (n\theta_1 - n\theta_2)$$

$$= \frac{r_1^n}{r_2^n} \text{Cis } (\theta_1 - \theta_2)n \rightarrow \textcircled{2}$$

from $\textcircled{1}$ & $\textcircled{2}$

$$\left(\frac{z_1}{z_2} \right)^n = \frac{z_1^n}{z_2^n}$$

Exercise # 1.3

1. Show that

(i) e^z is never zero.

$$e^z \cdot \frac{1}{e^z} = 1$$

Multiplicative inverse of e^z is possible, so e^z can never be zero

OR

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x [\cos y + i \sin y]$$

$$\therefore e^x \neq 0$$

there is no angle at which $\cos$ & $\sin$ become zero at same time

$$\text{so } e^z \neq 0$$

(ii) $|e^{iz}| = 1$

$$|\cos z + i \sin z| = 1$$

$$\sqrt{\cos^2 z + \sin^2 z} = 1$$

$$\sqrt{1} = 1$$

$$1 = 1$$

(iii) $e^z = 1$ iff z is integral multiple of $2\pi i$.

Proof: Suppose $e^z = 1$

$$e^{x+iy} = 1 = e^x e^{iy}$$

$$e^x [\cos y + i \sin y] = 1$$

$$e^x \cos y + i e^x \sin y = 1 + 0i$$

Equating real and imaginary part.

$$e^x \cos y = 1$$

$$e^x \sin y = 0$$

$$e^x = 1; \cos y = 1$$

$\Rightarrow x = 0$; from (i)

$$\cos(k\pi) = 1$$

$$(-1)^k = 1$$

$$k = 2n \rightarrow \text{put in (i)}$$

$$y = 2n\pi$$

$$\text{Now } z = x + iy = 0 + i2n\pi$$

$\therefore z = (2\pi i)n$ So z is integral multiple of $2\pi i$.

Conversely;

$$z = 2\pi ni$$

$$e^z = e^{2\pi ni}$$

$$= \cos(2n\pi) + i \sin 2n\pi$$

$$e^z = 1 + 0 \Rightarrow e^z = 1$$

Euler formula.

$$e^{i\theta} = \cos \theta + i \sin \theta = \text{cis}(\theta)$$

(iv) $e^{z_1} = e^{z_2}$ iff $z_1 - z_2 = 2k\pi i$

where k is integer.

Proof: $e^{z_1} = e^{z_2} \Rightarrow \frac{e^{z_1}}{e^{z_2}} = 1$

$$e^{z_1 - z_2} = 1$$

$$e^z = 1$$

put $z_1 - z_2 = z$

Now question is same as Q1 part (iii)

(v) $|e^z| = e^x$ where $z = x + iy$

$$|e^z| = |e^{x+iy}|$$

$$= |e^x e^{iy}|$$

$$= e^x |e^{iy}|$$

$$= e^x \sqrt{\cos^2 y + \sin^2 y}$$

$$= e^x \sqrt{1}$$

$$|e^z| = e^x$$

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$$(vi) e^{z_1} \cdot e^{z_2} \dots e^{z_n} = e^{z_1 + z_2 + \dots + z_n} \rightarrow (1)$$

We'll prove this by Mathematical induction

Case I put $n=1$
 $e^{z_1} = e^{z_1}$

put $n=2$
 $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$

Condition is true for $n=1$ & 2 .

Case II Suppose this is true for $n=k$ $k \in \mathbb{Z}^+$
 $e^{z_1} \cdot e^{z_2} \dots e^{z_k} = e^{z_1 + z_2 + \dots + z_k}$

Case III Multiply both sides with $e^{z_{k+1}}$

$$e^{z_1} \cdot e^{z_2} \dots e^{z_k} \cdot e^{z_{k+1}} = e^{z_1 + z_2 + \dots + z_k} \cdot e^{z_{k+1}}$$

$$= e^{z_1 + z_2 + \dots + z_k + z_{k+1}}$$

This is true for two numbers C-I

Hence This is true for $n=k+1$

So by Induction (1) is true.

(vii) $(e^z)^n = e^{nz}$, n being an integer

$$\begin{aligned} \text{L.H.S} &= (e^z)^n = (e^{x+iy})^n \\ &= (e^x e^{iy})^n \\ &= e^{xn} (e^{iy})^n \\ &= e^{xn} (\cos y + i \sin y)^n \\ &= e^{xn} [\cos ny + i \sin ny] \\ &= e^{xn} e^{iny} \\ &= e^{xn + iny} = e^{n(x+iy)} = e^{nz} \end{aligned}$$

Euler formula

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Formulae:

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}, \quad \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

Q2. Prove that $(\forall z_1, z_2, z \in \mathbb{C})$

(i) $\tan^2 z + 1 = \sec^2 z$

$$\begin{aligned} \text{L.H.S} &= 1 + \tan^2 z \\ &= 1 + \left(\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} \right)^2 \\ &= 1 - \frac{(e^{iz} - e^{-iz})^2}{(e^{iz} + e^{-iz})^2} \end{aligned}$$

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$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\tan z = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

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$$e^{iz} \cdot e^{-iz} = e^{iz-iz} = e^0 = 1$$

$$\begin{aligned}
 &= \frac{(e^{iz} + e^{-iz})^2 - (e^{iz} - e^{-iz})^2}{(e^{iz} + e^{-iz})^2} \\
 &= \frac{e^{2iz} + e^{-2iz} + 2 - (e^{2iz} + e^{-2iz} - 2)}{(e^{iz} + e^{-iz})^2} \\
 &= \frac{2+2-4}{(e^{iz} + e^{-iz})^2} \\
 &= \left[\frac{2}{e^{iz} + e^{-iz}} \right]^2 \\
 &= \left[\frac{1}{\cos z} \right]^2 \\
 &= (\sec z)^2 \\
 &= \sec^2 z
 \end{aligned}$$

$$(iii) \sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

$$\begin{aligned}
 \text{R.H.S} &= \sin z_1 \cos z_2 - \cos z_1 \sin z_2 \\
 &= \left(\frac{e^{iz_1} - e^{-iz_1}}{2i} \right) \left(\frac{e^{iz_2} + e^{-iz_2}}{2} \right) - \left(\frac{e^{iz_1} + e^{-iz_1}}{2} \right) \left(\frac{e^{iz_2} - e^{-iz_2}}{2i} \right) \\
 &= \frac{1}{4i} \left((e^{iz_1} - e^{-iz_1})(e^{iz_2} + e^{-iz_2}) - (e^{iz_1} + e^{-iz_1})(e^{iz_2} - e^{-iz_2}) \right) \\
 &= \frac{1}{4i} \left[e^{iz_1} e^{iz_2} + e^{iz_1} e^{-iz_2} - e^{-iz_1} e^{iz_2} + e^{-iz_1} e^{-iz_2} - e^{iz_1} e^{iz_2} + e^{iz_1} e^{-iz_2} \right. \\
 &\quad \left. - e^{-iz_1} e^{iz_2} + e^{-iz_1} e^{-iz_2} \right] \\
 &= \frac{1}{4i} (2e^{iz_1} e^{-iz_2} - 2e^{-iz_1} e^{iz_2}) \\
 &= \frac{1}{2i} \frac{2(e^{i(z_1 - z_2)} - e^{-i(z_1 - z_2)})}{e^{i(z_1 - z_2)} - e^{-i(z_1 - z_2)}} \\
 &= \sin(z_1 - z_2) = \text{L.H.S}
 \end{aligned}$$

$$(iv) \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\begin{aligned}
 \text{R.H.S} &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\
 &= \left(\frac{e^{iz_1} + e^{-iz_1}}{2} \right) \left(\frac{e^{iz_2} + e^{-iz_2}}{2} \right) - \left(\frac{e^{iz_1} - e^{-iz_1}}{2i} \right) \left(\frac{e^{iz_2} - e^{-iz_2}}{2i} \right) \\
 &= \frac{1}{4} \left((e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2}) + (e^{iz_1} - e^{-iz_1})(e^{iz_2} - e^{-iz_2}) \right) \\
 &= \frac{1}{4} \left[e^{iz_1} e^{iz_2} + e^{iz_1} e^{-iz_2} + e^{-iz_1} e^{iz_2} + e^{-iz_1} e^{-iz_2} + e^{iz_1} e^{iz_2} - e^{iz_1} e^{-iz_2} \right. \\
 &\quad \left. - e^{-iz_1} e^{iz_2} + e^{-iz_1} e^{-iz_2} \right] \\
 &= \frac{1}{4} [2e^{iz_1} e^{iz_2} + 2e^{-iz_1} e^{-iz_2}] \\
 &= \frac{1}{2} [e^{i(z_1 + z_2)} + e^{-i(z_1 + z_2)}] \\
 &= \cos(z_1 + z_2)
 \end{aligned}$$

$$(V) \cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2.$$

$$\begin{aligned} \text{R.H.S} &= \cos z_1 \cos z_2 + \sin z_1 \sin z_2 \\ &= \left(\frac{e^{iz_1} + e^{-iz_1}}{2} \right) \left(\frac{e^{iz_2} + e^{-iz_2}}{2} \right) + \left(\frac{e^{iz_1} - e^{-iz_1}}{2i} \right) \left(\frac{e^{iz_2} - e^{-iz_2}}{2i} \right) \\ &= \frac{1}{4} \left((e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2}) + (e^{iz_1} - e^{-iz_1})(e^{iz_2} - e^{-iz_2}) \right) \\ &= \frac{1}{4} \left(e^{iz_1} e^{iz_2} + e^{iz_1} e^{-iz_2} + e^{-iz_1} e^{iz_2} + e^{-iz_1} e^{-iz_2} \right. \\ &\quad \left. - e^{iz_1} e^{iz_2} - e^{iz_1} e^{-iz_2} + e^{-iz_1} e^{iz_2} - e^{-iz_1} e^{-iz_2} \right) \\ &= \frac{1}{4} \left(2e^{iz_1} e^{-iz_2} + 2e^{-iz_1} e^{iz_2} \right) \\ &= \frac{1}{2} \left(e^{i(z_1 - z_2)} + e^{-i(z_1 - z_2)} \right) \\ &= \cos(z_1 - z_2) = \text{L.H.S.} \end{aligned}$$

$$(VI) \cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$$

$$\cos^2 z - \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2} \right)^2 - \left(\frac{e^{iz} - e^{-iz}}{2i} \right)^2$$

$$= \frac{(e^{iz} + e^{-iz})^2}{4} - \frac{(e^{iz} - e^{-iz})^2}{4i^2}$$

 $i^2 = -1$

$$= \frac{1}{4} \left[(e^{iz} + e^{-iz})^2 + (e^{iz} - e^{-iz})^2 \right]$$

$$= \frac{1}{4} \left[e^{2iz} + e^{2iz} + 2 + e^{2iz} + e^{-2iz} - 2 \right]$$

$$= \frac{1}{4} \left[2e^{2iz} + 2e^{-2iz} \right]$$

$$= \frac{1}{2} \left[e^{2iz} + e^{-2iz} \right]$$

$$= \frac{e^{2iz} + e^{-2iz}}{2}$$

$$= \cos 2z \quad \text{Prove}$$

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$$2\cos^2 z - 1 = 2 \left(\frac{e^{iz} + e^{-iz}}{2} \right)^2 - 1$$

$$= \frac{2}{4} (e^{iz} + e^{-iz})^2 - 1$$

$$= \frac{(e^{iz} + e^{-iz})^2 - 2}{2}$$

$$= \frac{e^{2iz} + e^{-2iz} + 2 - 2}{2}$$

$$= \frac{e^{2iz} + e^{-2iz}}{2}$$

$$= \cos 2z$$

$$1 - 2\sin^2 z = 1 - 2 \left(\frac{e^{iz} - e^{-iz}}{2i} \right)^2$$

$$= 1 + \frac{(e^{iz} - e^{-iz})^2}{i^2}$$

$$= \frac{2 + (e^{iz} - e^{-iz})^2}{2}$$

$$= \frac{2 + e^{2iz} + e^{-2iz} - 2}{2}$$

$$= \frac{e^{2iz} + e^{-2iz}}{2}$$

$$= \cos 2z$$

(vii) $\sin 2z = 2 \sin z \cos z$

$$\begin{aligned} 2 \sin z \cos z &= z \left(\frac{e^{iz} - e^{-iz}}{2i} \right) \left(\frac{e^{iz} + e^{-iz}}{2} \right) \\ &= \frac{1}{2i} (e^{iz} - e^{-iz})(e^{iz} + e^{-iz}) \\ &= \frac{1}{2i} (e^{2iz} - e^{-2iz}) \\ &= \sin 2z \end{aligned}$$

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viii) $\cos z_1 - \cos z_2 = 2 \sin \frac{z_1 + z_2}{2} \sin \frac{z_2 - z_1}{2}$

$$\begin{aligned} \text{R.H.S} &= 2 \sin \left(\frac{z_1 + z_2}{2} \right) \sin \left(\frac{z_2 - z_1}{2} \right) \\ &= z \left[\frac{e^{i(\frac{z_1 + z_2}{2})} - e^{-i(\frac{z_1 + z_2}{2})}}{2i} \right] \left[\frac{e^{i(\frac{z_2 - z_1}{2})} - e^{-i(\frac{z_2 - z_1}{2})}}{2i} \right] \\ &= \frac{1}{2i^2} \left[(e^{i(\frac{z_1 + z_2}{2})} - e^{-i(\frac{z_1 + z_2}{2})}) (e^{i(\frac{z_2 - z_1}{2})} - e^{-i(\frac{z_2 - z_1}{2})}) \right] \\ &= -\frac{1}{2} \left[e^{i(\frac{z_1 + z_2}{2})} e^{i(\frac{z_2 - z_1}{2})} - e^{i(\frac{z_1 + z_2}{2})} e^{-i(\frac{z_2 - z_1}{2})} - e^{-i(\frac{z_1 + z_2}{2})} e^{i(\frac{z_2 - z_1}{2})} + e^{-i(\frac{z_1 + z_2}{2})} e^{-i(\frac{z_2 - z_1}{2})} \right] \\ &= \frac{1}{2} \left[e^{i(\frac{z_1 + z_2}{2} + i(\frac{z_2 - z_1}{2})} - e^{i(\frac{z_1 + z_2}{2}) + i(\frac{z_2 - z_1}{2})} - e^{-i(\frac{z_1 + z_2}{2}) + i(\frac{z_2 - z_1}{2})} + e^{-i(\frac{z_1 + z_2}{2}) - i(\frac{z_2 - z_1}{2})} \right] \\ &= \frac{1}{2} \left[e^{i(\frac{z_1 + z_2 + z_2 - z_1}{2})} - e^{i(\frac{z_1 + z_2 - z_2 + z_1}{2})} - e^{-i(\frac{z_1 + z_2 - z_2 + z_1}{2})} + e^{-i(\frac{z_1 + z_2 + z_2 - z_1}{2})} \right] \\ &= \frac{1}{2} \left[(e^{iz_2} - e^{iz_1} - e^{-iz_1} + e^{-iz_2}) \right] \\ &= -\frac{1}{2} \left[(e^{iz_2} + e^{-iz_2}) - (e^{iz_1} + e^{-iz_1}) \right] \\ &= - \left(\frac{e^{iz_2} + e^{-iz_2}}{2} \right) + \left(\frac{e^{iz_1} + e^{-iz_1}}{2} \right) \\ &= -\cos z_2 + \cos z_1 = \text{L.H.S. proved} \end{aligned}$$

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(ix) $\sin z_1 - \sin z_2 = 2 \cos \left(\frac{z_1 + z_2}{2} \right) \sin \left(\frac{z_1 - z_2}{2} \right)$

$$\begin{aligned} \text{R.H.S} &= 2 \cos \left(\frac{z_1 + z_2}{2} \right) \sin \left(\frac{z_1 - z_2}{2} \right) \\ &= z \left[\frac{e^{i(\frac{z_1 + z_2}{2})} + e^{-i(\frac{z_1 + z_2}{2})}}{2} \right] \left[\frac{e^{i(\frac{z_1 - z_2}{2})} - e^{-i(\frac{z_1 - z_2}{2})}}{2i} \right] \end{aligned}$$

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$$\begin{aligned}
 &= \frac{1}{2i} \left[\left(e^{i\left(\frac{z_1+z_2}{2}\right)} + e^{-i\left(\frac{z_1+z_2}{2}\right)} \right) \left(e^{i\left(\frac{z_1-z_2}{2}\right)} - e^{-i\left(\frac{z_1-z_2}{2}\right)} \right) \right] \\
 &= \frac{1}{2i} \left[e^{i\left(\frac{z_1+z_2}{2}\right)} e^{i\left(\frac{z_1-z_2}{2}\right)} - e^{i\left(\frac{z_1+z_2}{2}\right)} e^{-i\left(\frac{z_1-z_2}{2}\right)} + e^{-i\left(\frac{z_1+z_2}{2}\right)} e^{i\left(\frac{z_1-z_2}{2}\right)} - e^{-i\left(\frac{z_1+z_2}{2}\right)} e^{-i\left(\frac{z_1-z_2}{2}\right)} \right] \\
 &= \frac{1}{2i} \left[e^{i\left(\frac{z_1+z_2+z_1-z_2}{2}\right)} - e^{i\left(\frac{z_1+z_2-z_1+z_2}{2}\right)} + e^{-i\left(\frac{z_1+z_2-z_1+z_2}{2}\right)} - e^{-i\left(\frac{z_1+z_2+z_1-z_2}{2}\right)} \right] \\
 &= \frac{1}{2i} \left[e^{iz_1} - e^{iz_2} + e^{-iz_2} - e^{-iz_1} \right] \\
 &= \frac{1}{2i} \left[(e^{iz_1} - e^{-iz_1}) - (e^{iz_2} - e^{-iz_2}) \right] \\
 &= \frac{e^{iz_1} - e^{-iz_1}}{2i} - \frac{e^{iz_2} - e^{-iz_2}}{2i} = \sin z_1 - \sin z_2 = \text{L.H.S.}
 \end{aligned}$$

(X) $\sin 3z = 3\sin z - 4\sin^3 z$

$$\begin{aligned}
 3\sin z - 4\sin^3 z &= 3 \left(\frac{e^{iz} - e^{-iz}}{2i} \right) - 4 \left(\frac{e^{iz} - e^{-iz}}{2i} \right)^3 \\
 &= 3 \left(\frac{e^{iz} - e^{-iz}}{2i} \right) - \frac{4}{8i^3} (e^{iz} - e^{-iz})^3 \\
 &= \frac{3}{2i} (e^{iz} - e^{-iz}) - \frac{1}{-2i} (e^{3iz} - e^{-3iz} - 3e^{iz}e^{-iz} + 3e^{iz}e^{-iz}) \\
 &= \frac{1}{2i} [3e^{iz} - 3e^{-iz} + e^{3iz} - e^{-3iz} - 3e^{iz} + 3e^{-iz}] \\
 &= \frac{e^{3iz} - e^{-3iz}}{2i} \\
 &= \sin 3z = \text{R.H.S}
 \end{aligned}$$

x) $\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$

$$\tan(z_1 + z_2) = \frac{\sin(z_1 + z_2)}{\cos(z_1 + z_2)}$$

$$= \frac{\sin z_1 \cos z_2 + \cos z_1 \sin z_2}{\cos z_1 \cos z_2 - \sin z_1 \sin z_2}$$

$$= \frac{\sin z_1 \cos z_2}{\cos z_1 \cos z_2} + \frac{\cos z_1 \sin z_2}{\cos z_1 \cos z_2}$$

$$\frac{\sin z_1 \cos z_2}{\cos z_1 \cos z_2} + \frac{\sin z_1 \sin z_2}{\cos z_1 \cos z_2}$$

$$= \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$$

(xii) $\tan(z_1 - z_2) = \frac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2}$

$$\tan(z_1 - z_2) = \frac{\sin(z_1 - z_2)}{\cos(z_1 - z_2)}$$

$$= \frac{\sin z_1 \cos z_2 - \cos z_1 \sin z_2}{\cos z_1 \cos z_2 + \sin z_1 \sin z_2}$$

$$= \frac{\sin z_1 \cos z_2}{\cos z_1 \cos z_2} - \frac{\cos z_1 \sin z_2}{\cos z_1 \cos z_2}$$

$$\frac{\sin z_1 \cos z_2}{\cos z_1 \cos z_2} - \frac{\sin z_1 \sin z_2}{\cos z_1 \cos z_2}$$

$$\tan(z_1 - z_2) = \frac{\tan z_1 - \tan z_2}{1 + \tan z_1 \tan z_2}$$

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3. Show that.

Formulae:

- * $\sin(iz) = i \sinh z$
- * $\cos(iz) = \cosh z$
- * $\tan(iz) = i \tanh z$

(i) $\sin z = \sin \bar{z}$

$$\begin{aligned} \sin z &= \sin(x+iy) \\ &= \sin x \cos(iy) + \cos x \sin(iy) \\ &= \sin x \cosh y + \cos x i \sinh y \\ \sin \bar{z} &= \sin(x-iy) \\ &= \sin x \cosh y - \cos x i \sinh y \rightarrow (ii) \\ \text{From (i) \& (ii)} \quad \sin z &= \sin \bar{z} \end{aligned}$$

$$\begin{aligned} \sin \bar{z} &= \sin(\overline{x+iy}) \\ &= \sin(x-iy) \\ &= \sin x \cos(iy) - \cos x \sin(iy) \\ &= \sin x \cosh y - \cos x i \sinh y \\ \sin \bar{z} &= \cosh y \sin x - i \cos x \sinh y \rightarrow (iii) \end{aligned}$$

(ii) $\cos z = \cos \bar{z}$

$$\begin{aligned} \cos z &= \cos(x+iy) \\ &= \cos x \cos(iy) - \sin x \sin(iy) \\ &= \cos x \cosh y - \sin x i \sinh y \\ \cos \bar{z} &= \cos(x-iy) \\ &= \cos x \cosh y + \sin x i \sinh y \rightarrow (i) \\ \text{From (i) \& (ii)} \quad \cos z &= \cos \bar{z} \end{aligned}$$

$$\begin{aligned} \cos \bar{z} &= \cos(\overline{x+iy}) \\ &= \cos(x-iy) \\ &= \cos x \cos(iy) + \sin x \sin(iy) \\ \cos \bar{z} &= \cos x \cosh y + i \sin x \sinh y \rightarrow (ii) \end{aligned}$$

(iii) $\tan z = \tan \bar{z}$

$$\begin{aligned} \tan z &= \tan(x+iy) \\ &= \frac{\tan x + \tan(iy)}{1 - \tan x \tan(iy)} \\ &= \frac{\tan x + i \tanh y}{1 - \tan x i \tanh y} \\ \tan \bar{z} &= \left(\frac{\tan x + i \tanh y}{1 - i \tan x \tanh y} \right) \cdot \left(\frac{\bar{z}}{z} \right) = \frac{\bar{z}}{z} \\ &= \frac{\tan x + i \tanh y}{1 - i \tan x \tanh y} \end{aligned}$$

$$\begin{aligned} \tan \bar{z} &= \tan(\overline{x+iy}) \\ &= \tan(x-iy) \\ \tan \bar{z} &= \frac{\tan x - \tanh y}{1 + \tan x \tanh y} \\ \tan \bar{z} &= \frac{\tan x - i \tanh y}{1 + i \tan x \tanh y} \rightarrow (2) \end{aligned}$$

$$\tan z = \frac{\tan x - i \tanh y}{1 + i \tan x \tanh y} \rightarrow (1)$$

From (1) \& (2)
 $\tan z = \tan \bar{z}$

(iv) $\sin(-z) = -\sin z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \rightarrow (1)$$

$$\sin(-z) = \frac{e^{-iz} - e^{iz}}{2i}$$

$$\sin(-z) = - \left(\frac{e^{iz} - e^{-iz}}{2i} \right) = -\sin z \text{ from (1)}$$

$$\sin(-z) = -\sin z$$

$$(v) \quad \cos(-z) = +\cos(z)$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \rightarrow (i)$$

replace z by $-z$

$$\cos(-z) = \frac{e^{-iz} + e^{iz}}{2}$$

$$\cos(-z) = \cos z \quad \text{from (i)}$$

$$(vii) \quad \sinh(-z) = -\sinh z$$

$$\sinh z = \frac{e^z - e^{-z}}{2} \rightarrow (i)$$

replace z by $-z$

$$\sinh(-z) = \frac{e^{-z} - e^z}{2}$$

$$= -\left(\frac{e^z - e^{-z}}{2}\right)$$

$$\sinh(-z) = -\sinh z \quad \text{from (i)}$$

$$(ix) \quad \tanh(-z) = -\tanh z$$

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} \rightarrow (i)$$

replace z by $-z$

$$\tanh(-z) = \frac{e^{-z} - e^z}{e^{-z} + e^z}$$

$$\tanh(-z) = -\left(\frac{e^z - e^{-z}}{e^z + e^{-z}}\right)$$

$$\tanh(-z) = -\tanh z \quad \text{from (i)}$$

4. Prove the following identities

$$(i) \quad \cosh^2 z - \sinh^2 z = 1$$

$$\therefore \cos^2 z + \sin^2 z = 1$$

$$[\cos(iz)]^2 + [\sin(iz)]^2 = 1$$

$$(\cosh z)^2 + (i \sinh z)^2 = 1$$

$$\cosh^2 z + i^2 \sinh^2 z = 1$$

$$\cosh^2 z - \sinh^2 z = 1$$

proved.

$$(vi) \quad \tan(-z) = -\tan z$$

$$\tan(z) = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} \rightarrow (i)$$

replace z by $-z$

$$\tan(-z) = \frac{e^{-iz} - e^{iz}}{i(e^{-iz} + e^{iz})}$$

$$\tan(-z) = -\frac{(e^{iz} - e^{-iz})}{i(e^{iz} + e^{-iz})}$$

$$\tan(-z) = -\tan(z) \quad \text{from (i)}$$

$$(viii) \quad \cosh(z) = \cosh \bar{z}$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \rightarrow (i)$$

replace z by $-z$

$$\cosh(-z) = \frac{e^{-z} + e^z}{2}$$

$$\cosh(-z) = \cosh z \quad \text{from (i)}$$

$$(x) \quad \tanh z = \tanh \bar{z}$$

By formula

$$i \tanh z = \tan(iz)$$

$$\tanh z = \frac{\tan(iz)}{i}$$

$$\overline{\tanh z} = \overline{\left(\frac{\tan(iz)}{i}\right)} = \frac{\overline{\tan(iz)}}{\overline{i}}$$

$$= \frac{\tan(\bar{i}z)}{-i} \quad \because \overline{\tan z} = \tan \bar{z}$$

$$= \frac{-i \tan(-i\bar{z})}{-i}$$

$$= -\frac{\tan(i\bar{z})}{-i}$$

$$= \frac{\tan(i\bar{z})}{i}$$

$$= \tanh(\bar{z})$$

$$\overline{\tanh z} = \tanh \bar{z} \quad \text{proved}$$

(ii) $\text{sech}^2 z = 1 - \tanh^2 z$

$\therefore \sec^2 z = 1 + \tan^2 z$
replace z by iz

$[\sec(iz)]^2 = 1 + [\tan(iz)]^2$

$(\text{sech } z)^2 = 1 + (i \tanh z)^2$

$\text{sech}^2 z = 1 - \tanh^2 z$

(iii) $\text{cosech}^2 z = \coth^2 z - 1$

$\therefore \text{cosec}^2 z = 1 + \cot^2 z$

replace z by iz

$[\text{cosec}(iz)]^2 = 1 + [\cot(iz)]^2$

$[i \text{cosech } z]^2 = 1 + [i \coth z]^2$

$i^2 \text{cosech}^2 z = 1 + i^2 \coth^2 z$

$-\text{cosech}^2 z = 1 - \coth^2 z$

$\text{cosech}^2 z = \coth^2 z - 1$ proved.

(iv) $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 1 + 2\sinh^2 z$

$\therefore \cos 2z = \cos^2 z - \sin^2 z$
replace z by iz

$\cos(2iz) = [\cos(iz)]^2 - [\sin(iz)]^2$

$\cosh(2z) = (\cosh z)^2 - (i \sinh z)^2$ $\therefore i^2 = -1$

$\cosh 2z = \cosh^2 z + \sinh^2 z \rightarrow (i)$ **Proved**

$\cosh 2z = 1 + \sinh^2 z + \sinh^2 z$

$= 1 + 2\sinh^2 z$

Proved

$\therefore \cosh^2 z = 1 + \sinh^2 z$
(put in (i))

$\cosh 2z = \cosh^2 z + \cosh^2 z - 1$
 $= 2\cosh^2 z - 1$

Proved

(v) $\sinh 2z = 2 \sinh z \cosh z$

$\therefore \sin 2z = 2 \sin z \cos z$

replace z by iz

$\sin(2iz) = 2 \sin(iz) \cos iz$

$i \sinh 2z = 2 i \sinh z \cosh z$

$\sinh 2z = 2 \sinh z \cosh z$

(vi) $\sinh 3z = 3 \sinh z + 4 \sinh^3 z$

$\therefore \sin 3z = 3 \sin z - 4 \sin^3 z$

replace z by iz

$\sin(3iz) = 3 \sin(iz) - 4 (\sin(iz))^3$

$i \sinh 3z = i 3 \sinh z - 4 (i \sinh z)^3$

$i \sinh 3z = i 3 \sinh z - 4 i^2 \sinh^3 z$

$\sinh 3z = 3 \sinh z + 4 \sinh^3 z$

Proved

(vii) $\cosh 3z = 4 \cosh^3 z - 3 \cosh z$

$\therefore \cos 3z = 4 \cos^3 z - 3 \cos z$

replace z by iz

$\cos(3iz) = 4 (\cos(iz))^3 - 3 \cos(iz)$

$\cosh 3z = 4 (\cosh z)^3 - 3 \cosh z$

$= 4 \cosh^3 z - 3 \cosh z$

5. if $z = x+iy$, prove that

(i) $\sin z = \sin x \cosh y + i \cos x \sinh y$

$$\begin{aligned} \sin z &= \sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy) \\ \sin z &= \sin x \cosh y + i \cos x \sinh y \quad \text{proved} \end{aligned}$$

(ii) $\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$

$$\begin{aligned} \tan z &= \tan(x+iy) \\ &= \frac{\sin(x+iy)}{\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)} \\ &= \frac{2 \sin(x+iy) \cos(x-iy)}{2 \cos(x+iy) \cos(x-iy)} \end{aligned}$$

$$\begin{aligned} &= \frac{\sin(x+iy+x-iy) + \sin(x+iy-x-iy)}{\cos(x+iy+x-iy) + \cos(x+iy-x-iy)} \\ &= \frac{\sin(2x) + \sin(2iy)}{\cos(2x) + \cosh(2y)} \end{aligned}$$

$$\therefore = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y} \quad \text{proved}$$

Formula used

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

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6. if $\sin(A+iB) = x+iy$, show that

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1 \quad \& \quad \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

$$\sin(A+iB) = x+iy$$

$$\sin A \cos(iB) + \cos A \sin(iB) = x+iy$$

$$\sin A \cosh B + i \cos A \sinh B = x+iy$$

equating real and imaginary parts

$$\sin A \cosh B = x$$

$$\cos A \sinh B = y$$

$$\cosh B = \frac{x}{\sin A} \rightarrow (i)$$

$$\sinh B = \frac{y}{\cos A} \rightarrow (ii)$$

$$\& \sin A = \frac{x}{\cosh B} \rightarrow (iii)$$

$$\cos A = \frac{y}{\sinh B} \rightarrow (iv)$$

Squaring & Subtracting (i) & (iii)

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

Proved

Squaring & adding (ii) & (iv)

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \cos^2 A + \sin^2 A$$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

Proved

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$$(viii) \sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$$

$$\therefore \sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

replace z_1 by iz_1 and z_2 by iz_2

$$\sin(iz_1 - iz_2) = \sin(iz_1) \cos(iz_2) - \cos(iz_1) \sin(iz_2)$$

$$\sin i(z_1 - z_2) = i \sinh z_1 \cosh z_2 - \cosh z_1 i \sinh z_2$$

$$i \sinh(z_1 - z_2) = i [\sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2]$$

$$\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2 \quad \text{proved}$$

$$(ix) \tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \mp \tanh z_1 \tanh z_2}$$

$$\therefore \tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

$$\tan(iz_1 \pm iz_2) = \frac{\tan(iz_1) \pm \tan(iz_2)}{1 \mp \tan(iz_1) \tan(iz_2)} \quad \begin{array}{l} \text{replace } z_1 \text{ by } iz_1 \\ \text{and } z_2 \text{ by } iz_2 \end{array}$$

$$\tan i(z_1 \pm z_2) = \frac{i \tanh z_1 \pm i \tanh z_2}{1 \mp i \tanh z_1 i \tanh z_2} = \frac{i \tanh z_1 \pm i \tanh z_2}{1 \mp i^2 \tanh z_1 \tanh z_2}$$

$$i \tanh(z_1 \pm z_2) = i \frac{(\tanh z_1 \pm \tanh z_2)}{1 \pm \tanh z_1 \tanh z_2}$$

$$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2} \quad \text{proved}$$

$$(x) \tanh 3z = \frac{3 \tanh z + \tanh^3 z}{1 + 3 \tanh^2 z}$$

$$\therefore \tan 3z = \frac{3 \tan z - \tan^3 z}{1 - 3 \tan^2 z}$$

$$\tan(3iz) = \frac{3 \tan(iz) - (\tan(iz))^3}{1 - 3(\tan(iz))^2} \quad \text{replace } z \text{ by } iz$$

$$= \frac{i 3 \tanh z - i^3 \tanh^3 z}{1 - 3 i^2 \tanh^2 z}$$

$$i^3 = i^2 \cdot i = -i$$

$$= \frac{i 3 \tanh z + i \tanh^3 z}{1 + 3 \tanh^2 z}$$

$$i \tanh 3z = i \frac{(3 \tanh z + \tanh^3 z)}{1 + 3 \tanh^2 z}$$

$$\tanh 3z = \frac{3 \tanh z + \tanh^3 z}{1 + 3 \tanh^2 z} \quad \text{Proved}$$

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7. if $\tan(\alpha + i\beta) = x + iy$, show that

$$x^2 + y^2 + 2x \cot 2\alpha = 1 \quad \text{and} \quad x^2 + y^2 - 2y \coth 2\beta = -1$$

Sol. $\tan(\alpha + i\beta) = x + iy \Rightarrow \alpha + i\beta = \tan^{-1}(x + iy) \rightarrow (i)$

$$\overline{\tan(\alpha + i\beta)} = \overline{x + iy}$$

$$\tan(\overline{\alpha + i\beta}) = \overline{x + iy} = x - iy = \tan(\alpha - i\beta)$$

$$\alpha - i\beta = \tan^{-1}(x - iy) \rightarrow (ii)$$

Adding (i) & (ii)

$$\alpha + i\beta + \alpha - i\beta = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)$$

$$2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)$$

$$2\alpha = \tan^{-1} \left[\frac{x + iy + x - iy}{1 - (x + iy)(x - iy)} \right]$$

$$\tan^{-1}(A) + \tan^{-1}(B) = \tan^{-1} \left(\frac{A+B}{1-AB} \right)$$

$$\tan(2\alpha) = \frac{2x}{1 - (x^2 + y^2)}$$

$$\therefore (x + iy)(x - iy) = x^2 - i^2 y^2 = x^2 + y^2$$

$$\frac{1}{\cot 2\alpha} = \frac{2x}{1 - (x^2 + y^2)}$$

$$1 - (x^2 + y^2) = 2x \cot 2\alpha$$

$$\Rightarrow x^2 + y^2 + 2x \cot 2\alpha = 1 \quad \text{proved.}$$

Subtracting (i) & (ii)

$$\alpha + i\beta - \alpha + i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy)$$

$$2i\beta = \tan^{-1} \left(\frac{x + iy - x + iy}{1 + (x + iy)(x - iy)} \right)$$

$$\tan(2i\beta) = \frac{2iy}{1 + x^2 + y^2}$$

$$\tanh(2\beta) = \frac{2y}{1 + x^2 + y^2}$$

$$\frac{1}{\coth(2\beta)} = \frac{2y}{1 + x^2 + y^2}$$

$$1 + x^2 + y^2 = 2y \coth(2\beta)$$

$$x^2 + y^2 - 2y \coth(2\beta) = -1 \quad \text{proved}$$

8. if $\sin(\theta + i\phi) = x + iy$, show that $\cos^2 \theta = \pm \sin \alpha$.

Sol. $\sin(\theta + i\phi) = x + iy \quad \cos \alpha + i \sin \alpha$

$$\sin \theta \cos(i\phi) + \cos \theta \sin(i\phi) = x + iy \cos \alpha + i \sin \alpha$$

$$\sin \theta \cosh \phi + i \cos \theta \sinh \phi = x + iy \cos \alpha + i \sin \alpha$$



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Equating real & imaginary parts

$$\sin \theta \cosh \phi = \cos \alpha$$

$$\cos \theta \sinh \phi = \sin \alpha$$

$$\cosh \phi = \frac{\cos \alpha}{\sin \theta} \rightarrow (i)$$

$$\sinh \phi = \frac{\sin \alpha}{\cos \theta} \rightarrow (ii)$$

Squaring & subtracting (i) & (ii)

$$\cosh^2 \phi - \sinh^2 \phi = \frac{\cos^2 \alpha}{\sin^2 \theta} - \frac{\sin^2 \alpha}{\cos^2 \theta}$$

$$1 = \frac{\cos^2 \alpha \cos^2 \theta - \sin^2 \alpha \sin^2 \theta}{\sin^2 \theta \cos^2 \theta}$$

$$1 = \frac{(1 - \sin^2 \alpha) \cos^2 \theta - \sin^2 \alpha (1 - \cos^2 \theta)}{\sin^2 \theta \cos^2 \theta}$$

$$\sin^2 \theta \cos^2 \theta = \cos^2 \theta - \cos^2 \theta \sin^2 \alpha - \sin^2 \alpha + \sin^2 \alpha \cos^2 \theta$$

$$(1 - \cos^2 \theta) \cos^2 \theta = \cos^2 \theta - \cos^4 \theta = \cos^2 \theta - \sin^2 \alpha$$

$$-\cos^4 \theta = -\sin^2 \alpha$$

$$\cos^4 \theta = \sin^2 \alpha$$

$$\cos^2 \theta = \pm \sin \alpha \quad \text{Proved}$$

Cos

9. if $\tan(\theta + i\phi) = \tan \alpha + i \sec \alpha$, prove that
 $e^{2\phi} = \pm \cot \frac{\alpha}{2} \quad \& \quad 2\theta = n\pi + \frac{\pi}{2} + \alpha$

Sol

$$\tan(\theta + i\phi) = \tan \alpha + i \sec \alpha$$

$$\theta + i\phi = \tan^{-1}(\tan \alpha + i \sec \alpha) \rightarrow (i)$$

$$\tan(\theta - i\phi) = \tan \alpha - i \sec \alpha \quad \therefore \overline{\tan z} = \tan \bar{z}$$

$$\theta - i\phi = \tan^{-1}(\tan \alpha - i \sec \alpha) \rightarrow (ii)$$

adding (i) and (ii)

$$2\theta = \tan^{-1}(\tan \alpha + i \sec \alpha) + \tan^{-1}(\tan \alpha - i \sec \alpha)$$

$$2\theta = \tan^{-1} \left[\frac{\tan \alpha + i \sec \alpha + \tan \alpha - i \sec \alpha}{1 - (\tan \alpha + i \sec \alpha)(\tan \alpha - i \sec \alpha)} \right] = \tan^{-1} \left[\frac{2 \tan \alpha}{1 - (\tan^2 \alpha - i^2 \sec^2 \alpha)} \right] = \tan^{-1} \left[\frac{2 \tan \alpha}{1 - \tan^2 \alpha + \sec^2 \alpha} \right]$$

$$2\theta = \tan^{-1} \left[\frac{2 \tan \alpha}{1 - (\tan^2 \alpha - i^2 \sec^2 \alpha)} \right]$$

$$\therefore i^2 = -1$$

$$2\theta = \tan^{-1} \left[\frac{2 \tan \alpha}{1 - (\tan^2 \alpha + \sec^2 \alpha)} \right]$$

$$2\theta = \tan^{-1} \left[\frac{2 \tan \alpha}{1 - \tan^2 \alpha - \sec^2 \alpha} \right]$$

$$2\theta = \tan^{-1} \left[\frac{2 \tan \alpha}{1 - \tan^2 \alpha - 1 - \tan^2 \alpha} \right]$$

$$\therefore \sec^2 \alpha = 1 + \tan^2 \alpha$$

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$$2\theta = \tan^{-1} \left(\frac{2 \tan \alpha}{-2 \tan^2 \alpha} \right)$$

$$= \tan^{-1} \left(-\frac{1}{\tan \alpha} \right)$$

$$= \tan^{-1} (-\cot \alpha)$$

$$\therefore \tan \left(\frac{\pi}{2} + \alpha \right) = -\cot \alpha$$

$$2\theta = \tan^{-1} \left[\tan \left(\frac{\pi}{2} + \alpha + n\pi \right) \right]$$

$$2\theta = \frac{\pi}{2} + \alpha + n\pi$$

(ii) Subtracting (i) & (ii)

$$2i\phi = \tan^{-1} (\tan \alpha + i \sec \alpha) - \tan^{-1} (\tan \alpha - i \sec \alpha)$$

$$2i\phi = \tan^{-1} \left[\frac{\tan \alpha + i \sec \alpha - (\tan \alpha - i \sec \alpha)}{1 + (\tan \alpha + i \sec \alpha)(\tan \alpha - i \sec \alpha)} \right]$$

$$= \tan^{-1} \left[\frac{\tan \alpha + i \sec \alpha - \tan \alpha + i \sec \alpha}{1 + \tan^2 \alpha + \sec^2 \alpha} \right]$$

$$= \tan^{-1} \left[\frac{2i \sec \alpha}{\sec^2 \alpha + \sec^2 \alpha} \right]$$

$$= \tan^{-1} \left[\frac{2i \sec \alpha}{2 \sec^2 \alpha} \right]$$

$$\tan(2i\phi) = \frac{i}{\sec \alpha}$$

$$i \tanh 2\phi = i \cos \alpha$$

$$\tanh 2\phi = \cos \alpha$$

$$\frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \cos \alpha$$

$$\frac{e^{2\phi} + e^{-2\phi}}{e^{2\phi} - e^{-2\phi}} = \frac{1}{\cos \alpha}$$

$$\frac{e^{2\phi} + e^{-2\phi} + e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi} - e^{2\phi} + e^{-2\phi}} = \frac{1 + \cos \alpha}{1 - \cos \alpha}$$

$$\frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{2 \cos^2(\alpha/2)}{2 \sin^2(\alpha/2)}$$

$$e^{2\phi + 2\phi} = \cot^2 \alpha/2$$

$$e^{4\phi} = \cot^2 \alpha/2$$

$$e^{2\phi} = \pm \cot(\alpha/2)$$

Using componendo-dividendo Theorem.

Proved



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10. Prove that.

$$\sinh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x - 1}{2}} \quad \text{if } x \geq 0$$

$$= -\sqrt{\frac{\cosh x - 1}{2}} \quad \text{if } x < 0$$

Sol:

$$\begin{aligned} \sqrt{\frac{\cosh x - 1}{2}} &= \sqrt{\frac{\frac{e^x + e^{-x}}{2} - 1}{2}} = \sqrt{\frac{e^x + e^{-x} - 2}{4}} \\ &= \sqrt{\frac{(e^{x/2} - e^{-x/2})^2}{(2)^2}} \\ &= \sqrt{\left(\frac{e^{x/2} - e^{-x/2}}{2}\right)^2} \\ &= \sqrt{\left(\sinh\left(\frac{x}{2}\right)\right)^2} \\ &= \pm \sinh\left(\frac{x}{2}\right) \end{aligned}$$

$$\sinh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh x - 1}{2}} \quad \text{when } x \geq 0$$

$$\sinh\left(\frac{x}{2}\right) = -\sqrt{\frac{\cosh x - 1}{2}} \quad \text{when } x < 0$$

11. Show that multiplication of a vector z by $e^{i\alpha}$, where α is real no. rotates the vector z counterclockwise through an angle of measure α .

Sol. Let $z = r[\cos\theta + i\sin\theta]$
 $z = re^{i\theta}$
 $z(e^{i\alpha}) = re^{i\theta} \cdot r^{i\alpha}$
 $z(e^{i\alpha}) = re^{i(\theta+\alpha)}$
 $= r[\cos(\theta+\alpha) + i\sin(\theta+\alpha)]$ proved.

12. Show that

$$(i) \quad 2+i = \sqrt{5} e^{i \tan^{-1}(1/2)}$$

$$z = 2+i, \quad r = |z| = \sqrt{4+1}$$

$$\cos\theta = \frac{2}{\sqrt{5}}, \quad \sin\theta = \frac{1}{\sqrt{5}}$$

Ist quadrant.

$$\tan\theta = \frac{1/\sqrt{5}}{2/\sqrt{5}} = 1/2$$

$$\theta = \tan^{-1}(1/2)$$

$$z = r(\cos\theta + i\sin\theta)$$

$$= \sqrt{5} [\cos(\tan^{-1}(1/2)) + i\sin(\tan^{-1}(1/2))]$$

$$z = \sqrt{5} e^{i \tan^{-1}(1/2)}$$

$$(ii) \quad -3-4i = 5e^{i(\pi + \tan^{-1}(4/3))}$$

$$z = -3-4i, \quad r = \sqrt{9+16} = \sqrt{25}$$

$$r = 5$$

$$\cos\theta = -\frac{3}{5}, \quad \sin\theta = -\frac{4}{5}$$

$$\tan\theta = \pi + \alpha$$

$$\tan\alpha = 4/3$$

$$\theta = \pi + \tan^{-1}(4/3)$$

$$\alpha = \tan^{-1}(4/3)$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 5 [\cos(\pi + \tan^{-1}(4/3)) + i\sin(\pi + \tan^{-1}(4/3))]$$

$$z = 5 e^{i(\pi + \tan^{-1}(4/3))}$$

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Exercise 1.4

1. Prove that

(i) $\text{Log } i = \frac{\pi i}{2}$

Here

$$z = i$$

$$|r| = \sqrt{1+0} = 1$$

$$\cos \theta = 0, \sin \theta = 1$$

$$\text{Log } i = \log |z| + i\theta \Rightarrow \theta = \frac{\pi}{2}$$

$$= \ln(1) + i\left(\frac{\pi}{2}\right)$$

$$= 0 + i\left(\frac{\pi}{2}\right)$$

$$\text{Log } i = i\left(\frac{\pi}{2}\right)$$

(iii) $-\log(-1+i) = \frac{1}{2} \ln 2 + \frac{3\pi i}{4}$

Here $z = -1+i$

$$|z| = \sqrt{1+1} = \sqrt{2}$$

$$\cos \theta = -\frac{1}{\sqrt{2}}, \sin \theta = \frac{1}{\sqrt{2}}$$

reference angle = $\alpha = \frac{\pi}{4}$
 2nd quadrd

$$\theta = \pi - \alpha = \pi - \frac{\pi}{4}$$

$$\theta = \frac{3\pi}{4}$$

$$\log(-1+i) = \ln(\sqrt{2}) + i\left(\frac{3\pi}{4}\right)$$

$$= \frac{1}{2} \ln 2 + i\left(\frac{3\pi}{4}\right)$$

Proved

vii) $\log(1-i) = \frac{1}{2} \ln 2 - \frac{\pi i}{4}$

Here $z = 1-i$, $|z| = \sqrt{2}$

$$\cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = -\frac{1}{\sqrt{2}}$$

reference angle = $\alpha = \pi/4$

4th quadrant,

$$\theta = -\alpha = -\pi/4$$

$$\text{Log}(1-i) = \ln(2^{1/2}) + i\left(-\frac{\pi}{4}\right)$$

$$= \frac{1}{2} \ln 2 - \frac{i\pi}{4}$$

Proved

$$\log z = \ln |z| + i\theta$$

(ii) $\text{Log}(-5) = \ln 5 + \pi i$

Here $z = -5$, $|z| = 5$

$$\cos \theta = -1, \sin \theta = 0$$

quadrantal angle
 $\theta = \pi$

$$\text{Log}(-5) = \ln |z| + i(\theta)$$

$$\text{Log}(-5) = \ln(5) + i\pi$$

Proved

(iv) $\text{Log}(1+i) = \frac{1}{2} \ln 2 + \frac{\pi i}{4}$

Here $z = 1+i$, $|z| = \sqrt{2}$

$$\cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

$$\log(1+i) = \ln(2^{1/2}) + i\left(\frac{\pi}{4}\right)$$

$$\log(1+i) = \frac{1}{2} \ln 2 + \frac{\pi i}{4}$$

proved

(v) $\log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{2\pi i}{3}$

Here

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, |z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$|z| = 1$$

$$\cos \theta = -\frac{1}{2}, \sin \theta = -\frac{\sqrt{3}}{2}$$

reference angle = $\alpha = \pi/3$

3rd quadrant

$$\theta = \alpha - \pi = \frac{\pi}{3} - \pi$$

$$\theta = -\frac{2\pi}{3}$$

$$\log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \ln(1) + i\left(-\frac{2\pi}{3}\right)$$

$$= 0 - \frac{2\pi i}{3}$$

$$\log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{2\pi i}{3}$$

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$$\therefore 1 - \tanh^2 z = \operatorname{sech}^2 z$$

$$1 - \operatorname{sech}^2 z = \tanh^2 z$$

2. Show that

$$(i) \operatorname{Coth}^{-1} z = \frac{1}{2} \log \left(\frac{z+1}{z-1} \right)$$

$$\text{Let } \operatorname{Coth}^{-1} z = w$$

$$z = \operatorname{Coth} w$$

Now

$$\frac{z+1}{z-1} = \frac{\operatorname{Coth} w + 1}{\operatorname{Coth} w - 1}$$

$$= \frac{\frac{e^w + e^{-w}}{e^w - e^{-w}} + 1}{\frac{e^w + e^{-w}}{e^w - e^{-w}} - 1}$$

$$= \frac{e^w + e^{-w} + e^w - e^{-w}}{e^w + e^{-w} - e^w + e^{-w}}$$

$$= \frac{2e^w}{2e^{-w}}$$

$$\frac{z+1}{z-1} = e^{w+w} = e^{2w}$$

$$2w = \log \left(\frac{z+1}{z-1} \right)$$

$$w = \frac{1}{2} \log \left(\frac{z+1}{z-1} \right)$$

$$\operatorname{Coth}^{-1} z = \frac{1}{2} \log \left(\frac{z+1}{z-1} \right)$$

$$(ii) \operatorname{Sech}^{-1} z = \log \left(\frac{1 + \sqrt{1-z^2}}{z} \right)$$

$$\text{Let } \operatorname{Sech}^{-1} z = w$$

$$\Rightarrow z = \operatorname{Sech} w$$

Now

$$\frac{1 + \sqrt{1-z^2}}{z} = \frac{1 + \sqrt{1 - \operatorname{sech}^2 w}}{\operatorname{sech} w}$$

$$= \frac{1 + \tanh w}{\operatorname{sech} w}$$

$$= \frac{1 + \tanh w}{\operatorname{sech} w}$$

$$= \frac{1 + \frac{\sinh w}{\cosh w}}{\frac{1}{\cosh w}}$$

$$= \cosh w + \sinh w$$

$$= \frac{e^w + e^{-w}}{2} + \frac{e^w - e^{-w}}{2}$$

$$= \frac{1}{2} (e^w + e^{-w} + e^w - e^{-w})$$

$$= \frac{1}{2} (2e^w)$$

$$\frac{1 + \sqrt{1-z^2}}{z} = e^w$$

$$w = \log \left(\frac{1 + \sqrt{1-z^2}}{z} \right)$$

$$(iii) \operatorname{Cosech}^{-1} z = \log \left(\frac{1 + \sqrt{z^2 + 1}}{z} \right)$$

Let

$$\operatorname{Cosech}^{-1} z = w \Rightarrow \operatorname{Cosech} w = z$$

$$\frac{1 + \sqrt{z^2 + 1}}{z} = \frac{1 + \sqrt{\operatorname{Cosech}^2 w + 1}}{\operatorname{Cosech} w}$$

$$= \frac{1 + \sqrt{\operatorname{Coth}^2 w}}{\operatorname{Cosech} w}$$

$$= \frac{1 + \operatorname{Coth} w}{\operatorname{Cosech} w}$$

$$= 1 + \frac{\cosh w}{\sinh w}$$

$$\frac{1}{\sinh w}$$

$$= \sinh w + \cosh w$$

$$= \frac{e^w - e^{-w}}{2} + \frac{e^w + e^{-w}}{2}$$

$$= \frac{1}{2} (e^w - e^{-w} + e^w + e^{-w})$$

$$= \frac{1}{2} (2e^w) = e^w$$

$$\frac{1 + \sqrt{z^2 + 1}}{z}$$

$$w = \log \left(\frac{1 + \sqrt{z^2 + 1}}{z} \right)$$

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$$\begin{aligned} \therefore 1 + \tan^2 z &= \sec^2 z \\ 1 - \sec^2 z &= -\tan^2 z \\ \Rightarrow \sec^2 z - 1 &= \tan^2 z \end{aligned}$$

3. Prove that:

$$(i) \cos^{-1} z = \frac{1}{i} \log(z + \sqrt{z^2 - 1})$$

$$\text{Let } \cos^{-1} z = w \\ \Rightarrow \cos w = z$$

$$\begin{aligned} z + \sqrt{z^2 - 1} &= \cos w + \sqrt{\cos^2 w - 1} \\ &= \cos w + \sqrt{-(1 - \cos^2 w)} \\ &= \cos w + \sqrt{-\sin^2 w} \\ &= \cos w + i \sin w \\ &= \frac{e^{iw} + e^{-iw}}{2} + i \cdot \frac{e^{iw} - e^{-iw}}{2i} \\ &= \frac{1}{2} (e^{iw} + e^{-iw} + e^{iw} - e^{-iw}) \\ &= \frac{1}{2} (2e^{iw}) \\ &= e^{iw} \end{aligned}$$

$$z + \sqrt{z^2 - 1} = e^{iw}$$

$$iw = \log(z + \sqrt{z^2 - 1})$$

$$w = \frac{1}{i} \log(z + \sqrt{z^2 - 1})$$

$$\cos^{-1} z = \frac{1}{i} \log(z + \sqrt{z^2 - 1})$$

$$(iii) \cot^{-1} z = \frac{1}{2i} \log\left(\frac{z+i}{z-i}\right)$$

$$\text{Let } \cot^{-1} z = w \Rightarrow \cot w = z$$

$$\frac{z+i}{z-i} = \frac{\cot w + i}{\cot w - i}$$

$$= \frac{\cos w}{\sin w} + i$$

$$\frac{\cos w}{\sin w} - i$$

$$= \frac{\cos w + i \sin w}{\cos w - i \sin w}$$

$$= \frac{e^{iw} + e^{-iw}}{2} + i \frac{e^{iw} - e^{-iw}}{2i}$$

$$\frac{e^{iw} + e^{-iw}}{2} + i \frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{1}{2} (e^{iw} + e^{-iw} + e^{iw} - e^{-iw})$$

$$= \frac{1}{2} (e^{iw} + e^{-iw} - e^{iw} + e^{-iw})$$

$$= \frac{2e^{-iw}}{2} = e^{-iw}$$

$$\frac{z+i}{z-i} = e^{-2iw}$$

$$2iw = \log\left(\frac{z+i}{z-i}\right)$$

$$w = \cot^{-1} z = \frac{1}{2i} \log\left(\frac{z+i}{z-i}\right)$$

$$(ii) \tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$$

$$\text{Let } \tan^{-1} z = w \Rightarrow \tan w = z$$

$$\frac{1+iz}{1-iz} = \frac{1+i \tan w}{1-i \tan w}$$

$$= \frac{1 + i \sin w}{\cos w}$$

$$\frac{1 - i \sin w}{\cos w}$$

$$= \frac{\cos w + i \sin w}{\cos w - i \sin w}$$

$$= \frac{e^{iw} + e^{-iw}}{2} + i \frac{e^{iw} - e^{-iw}}{2i}$$

$$\frac{e^{iw} + e^{-iw}}{2} + i \frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{1}{2} (e^{iw} + e^{-iw} + e^{iw} - e^{-iw})$$

$$= \frac{1}{2} (e^{iw} + e^{-iw} - e^{iw} + e^{-iw})$$

$$= \frac{2e^{iw}}{2e^{-iw}} = e^{2iw}$$

$$\frac{1+iz}{1-iz} = e^{2iw}$$

$$\log\left(\frac{1+iz}{1-iz}\right) = 2iw$$

$$w = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$$

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$$

$$(iv) \sec^{-1} z = \frac{1}{i} \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\text{Let } \sec^{-1} z = w \Rightarrow \sec w = z$$

$$\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\sec^2 w}}{\sec w} = \frac{1+\sqrt{-(\sec^2 w - 1)}}{\sec w}$$

$$= \frac{1+\sqrt{-\tan^2 w}}{\sec w} = \frac{1+\sqrt{-1} \tan w}{\sec w}$$

$$= \frac{1+i \tan w}{\sec w} = 1 + \frac{\sin w}{\cos w}$$

$$= \frac{\cos w + i \sin w}{\cos w}$$

$$= \frac{\cos w + i \sin w}{1/\cos w}$$

$$= \cos w + i \sin w$$

$$= \frac{e^{iw} + e^{-iw}}{2} + i \frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{1}{2} (e^{iw} + e^{-iw} + e^{iw} - e^{-iw})$$

$$\frac{1+\sqrt{1-z^2}}{z} = \frac{1}{2} (2e^{iw}) = e^{iw}$$

$$iw = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$w = \frac{1}{i} \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$(v) \operatorname{Cosec}^{-1} z = \frac{1}{i} \log \left(\frac{i + \sqrt{z^2 - 1}}{z} \right)$$

$$\text{Let } \operatorname{Cosec}^{-1} z = w \Rightarrow \operatorname{Cosec} w = z$$

$$\begin{aligned} \frac{i + \sqrt{z^2 - 1}}{z} &= \frac{i + \sqrt{\operatorname{Cosec}^2 w - 1}}{\operatorname{Cosec} w} \\ &= \frac{i + \sqrt{\cot^2 w}}{\operatorname{Cosec} w} = \frac{i + \cot w}{\operatorname{Cosec} w} \\ &= \frac{i + \frac{\cos w}{\sin w}}{1/\sin w} = \frac{i \sin w + \cos w}{\sin w / \sin w} \\ &= \cos w + i \sin w \end{aligned}$$

By Euler formula

$$e^{iw} = \cos w + i \sin w$$

$$\frac{i + \sqrt{z^2 - 1}}{z} = e^{iw}$$

$$iw = \log \left(\frac{i + \sqrt{z^2 - 1}}{z} \right)$$

$$\operatorname{Cosec}^{-1} w = w = \frac{1}{i} \log \left(\frac{i + \sqrt{z^2 - 1}}{z} \right)$$

4. Prove that

$$(i) \quad i^i = e^{-\pi/2}$$

$$\begin{aligned} i^i &= e^{\log i^i} \\ &= e^{i \log i} \\ &= e^{i [\ln|i| + i\theta]} \end{aligned}$$

$$\text{here } z = i, |z| = 1$$

$$\cos \theta = 0, \sin \theta = 1$$

$$\theta = \pi/2$$

$$i^i = e^{i [\ln(1) + i\pi/2]}$$

$$= e^{i(0 + i\pi/2)}$$

$$= e^{-\pi/2}$$

$$i^i = e^{-\pi/2}$$

$$(ii) \quad (-1)^i = e^{-\pi}$$

$$\begin{aligned} (-1)^i &= e^{\log(-1)^i} \\ &= e^{i \log(-1)} \\ &= e^{i [\ln|-1| + i\theta]} \end{aligned}$$

$$\text{here } z = -1, |z| = 1$$

$$\log|z| = \ln(1) = 0$$

$$\cos \theta = -1, \sin \theta = 0$$

$$\theta = \pi$$

$$(-1)^i = e^{i [0 + i\pi]}$$

$$= e^{-\pi}$$

$$(-1)^i = e^{-\pi}$$

$$(iii) \quad (-i)^{-i} = e^{-\pi/2}$$

$$\begin{aligned} (-i)^{-i} &= e^{\log(-i)^{-i}} \\ &= e^{-i \log(-i)} \\ &= e^{-i [\ln|-i| + i\theta]} \\ &= e^{-i [0 - i\pi/2]} \end{aligned}$$

$$= e^{-\pi/2}$$

$$(-i)^{-i} = e^{-\pi/2}$$

$$\text{here } z = -i, |z| = 1$$

$$\cos \theta = 0, \sin \theta = -1, \log|z| = \ln|z| = 0$$

$$\theta = -\pi/2$$

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$$(iv) a^i = \cos(\ln a) + i \sin(\ln a), a > 0$$

$$a^i = e^{\log a^i} = e^{i \log a}$$

$$a^i = e^{i[\ln|a| + i\theta]}$$

$$a^i = e^{i[\ln a + 0]}$$

$$a^i = e^{i(\ln a)}$$

$$= \cos(\ln a) + i \sin(\ln a), a > 0$$

here

$$z = a, |z| = a$$

$$\cos \theta = \frac{a}{a} = 1, \sin \theta = 0$$

$$\theta = 0^\circ$$

by euler formula

$$5. \text{ Prove that } \tanh^{-1} z = \sinh^{-1} \left(\frac{z}{\sqrt{1-z^2}} \right)$$

$$\text{Let } \tanh^{-1} z = w \Rightarrow z = \tanh w$$

$$\frac{z}{\sqrt{1-z^2}} = \frac{\tanh w}{\sqrt{1-\tanh^2 w}} = \frac{\tanh w}{\sqrt{\text{sech}^2 w}} = \frac{\tanh w}{\text{sech} w} = \frac{\sinh w \cdot \cosh w}{\cosh w}$$

$$= \sinh w$$

$$w = \sinh^{-1} \left(\frac{z}{\sqrt{1-z^2}} \right)$$

$$\tanh^{-1} z = \sinh^{-1} \left(\frac{z}{\sqrt{1-z^2}} \right)$$

$$6. \text{ Show that if } z = x + iy, \text{ then } \log \left(\frac{z}{\bar{z}} \right) = 2i \tan^{-1} \left(\frac{y}{x} \right)$$

$$\log \left(\frac{z}{\bar{z}} \right) = \log z - \log \bar{z}$$

$$= \log(x + iy) - \log(x - iy)$$

$$\text{for } z = x + iy$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

&

$$\text{For } \bar{z} = x - iy$$

$$|\bar{z}| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(-y/x) = -\tan^{-1}(y/x)$$

$$\log z = \ln|z| + i\theta$$

$$\log \left(\frac{z}{\bar{z}} \right) = \ln(\sqrt{x^2 + y^2}) + i \tan^{-1}(y/x) - \ln(\sqrt{x^2 + y^2}) + i \tan^{-1}(y/x)$$

$$\log \left(\frac{z}{\bar{z}} \right) = 2i \tan^{-1}(y/x)$$

$$7. \text{ if } \alpha^{a+ib} = (x+iy)^{p+iq}, a > 0 \text{ prove that}$$

$$(i) \alpha = \frac{1}{2} p \log_a(x^2 + y^2) - q \tan^{-1}(y/x) \log_a e$$

$$(ii) \log_a(x^2 + y^2) = \frac{2(\alpha p + \beta q)}{p^2 + q^2}$$

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Sol:

$$a^{\alpha+i\beta} = (x+iy)^{p+iq}$$

$$\log a^{\alpha+i\beta} = \log (x+iy)^{p+iq}$$

$$(\alpha+i\beta) \log a = (p+iq) \log (x+iy)$$

$$(\alpha+i\beta) [\ln|a| + i(0)] = (p+iq) [\ln [\sqrt{x^2+y^2}] + i \tan^{-1}(y/x)]$$

$$(\alpha+i\beta)(\ln a) = (p+iq) \left[\frac{1}{2} \ln(x^2+y^2) + i \tan^{-1}(y/x) \right]$$

$$\alpha \ln a + i\beta \ln a = \frac{p}{2} \ln(x^2+y^2) + i p \tan^{-1}(y/x) + i \frac{q}{2} \ln(x^2+y^2) + i^2 q \tan^{-1}(y/x)$$

$$\alpha \ln a + \beta \ln a = \frac{p}{2} \ln(x^2+y^2) - q \tan^{-1}(y/x) + i \left(p \tan^{-1}(y/x) + \frac{q}{2} \ln(x^2+y^2) \right)$$

(i) Equating real part.

$$\alpha \ln a = \frac{p}{2} \ln(x^2+y^2) - q \tan^{-1}(y/x) \cdot 1$$

$$= \frac{p}{2} \ln(x^2+y^2) - q \tan^{-1}(y/x) \ln e$$

$$\alpha = \frac{p}{2} \frac{\ln(x^2+y^2)}{\ln a} - q \tan^{-1}(y/x) \frac{\ln e}{\ln a}$$

$$\alpha = \frac{p}{2} \log_a(x^2+y^2) - q \tan^{-1}(y/x) \log_a e \quad \text{Proved} \rightarrow \textcircled{1}$$

(ii) Equating imaginary part.

$$\beta \ln a = p \tan^{-1}(y/x) + \frac{q}{2} \ln(x^2+y^2) \cdot 1$$

$$\beta \ln a = p \tan^{-1}(y/x) \cdot 1 + \frac{q}{2} \ln(x^2+y^2)$$

$$= \frac{q}{2} \ln(x^2+y^2) + p \tan^{-1}(y/x) \ln e$$

$$\beta = \frac{q}{2} \frac{\ln(x^2+y^2)}{\ln a} + p \tan^{-1}(y/x) \frac{\ln e}{\ln a}$$

$$= \frac{q}{2} \log_a(x^2+y^2) + p \tan^{-1}(y/x) \log_a e \rightarrow \textcircled{2}$$

multiplying (i) with p and (ii) with q & add

$$p\alpha = \frac{p^2}{2} \log_a(x^2+y^2) - pq \tan^{-1}(y/x) \log_a e$$

$$q\beta = \frac{q^2}{2} \log_a(x^2+y^2) + pq \tan^{-1}(y/x) \log_a e$$

$$p\alpha + q\beta = \frac{(p^2+q^2) \log_a(x^2+y^2)}{2}$$

$$\log_a(x^2+y^2) = \frac{2(p\alpha + q\beta)}{p^2+q^2}$$

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8. if $\text{Log Sin}(x+iy) = u+iv$, show that

$$\text{ii) } \cosh 2y = \cos 2x + 2e^{2u}$$

$$\text{ii) } e^{2v} = \frac{\cos(x-v)}{\cos(x+v)}$$

Sol.

$$\text{Log Sin}(x+iy) = u+iv$$

$$\text{Sin}(x+iy) = e^{u+iv} = e^u e^{iv}$$

$$\therefore \text{Sin} x \cosh y + i \cos x \sin y = e^u (\cos v + i \sin v)$$

$$\text{Sin} x \cosh y + i \cos x \sin y = e^u \cos v + i e^u \sin v$$

Equating Real & imaginary parts;

$$\text{Sin} x \cosh y = e^u \cos v \rightarrow \text{i) ; } \cos x \sin y = e^u \sin v \rightarrow \text{ii)}$$

Squaring & adding i) & ii)

dividing i) & ii)

$$\text{Sin}^2 x \cosh^2 y + \cos^2 x \sin^2 y = e^{2u} \cos^2 v + e^{2u} \sin^2 v$$

$$(1 - \cos^2 x) \cosh^2 y + \cos^2 x (\cosh^2 y - 1) = e^{2u} (\cos^2 v + \sin^2 v)$$

$$\cosh^2 y - \cos^2 x \cosh^2 y + \cos^2 x \cosh^2 y - \cos^2 x = e^{2u} (1)$$

$$\cosh^2 y - \cos^2 x = e^{2u}$$

$$\cosh^2 y = \cos^2 x + e^{2u}$$

$$\frac{1 + \cosh 2y}{2} = \frac{1 + \cos 2x}{2} + e^{2u}$$

$$1 + \cosh 2y = 1 + \cos 2x + 2e^{2u}$$

$$\cosh 2y = \cos 2x + 2e^{2u} \quad \text{Proved}$$

$$\frac{\text{Sin} x \cosh y}{\cos x \sin y} = \frac{e^u \cos v}{e^u \sin v}$$

$$\frac{\text{Sin} x \cosh y}{\cos x \sin y} = \frac{\cos v}{\sin v}$$

$$\frac{\cos x \cos v}{\text{Sin} x \sin v} = \frac{\cosh y}{\sinh y}$$

componendo-dividendo Theorem

$$\frac{\cos x \cos v + \text{Sin} x \sin v}{\cos x \cos v - \text{Sin} x \sin v} = \frac{\cosh y + \sinh y}{\cosh y - \sinh y}$$

$$\frac{\cos(x+v)}{\cos(x-v)} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y - e^{-y}}$$

$$= \frac{1}{2} \left(\frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y - e^{-y}} \right) = \frac{2e^y}{2e^{-y}} = e^{y+y} = e^{2y}$$

$$\frac{\cos(x-v)}{\cos(x+v)} = e^{2y} \quad \text{Proved}$$

9. Show that

$$\text{Log}(1 + \cos \theta + i \sin \theta) = \ln(2 \cos(\theta/2)) + i \theta/2$$

Sol.

$$\text{Log}((1 + \cos \theta) + i \sin \theta)$$

$$= \ln(\sqrt{2 + 2 \cos \theta}) + i \tan^{-1} \left(\frac{\sin \theta}{1 + \cos \theta} \right)$$

$$= \frac{1}{2} \ln(2(1 + \cos \theta)) + i \tan^{-1} \left(\frac{\sin \theta}{1 + \cos \theta} \right)$$

put i) & ii)

$$= \frac{1}{2} \ln(2 \cos^2 \theta/2) + i \tan^{-1} \left(\frac{2 \sin \theta/2 \cos \theta/2}{2 \cos^2 \theta/2} \right)$$

$$= \frac{2}{2} \ln(2 \cos^2 \theta/2) + i \tan^{-1}(\tan(\theta/2))$$

$$= \ln(2 \cos(\theta/2)) + i(\theta/2)$$

Proved

$$\text{here } z = 1 + \cos \theta + i \sin \theta$$

$$|z|^2 = (1 + \cos \theta)^2 + \sin^2 \theta$$

$$= 1 + \cos^2 \theta + 2 \cos \theta + \sin^2 \theta$$

$$= 1 + 1 + 2 \cos \theta$$

$$|z| = \sqrt{2 + 2 \cos \theta}$$

$$\theta = \tan^{-1} \left(\frac{\sin \theta}{1 + \cos \theta} \right)$$

$$\therefore \frac{1 + \cos \theta}{2} = \cos^2 \theta/2$$

$$1 + \cos \theta = 2 \cos^2 \theta/2$$

$$4(1 + \cos \theta) = 4 \cos^2 \theta/2$$

$$2(1 + \cos \theta) = 4 \cos^2 \theta/2 = (2 \cos \theta/2)^2$$

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10. Prove that.

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4} + \frac{i}{2} \ln\left(\frac{x+y}{x-y}\right) \quad \text{if } x > y > 0$$

Sol. Let $\alpha + i\beta = \tan^{-1}\left(\frac{x+iy}{x-iy}\right) \rightarrow (i)$ $\therefore \overline{\tan^{-1} z} = \tan^{-1} \bar{z}$

$$\overline{\alpha + i\beta} = \tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \tan^{-1}\left(\frac{x-iy}{x+iy}\right) \quad \therefore \left(\frac{\bar{z}_1}{\bar{z}_2}\right) = \frac{\bar{z}_1}{\bar{z}_2}$$

$$\alpha - i\beta = \tan^{-1}\left(\frac{x-iy}{x+iy}\right) \rightarrow (ii)$$

adding (i) & (ii)

$$2\alpha = \tan^{-1}\left(\frac{x+iy}{x-iy}\right) + \tan^{-1}\left(\frac{x-iy}{x+iy}\right) \quad \therefore \tan^{-1}(A+B) = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x+iy}{x-iy} + \frac{x-iy}{x+iy}}{1 - \left(\frac{x+iy}{x-iy}\right)\left(\frac{x-iy}{x+iy}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{(x+iy)^2 + (x-iy)^2}{(x+iy)(x-iy)}}{0}\right) \quad \frac{\text{any no}}{0} = \infty$$

$$= \tan^{-1}(\infty)$$

$$2\alpha = \frac{\pi}{2} \Rightarrow$$

$$\boxed{\alpha = \frac{\pi}{4}}$$

Subtracting (i) & (ii)

$$i\beta + i\beta = \tan^{-1}\left(\frac{x+iy}{x-iy}\right) - \tan^{-1}\left(\frac{x-iy}{x+iy}\right)$$

$$2i\beta = \tan^{-1}\left[\frac{\frac{x+iy}{x-iy} - \frac{x-iy}{x+iy}}{1 + \frac{x+iy}{x-iy} \cdot \frac{x-iy}{x+iy}}\right]$$

$$\tan^{-1}(A-B) = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$2i\beta = \tan^{-1}\left[\frac{\frac{(x+iy)^2 - (x-iy)^2}{(x+iy)(x-iy)}}{1+1}\right]$$

$$2i\beta = \tan^{-1}\left[\frac{x^2 - y^2 + 2xyi - x^2 + y^2 + 2xyi}{2(x^2 + y^2)}\right]$$

$$= \tan^{-1}\left(\frac{i^2 4xy}{2(x^2 + y^2)}\right)$$

$$2i\beta = \tan^{-1}\left(i \frac{2xy}{(x^2 + y^2)}\right)$$

$$\tan(2i\beta) = i \left(\frac{2xy}{x^2 + y^2}\right)$$

$$i \tanh 2\beta = i \left(\frac{2xy}{x^2 + y^2}\right)$$

$$\frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}} = \frac{2xy}{x^2 + y^2} \Rightarrow$$

$$\frac{e^{2\beta} + e^{-2\beta}}{e^{2\beta} - e^{-2\beta}} = \frac{x^2 + y^2}{2xy}$$

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Componendo-dividendo Theorem.

$$\frac{e^{2\beta} + e^{-2\beta} + e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta} - e^{2\beta} + e^{-2\beta}} = \frac{x^2 + y^2 + 2xy}{x^2 + y^2 - 2xy} = \frac{(x+y)^2}{(x-y)^2}$$

$$\frac{2e^{2\beta}}{2e^{-2\beta}} = \frac{(x+y)^2}{(x-y)^2}$$

$$e^{4\beta} = \left(\frac{x+y}{x-y}\right)^2$$

$$24\beta = \ln\left(\frac{x+y}{x-y}\right)^2 = 2\ln\left(\frac{x+y}{x-y}\right)$$

$$2\beta = \ln\left(\frac{x+y}{x-y}\right)$$

$$\beta = \frac{1}{2} \ln\left(\frac{x+y}{x-y}\right)$$



$$\therefore \tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \alpha + i\beta$$

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4} + \frac{i}{2} \ln\left(\frac{x+y}{x-y}\right) \quad \text{Proved}$$

11. Prove that

$$(i) \sec(x+iy) = 2 \left[\frac{\cos x \cosh y + i \sin x \sinh y}{\cos 2x + \cosh 2y} \right]$$

$$\text{Sol: } \sec(x+iy) = \frac{1}{\cos(x+iy)} \cdot \frac{\cos(x-iy)}{\cos(x-iy)}$$

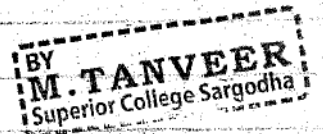
$$= \frac{\cos(x-iy)}{\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$$

$$= \frac{2\cos(x-iy)}{\cos(x+iy+x-iy) + \cos(x+iy-x+iy)}$$

$$= 2 \left[\frac{\cos x \cos(iy) + \sin x \sin(iy)}{\cos(2x) + \cos(2iy)} \right]$$

$$= 2 \left[\frac{\cos x \cosh y + i \sin x \sinh y}{\cos 2x + \cosh 2y} \right] \quad \text{proved}$$



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(ii) $\cos^{-1}(\cos \theta + i \sin \theta) = \sin^{-1} \sqrt{\sin \theta} + i \ln(\sqrt{1 + \sin \theta} - \sqrt{\sin \theta})$

Sol:

$$\alpha + i\beta = \cos^{-1}(\cos \theta + i \sin \theta)$$

$$\cos(\alpha + i\beta) = \cos \theta + i \sin \theta$$

$$\cos \alpha \cos(i\beta) - \sin \alpha \sin(i\beta) = \cos \theta + i \sin \theta$$

$$\cos \alpha \cosh \beta - i \sin \alpha \sinh \beta = \cos \theta + i \sin \theta$$

Equating real and imaginary parts

$$- \cos \alpha \cosh \beta = \cos \theta \quad ; \quad - \sin \alpha \sinh \beta = \sin \theta$$

$$\Rightarrow \cosh \beta = \frac{\cos \theta}{\cos \alpha} \quad ; \quad \sinh \beta = - \frac{\sin \theta}{\sin \alpha}$$

Squaring & Subtracting

$$\begin{aligned} \cosh^2 \beta - \sinh^2 \beta &= \frac{\cos^2 \theta}{\cos^2 \alpha} - \frac{\sin^2 \theta}{\sin^2 \alpha} \\ &= \frac{\cos^2 \theta \sin^2 \alpha - \sin^2 \theta \cos^2 \alpha}{\cos^2 \alpha \sin^2 \alpha} \\ &= \frac{(1 - \sin^2 \theta) \sin^2 \alpha - \sin^2 \theta (1 - \sin^2 \alpha)}{\cos^2 \alpha \sin^2 \alpha} \\ &= \frac{\sin^2 \alpha - \sin^2 \theta \sin^2 \alpha - \sin^2 \theta + \sin^2 \theta \sin^2 \alpha}{\cos^2 \alpha \sin^2 \alpha} \end{aligned}$$

$$\cos^2 \alpha \sin^2 \alpha = \sin^2 \alpha - \sin^2 \theta$$

$$(1 - \sin^2 \alpha) \sin^2 \alpha = \sin^2 \alpha - \sin^2 \theta \Rightarrow \sin^4 \alpha = \sin^2 \theta$$

$$\Rightarrow \sin^2 \alpha = \sin^2 \theta$$

$$\Rightarrow \sin^2 \alpha = \sin^2 \theta$$

$$\Rightarrow \sin \alpha = \sqrt{\sin \theta} \Rightarrow \boxed{\alpha = \sin^{-1} \sqrt{\sin \theta}}$$

Now

$$\cos \alpha = \frac{\cos \theta}{\cosh \beta} \quad ; \quad \sin \alpha = - \frac{\sin \theta}{\sinh \beta}$$

Squaring & adding

$$\begin{aligned} \cos^2 \alpha + \sin^2 \alpha &= \frac{\cos^2 \theta}{\cosh^2 \beta} + \frac{\sin^2 \theta}{\sinh^2 \beta} \\ 1 &= \frac{\cos^2 \theta \sinh^2 \beta + \sin^2 \theta \cosh^2 \beta}{\cosh^2 \beta \sinh^2 \beta} \end{aligned}$$

$$\cosh^2 \beta \sinh^2 \beta = \cos^2 \theta \sinh^2 \beta + \sin^2 \theta \cosh^2 \beta$$

$$(1 + \sinh^2 \beta) \sinh^2 \beta = (1 - \sin^2 \theta) \sinh^2 \beta + \sin^2 \theta (1 + \sinh^2 \beta)$$

$$\sinh^2 \beta + \sinh^4 \beta = \sinh^2 \beta - \sin^2 \theta \sinh^2 \beta + \sin^2 \theta + \sin^2 \theta \sinh^2 \beta$$

$$\sinh^4 \beta = \sin^2 \theta$$

$$\sinh^2 \beta = \sin \theta$$

$$\sinh \beta = \sqrt{\sin \theta}$$

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$$\sinh \beta = \sqrt{\sin \theta}$$

$$e^{\beta} - e^{-\beta} = \sqrt{\sin \theta} \Rightarrow e^{\beta} - e^{-\beta} = 2\sqrt{\sin \theta}$$

$$e^{\beta} - \frac{1}{e^{\beta}} = 2\sqrt{\sin \theta}$$

$$\frac{e^{2\beta} - 1}{e^{\beta}} = 2\sqrt{\sin \theta} \Rightarrow e^{2\beta} - 1 = 2\sqrt{\sin \theta} e^{\beta}$$

$$e^{2\beta} - 2e^{\beta}\sqrt{\sin \theta} - 1 = 0$$

quadratic in e^{β}

$$a = 1, b = -2\sqrt{\sin \theta}, c = -1$$

$$e^{\beta} = \frac{2\sqrt{\sin \theta} \pm \sqrt{4\sin \theta + 4}}{2}$$

$$e^{\beta} = \frac{2\sqrt{\sin \theta} \pm 2\sqrt{\sin \theta + 1}}{2} = \frac{2(\sqrt{\sin \theta} \pm \sqrt{\sin \theta + 1})}{2}$$

$$e^{\beta} = \sqrt{\sin \theta} \pm \sqrt{\sin \theta + 1}$$

$$\beta = \ln |\sqrt{\sin \theta + 1} \pm \sqrt{\sin \theta}| \Rightarrow \beta = \ln \left| \frac{\sqrt{\sin \theta} - \sqrt{\sin \theta - 1}}{\sqrt{\sin \theta - 1} - \sqrt{\sin \theta}} \right|$$

$$\cos^{-1}(\cos \theta + i \sin \theta) = \alpha + i\beta = \sin^{-1} \sqrt{\sin \theta} + i \ln (\sqrt{\sin \theta + 1} - \sqrt{\sin \theta}) \text{ Proved.}$$

$$(iii) \tan^{-1}(\cos \theta + i \sin \theta) = \pm \frac{\pi}{4} + \frac{i}{4} \ln \frac{1 + \sin \theta}{1 - \sin \theta}$$

Sol

$$\alpha + i\beta = \tan^{-1}(\cos \theta + i \sin \theta) \rightarrow i,$$

$$\alpha - i\beta = \tan^{-1}(\cos \theta - i \sin \theta) \rightarrow ii,$$

adding (i) & (ii)

$$2\alpha = \tan^{-1}(\cos \theta + i \sin \theta) + \tan^{-1}(\cos \theta - i \sin \theta)$$

$$= \tan^{-1} \left[\frac{\cos \theta + i \sin \theta + \cos \theta - i \sin \theta}{1 - (\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)} \right]$$

$$= \tan^{-1} \left[\frac{2\cos \theta}{1 - (\cos^2 \theta + \sin^2 \theta)} \right]$$

$$= \tan^{-1} \left[\frac{2\cos \theta}{1 - 1} \right]$$

$$= \tan^{-1} \left(\frac{2\cos \theta}{0} \right)$$

$$= \tan^{-1}(\infty)$$

$$2\alpha = \pm \pi/2 \Rightarrow \alpha = \pm \pi/4$$

Subtracting (i) & (ii)

$$2i\beta = \tan^{-1}(\cos \theta + i \sin \theta) - \tan^{-1}(\cos \theta - i \sin \theta)$$

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$$2i\beta = \tan^{-1} \left[\frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{1 + (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} \right]$$

$$= \tan^{-1} \left(\frac{2i\sin\theta}{1 + \cos^2\theta + \sin^2\theta} \right)$$

$$\tan(2i\beta) = \frac{2i\sin\theta}{1+1}$$

$$i \tanh 2\beta = \frac{2i\sin\theta}{2}$$

$$\tanh 2\beta = \sin\theta$$

$$\frac{1}{\tanh 2\beta} = \frac{1}{\sin\theta}$$

$$\frac{e^{2\beta} + e^{-2\beta}}{e^{2\beta} - e^{-2\beta}} = \frac{1}{\sin\theta}$$

$$\frac{e^{2\beta} + e^{-2\beta} + e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta} - e^{2\beta} + e^{-2\beta}} = \frac{1 + \sin\theta}{1 - \sin\theta}$$

$$\frac{2e^{2\beta}}{2e^{-2\beta}} = \frac{1 + \sin\theta}{1 - \sin\theta} = e^{2\beta + 2\beta}$$

$$e^{4\beta} = \frac{1 + \sin\theta}{1 - \sin\theta}$$

$$4\beta = \ln \left(\frac{1 + \sin\theta}{1 - \sin\theta} \right)$$

$$\beta = \frac{1}{4} \ln \left(\frac{1 + \sin\theta}{1 - \sin\theta} \right)$$

Putting value of α and β .

$$\tan^{-1}(\cos\theta + i\sin\theta) = \alpha + i\beta = \frac{+i\pi}{4} + \frac{i}{4} \ln \left(\frac{1 + \sin\theta}{1 - \sin\theta} \right) \text{ proved}$$

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Exercise 1.5

In each of Problem 1-5, evaluate The indicated sum.

1. (i) $\sin A + \sin 2A + \sin 3A + \dots + \sin nA$

(ii) $\cos A + \cos 2A + \cos 3A + \dots + \cos nA$

Sol: Let $S = \sin A + \cos A + \sin 2A + \sin 3A + \dots + \sin nA$

$C = \cos A + \cos 2A + \dots + \cos nA$

$C + iS = (\cos A + i\sin A) + (\cos 2A + i\sin 2A) + \dots + (\sin nA + \cos nA)$

$= e^{iA} + e^{i2A} + \dots + e^{inA}$

This is geometric progression with $r = e^{iA}$
here $a = e^{iA}$, $r = e^{iA}$, $n = n$

$C + iS = \frac{a[r^n - 1]}{r - 1}$

$= \frac{e^{iA}(e^{inA} - 1)}{e^{iA} - 1}$

$= \frac{e^{iA} \cdot e^{i nA/2} (e^{i nA/2} - e^{-i nA/2})}{e^{iA/2} (e^{iA/2} - e^{-iA/2})}$

$= \frac{e^{iA - iA/2 + i nA/2} (e^{i nA/2} - e^{-i nA/2})}{e^{iA/2} (e^{iA/2} - e^{-iA/2})}$

$= \frac{e^{iA/2 + i nA/2} \sin(nA/2)}{\sin(A/2)}$

$= \frac{e^{iA/2(n+1)} \sin(nA/2)}{\sin(A/2)}$

$C + iS = \left(\cos\left(\frac{A(n+1)}{2}\right) + i \sin\left(\frac{A(n+1)}{2}\right) \right) \frac{\sin(nA/2)}{\sin(A/2)}$

Equating real and imaginary part

(ii) $C = \cos\left(\frac{A(n+1)}{2}\right) \cdot \frac{\sin(nA/2)}{\sin(A/2)}$ Ans

(i) $S = \sin\left(\frac{A(n+1)}{2}\right) \cdot \frac{\sin(nA/2)}{\sin(A/2)}$ Ans

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2. $\cos \theta + \cos 3\theta + \dots + \cos(2n-1)\theta = C$

Sol Let $C = \cos \theta + \cos 3\theta + \dots + \cos(2n-1)\theta$

$S = \sin \theta + \sin 3\theta + \dots + \sin(2n-1)\theta$

$C + iS = (\cos \theta + i \sin \theta) + (\cos 3\theta + i \sin 3\theta) + \dots + (\cos(2n-1)\theta + i \sin(2n-1)\theta)$

$C + iS = e^{i\theta} + e^{i3\theta} + e^{i5\theta} + \dots + e^{i(2n-1)\theta}$

geometric progression

$a = e^{i\theta}, r = e^{i2\theta}, n = n$

$C + iS = \frac{e^{i\theta} (e^{i2n\theta} - 1)}{e^{i2\theta} - 1}$

$\therefore \text{Sum} = \frac{a(r^n - 1)}{(r - 1)}$

$= \frac{e^{i\theta} e^{i2n\theta} (e^{i2\theta} - 1)}{e^{i2\theta} - 1}$

$= \frac{e^{i\theta} (e^{i2n\theta} - e^{-i2n\theta})}{e^{i2\theta} - 1}$

$= \frac{e^{i\theta} (e^{i2n\theta} - e^{-i2n\theta})}{2i}$

$= \frac{e^{i\theta} (e^{i2n\theta} - e^{-i2n\theta})}{2i}$

$= (\cos n\theta + i \sin n\theta) \cdot \frac{\sin n\theta}{\sin \theta}$

Equating Real part

$C = \cos n\theta \cdot \frac{\sin n\theta}{\sin \theta} \quad \text{Ans.}$

3. $1 + x \cos \theta + x^2 \cos 2\theta + \dots + x^n \cos n\theta$

Sol Let $C = 1 + x \cos \theta + x^2 \cos 2\theta + \dots + x^n \cos n\theta$

$S = x \sin \theta + x^2 \sin 2\theta + \dots + x^n \sin n\theta$

$C + iS = 1 + x(\cos \theta + i \sin \theta) + x^2(\cos 2\theta + i \sin 2\theta) + \dots$

$+ x^n(\cos n\theta + i \sin n\theta)$

$C + iS = 1 + x e^{i\theta} + x^2 e^{i2\theta} + \dots + e^{in\theta} \cdot x^n$

$a = 1, r = x e^{i\theta}, n = n+1$

$C + iS = \frac{1 [x^{n+1} e^{i(n+1)\theta} - 1]}{x e^{i\theta} - 1}$

$\therefore \frac{a(r^{n+1} - 1)}{r - 1}$

$= \frac{x^{n+1} (\cos(n+1)\theta + i \sin(n+1)\theta) - 1}{x \cos \theta - 1}$

$= \frac{x(\cos \theta + i \sin \theta) - 1}{x \cos \theta - 1}$

$= \frac{[x^{n+1} \cos(n+1)\theta - 1] + i x^{n+1} \sin(n+1)\theta}{(x \cos \theta - 1) + i x \sin \theta}$

$(x \cos \theta - 1) + i x \sin \theta$

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multiplying & dividing by $(x \cos \theta - 1) - i x \sin \theta$

$$C + iS = \frac{(x^{n+1} \cos(n+1)\theta - 1) + i x^{n+1} \sin(n+1)\theta}{(x \cos \theta - 1) + i x \sin \theta} \times \frac{(x \cos \theta - 1) - i x \sin \theta}{(x \cos \theta - 1) - i x \sin \theta}$$

Equating real part

$$C = \frac{(x^{n+1} \cos(n+1)\theta - 1)(x \cos \theta - 1) + x^{n+1} \sin(n+1)\theta \sin \theta \cdot x}{(x \cos \theta - 1)^2 + (x \sin \theta)^2}$$

$$C = \frac{x^{n+1} \cos(n+1)\theta x \cos \theta - x^{n+1} \cos(n+1)\theta - x \cos \theta + 1 + x^{n+2} \sin(n+1)\theta \sin \theta}{(x^2 \cos^2 \theta + 1 - 2x \cos \theta + x^2 \sin^2 \theta)}$$

$$C = \frac{x^{n+2} [\underbrace{\cos(n+1)\theta \cos \theta + \sin(n+1)\theta \sin \theta}_{\cos(\alpha - \beta)}] - x^{n+1} \cos(n+1)\theta - x \cos \theta + 1}{x^2 (\cos^2 \theta + \sin^2 \theta) + 1 - 2x \cos \theta}$$

$$C = \frac{x^{n+2} (\cos(n+1 - 1)\theta) - x^{n+1} \cos(n+1)\theta - x \cos \theta + 1}{x^2 + 1 - 2x \cos \theta}$$

$$C = \frac{x^{n+2} \cos n\theta - x^{n+1} \cos(n+1)\theta - x \cos \theta + 1}{x^2 + 1 - 2x \cos \theta} \quad \text{Ans}$$

4. Let

$$S = 3 \sin \alpha + 5 \sin 2\alpha + 7 \sin 3\alpha + \dots + (2n+1) \sin n\alpha$$

Sol

$$C = 3 \cos \alpha + 5 \cos 2\alpha + 7 \cos 3\alpha + \dots + (2n+1) \cos n\alpha$$

$$C + iS = 3(\cos \alpha + i \sin \alpha) + 5(\cos 2\alpha + i \sin 2\alpha) + 7(\cos 3\alpha + i \sin 3\alpha) + \dots + (2n+1)(\cos n\alpha + i \sin n\alpha)$$

$$C + iS = 3e^{i\alpha} + 5e^{i2\alpha} + 7e^{i3\alpha} + \dots + (2n+1)e^{in\alpha}$$

$$e^{i\alpha}(C + iS) = -3e^{i2\alpha} - 5e^{i3\alpha} + \dots - (2n-1)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$$

adding

$$(C + iS) - e^{i\alpha}(C + iS) = 3e^{i\alpha} + 2e^{i2\alpha} + 2e^{i3\alpha} + \dots + 2e^{in\alpha} - (2n+1)e^{i(n+1)\alpha}$$

$$(C + iS)(1 - e^{i\alpha}) = 3e^{i\alpha} + 2[e^{i2\alpha} + e^{i3\alpha} + \dots + e^{in\alpha}] - (2n+1)e^{i(n+1)\alpha}$$

g.p. $a = e^{i2\alpha}, r = e^{i\alpha}, n = n-1$

$$(C + iS)(1 - e^{i\alpha}) = 3e^{i\alpha} + 2 \left[\frac{e^{i2\alpha} [e^{i(n-1)\alpha} - 1]}{e^{i\alpha} - 1} \right] - (2n+1)e^{i(n+1)\alpha}$$

$$= 3e^{i\alpha}(e^{i\alpha} - 1) + 2e^{i2\alpha}(e^{i(n-1)\alpha} - 1) - (2n+1)e^{i(n+1)\alpha}(e^{i\alpha} - 1)$$

$$= 3e^{i\alpha}(e^{i\alpha} - 1) + 2 \left[\frac{e^{i\alpha} - 1}{e^{i\alpha} - 1} e^{i2\alpha + i(n-1)\alpha} - e^{i2\alpha} \right] - (2n+1)e^{i(n+1)\alpha}(e^{i\alpha} - 1)$$

(65)

$$= 3e^{i2\alpha} - 3e^{i\alpha} + 2e^{i(n+1)\alpha} - 2e^{i2\alpha} - (2n+1)e^{i(n+2)\alpha} + (2n+1)e^{i(n+1)\alpha}$$

$$= \frac{2e^{i2\alpha} - 3e^{i\alpha} + (2+2n+1)e^{i(n+1)\alpha} - (2n+1)e^{i(n+2)\alpha}}{e^{i\alpha} - 1}$$

$$= \frac{2e^{i2\alpha} - 3e^{i\alpha} + (2n+3)e^{i(n+1)\alpha} - (2n+1)e^{i(n+2)\alpha}}{e^{i\alpha} - 1}$$

$$(C+iS)(1-e^{i\alpha}) = e^{i\alpha}(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha})$$

$$(C+iS) = \frac{e^{i\alpha}(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha})}{e^{i\alpha} - 1}$$

$$= \frac{e^{i\alpha}(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha})}{-(e^{i\alpha} - 1)(e^{i\alpha} - 1)} = \frac{e^{i\alpha}(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha})}{-(e^{i\alpha} - 1)^2}$$

$$= \frac{e^{i\alpha}(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha})}{- \left(e^{i\alpha/2} \left(\frac{e^{i\alpha/2} - e^{-i\alpha/2}}{2i} \right) \right)^2 \cdot (2i)^2}$$

$$= \frac{e^{i\alpha}(e^{i\alpha} - 3 + (2n+3)e^{in\alpha} - (2n+1)e^{i(n+1)\alpha})}{4 e^{i\alpha} (\sin \alpha/2)^2}$$

$$C+iS = \frac{\cos \alpha + i \sin \alpha - 3 + (2n+3)(i \sin n\alpha + \cos n\alpha) - (2n+1)(\cos(n+1)\alpha + i \sin(n+1)\alpha)}{4 \sin^2 \alpha/2} = \frac{(1 - \cos \alpha)}{2}$$

Equating imaginary part

$$S = \frac{\sin \alpha + (2n+3) \sin n\alpha - (2n+1) \sin(n+1)\alpha}{2 \cdot \frac{(1 - \cos \alpha)}{2}}$$

$$S = \frac{\sin \alpha + (2n+3) \sin n\alpha - (2n+1) \sin(n+1)\alpha}{1 - \cos \alpha}$$

$$5. \cos^2 \theta + \cos^2 2\theta + \cos^2 3\theta + \dots + \cos^2 n\theta = C_1$$

$$\text{Sol } C_1 = \frac{1 + \cos 2\theta}{2} + \frac{1 + \cos 4\theta}{2} + \frac{1 + \cos 6\theta}{2} + \dots + \frac{1 + \cos 2n\theta}{2}$$

$$= \frac{1}{2} [1 + \cos 2\theta + 1 + \cos 4\theta + 1 + \cos 6\theta + \dots + 1 + \cos 2n\theta]$$

$$= \frac{1}{2} [(1+1+1+\dots+1 \text{ up to } n \text{ terms}) + (\cos 2\theta + \cos 4\theta + \dots + \cos 2n\theta)]$$

(66)

$$C_1 = \frac{1}{2} [n + (\cos 2\theta + \cos 4\theta + \cos 6\theta + \dots + \cos 2n\theta)] \rightarrow (1)$$

Let

$$C = \cos 2\theta + \cos 4\theta + \dots + \cos 2n\theta$$

$$S = \sin 2\theta + \sin 4\theta + \dots + \sin 2n\theta$$

$$C+iS = (\cos 2\theta + i\sin 2\theta) + (\cos 4\theta + i\sin 4\theta) + \dots + (\cos 2n\theta + i\sin 2n\theta)$$

$$C+iS = e^{i2\theta} + e^{i4\theta} + e^{i6\theta} + \dots + e^{i2n\theta}$$

$$a = e^{i2\theta}$$

$$C+iS = e^{i2\theta} \left[\frac{e^{i2n\theta} - 1}{e^{i2\theta} - 1} \right], \quad n = n$$

$$\frac{a(r^n - 1)}{r - 1} = \text{Sum}$$

$$= e^{i2\theta} \cdot \frac{e^{i2n\theta} - 1}{e^{i2\theta} - 1}$$

$$= e^{i2\theta} \cdot \frac{e^{i2n\theta} - e^{-i2n\theta}}{e^{i2\theta} - e^{-i2\theta}}$$

$$= e^{i2\theta} \cdot \frac{2i \sin n\theta}{2i \sin \theta} = e^{i(n+1)\theta} \frac{\sin n\theta}{\sin \theta}$$

$$C+iS = \frac{\sin n\theta}{\sin \theta} (\cos(n+1)\theta + i\sin(n+1)\theta)$$

Equating Real part

$$C = \frac{\sin n\theta}{\sin \theta} \cos(n+1)\theta$$

put in (1)

$$C_1 = \frac{1}{2} \left[n + \frac{\sin n\theta}{\sin \theta} \cos(n+1)\theta \right]$$

$$C_1 = \frac{n}{2} + \frac{\sin n\theta}{2\sin \theta} \cos(n+1)\theta$$

Proved.

In Problems 6-15. find sum of each infinite series.

$$6. \sin \theta + \frac{1}{2} \sin 3\theta + \frac{1 \cdot 3}{2 \cdot 4} \sin 5\theta + \dots = S$$

Sol

$$C = \cos \theta + \frac{1}{2} \cos 3\theta + \frac{1 \cdot 3}{2 \cdot 4} \cos 5\theta + \dots$$

$$C+iS = (\cos \theta + i\sin \theta) + \frac{1}{2} (\cos 3\theta + i\sin 3\theta) + \frac{1 \cdot 3}{2 \cdot 4} (\cos 5\theta + i\sin 5\theta) + \dots$$

(67)

$$= e^{i\theta} + \frac{1}{2} e^{i3\theta} + \frac{1 \cdot 3}{2 \cdot 4} e^{i5\theta} + \dots$$

$$C+iS = e^{i\theta} \left(1 + \frac{1}{2} e^{i2\theta} + \frac{1 \cdot 3}{2 \cdot 4} e^{i4\theta} + \dots \right)$$

$$= e^{i\theta} (1 - e^{i2\theta})^{-1/2}$$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \dots$$

$$C+iS = e^{i\theta} (1 - (\cos 2\theta + i \sin 2\theta))^{-1/2}$$

$$= e^{i\theta} ((1 - \cos 2\theta) - i \sin 2\theta)^{-1/2}$$

$$\frac{1 - \cos 2\theta}{2} = \sin^2 \theta$$

$$1 - \cos 2\theta = 2 \sin^2 \theta$$

$$= e^{i\theta} (2 \sin^2 \theta - i 2 \sin \theta \cos \theta)^{-1/2}$$

$$= e^{i\theta} [2 \sin \theta (\sin \theta - i \cos \theta)]^{-1/2}$$

$$= e^{i\theta} (2 \sin \theta)^{1/2} [\sin \theta - i \cos \theta]^{-1/2}$$

$$= e^{i\theta} (2 \sin \theta)^{1/2} \left[\cos \left(\frac{\pi}{2} - \theta \right) - i \sin \left(\frac{\pi}{2} - \theta \right) \right]^{-1/2}$$

De-Moivre's Theorem

$$= e^{i\theta} (2 \sin \theta)^{1/2} \left[\cos \left(-\frac{1}{2} \left(\frac{\pi}{2} - \theta \right) \right) - i \sin \left(-\frac{1}{2} \left(\frac{\pi}{2} - \theta \right) \right) \right]$$

$$= e^{i\theta} (2 \sin \theta)^{1/2} \left[\cos \left(\frac{1}{2} \left(\frac{\pi}{2} - \theta \right) \right) + i \sin \left(\frac{1}{2} \left(\frac{\pi}{2} - \theta \right) \right) \right]$$

$$= e^{i\theta} (2 \sin \theta)^{1/2} \left[\cos \left(\frac{\pi}{4} - \frac{\theta}{2} \right) + i \sin \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right]$$

Equating imaginary part

$$S = (2 \sin \theta)^{1/2} \left(\cos \theta \sin \left(\frac{\pi}{4} - \frac{\theta}{2} \right) + \sin \theta \cos \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right)$$

$$= (2 \sin \theta)^{1/2} \left[\sin \left(\theta + \frac{\pi}{4} - \frac{\theta}{2} \right) \right]$$

$$S = (2 \sin \theta)^{1/2} \sin \left(\frac{\theta}{2} + \frac{\pi}{4} \right) \quad \text{Ans}$$

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$$\sinh \theta = \frac{e^\theta - e^{-\theta}}{2}$$

$$7. \quad \sinh \theta + \frac{\sinh 2\theta}{2!} + \frac{\sinh 3\theta}{3!} + \dots$$

Sol.

$$= \frac{e^\theta - e^{-\theta}}{2} + \frac{e^{2\theta} - e^{-2\theta}}{2 \cdot 2!} + \frac{e^{3\theta} - e^{-3\theta}}{3! \cdot 2} + \dots$$

$$= \frac{1}{2} \left[e^\theta - e^{-\theta} + \frac{e^{2\theta}}{2} - \frac{e^{-2\theta}}{2} + \frac{e^{3\theta}}{3!} - \frac{e^{-3\theta}}{3!} + \dots \right]$$

$$= \frac{1}{2} \left[1 - 1 + e^\theta - e^{-\theta} + \frac{e^{2\theta}}{2} - \frac{e^{-2\theta}}{2} + \frac{e^{3\theta}}{3!} - \frac{e^{-3\theta}}{3!} + \dots \right]$$

$$= \frac{1}{2} \left[\left(1 + e^\theta + \frac{e^{2\theta}}{2!} + \frac{e^{3\theta}}{3!} + \dots \right) - \left(1 + \frac{e^{-\theta}}{1!} + \frac{e^{-2\theta}}{2!} + \frac{e^{-3\theta}}{3!} + \dots \right) \right]$$

(69)

$$9. \quad C = 1 - \frac{1}{2} \cos \theta + \frac{1 \cdot 3}{2 \cdot 4} \cos 2\theta - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cos 4\theta + \dots$$

$$S = -\frac{1}{2} \sin \theta + \frac{1 \cdot 3}{2 \cdot 4} \sin 2\theta - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \sin 4\theta + \dots$$

$$C + iS = 1 - \frac{1}{2} (\cos \theta + i \sin \theta) + \frac{1 \cdot 3}{2 \cdot 4} (\cos 2\theta + i \sin 2\theta) - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} (\cos 4\theta - i \sin 4\theta) + \dots$$

$$= 1 - \frac{1}{2} e^{i\theta} + \frac{1 \cdot 3}{2 \cdot 4} e^{i2\theta} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} e^{i4\theta} + \dots$$

$$C + iS = (1 + e^{i\theta})^{-1/2}$$

$$= (1 + \cos \theta + i \sin \theta)^{-1/2}$$

$$= (2 \cos^2 \theta/2 + i 2 \cos \theta/2 \sin \theta/2)^{-1/2}$$

$$= (2 \cos \theta/2 (\cos \theta/2 + i \sin \theta/2))^{-1/2}$$

$$= (2 \cos \theta/2)^{-1/2} \left[\cos \left(-\frac{1}{2} \cdot \frac{\theta}{2} \right) + i \sin \left(-\frac{1}{2} \cdot \frac{\theta}{2} \right) \right]$$

$$= (2 \cos \theta/2)^{-1/2} \left[\cos \left(\frac{\theta}{4} \right) - i \sin \left(\theta/4 \right) \right]$$

Equating real part.

$$C = (2 \cos \theta/2)^{-1/2} \cos(\theta/4)$$

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$$10. \quad n \sin \theta + \frac{n(n+1)}{2!} \sin 2\theta + \frac{n(n+1)(n+2)}{3!} \sin 3\theta + \dots = S$$

$$C = 1 + n \cos \theta + \frac{n(n+1)}{2!} \cos 2\theta + \frac{n(n+1)(n+2)}{3!} \cos 3\theta + \dots$$

$$C + iS = 1 + n(\cos \theta + i \sin \theta) + \frac{n(n+1)}{2!} (\cos 2\theta + i \sin 2\theta) + \frac{n(n+1)(n+2)}{3!} (\cos 3\theta + i \sin 3\theta) + \dots$$

$$C + iS = 1 + n e^{i\theta} + \frac{n(n+1)}{2!} e^{i2\theta} + \frac{n(n+1)(n+2)}{3!} e^{i3\theta} + \dots$$

$$= (1 - e^{i\theta})^{-n}$$

$$= (1 - \cos \theta - i \sin \theta)^{-n}$$

$$= \left(2 \sin^2 \frac{\theta}{2} - i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right)^{-n}$$

$$= \left[2 \sin \theta/2 \left(\sin \theta/2 - i \cos \theta/2 \right) \right]^{-n}$$

$$= (2 \sin \theta/2)^{-n} \left(\cos \left(\frac{\pi}{2} - \frac{\theta}{2} \right) - i \sin \left(\frac{\pi}{2} - \frac{\theta}{2} \right) \right)^{-n}$$

$$= (2 \sin \theta/2)^{-n} \left[\cos \left(\frac{\pi}{2} - \frac{\theta}{2} \right) + i \sin \left(\frac{\theta}{2} - \frac{\pi}{2} \right) \right]^{-n}$$

De-Moivre's Theorem

$$\frac{1 - \cos \theta}{2} = \sin^2 \theta/2$$

$$1 - \cos \theta = 2 \sin^2 \theta/2$$

(68)

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\begin{aligned}
 &= \frac{1}{2} (e^{e\theta} - e^{-e\theta}) \\
 &= \frac{1}{2} [e^{\cosh\theta + \sinh\theta} - e^{\cosh(-\theta) + \sinh(-\theta)}] \\
 &= \frac{1}{2} [e^{\cosh\theta} \cdot e^{\sinh\theta} - e^{\cosh\theta} \cdot e^{-\sinh\theta}] \\
 &= \frac{1}{2} [e^{\cosh\theta} \cdot e^{\sinh\theta} - e^{\cosh\theta} \cdot e^{-\sinh\theta}] \\
 &= \frac{1}{2} e^{\cosh\theta} [e^{\sinh\theta} - e^{-\sinh\theta}] \\
 &= e^{\cosh\theta} \left[\frac{e^{\sinh\theta} - e^{-\sinh\theta}}{2} \right] \\
 &= e^{\cosh\theta} \sinh(\sinh\theta) \quad \text{Ans}
 \end{aligned}$$

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$$8. \sin\alpha \sin\alpha + \sin^2\alpha \sin 2\alpha + \sin^3\alpha \sin 3\alpha + \dots = S$$

$$\text{Sol. } C = \sin\alpha \cos\alpha + \sin^2\alpha \cos 2\alpha + \sin^3\alpha \cos 3\alpha + \dots$$

$$C + iS = \sin\alpha (\cos\alpha + i\sin\alpha) + \sin^2\alpha (\cos 2\alpha + i\sin 2\alpha) + \sin^3\alpha (\cos 3\alpha + i\sin 3\alpha) + \dots$$

$$C + iS = \sin\alpha e^{i\alpha} + \sin^2\alpha e^{i2\alpha} + \sin^3\alpha e^{i3\alpha} + \dots$$

$$a = \sin\alpha e^{i\alpha}, \quad r = \sin\alpha e^{i\alpha} \quad \text{geometric series}$$

$$C + iS = \frac{\sin\alpha \cdot e^{i\alpha}}{1 - e^{i\alpha} \sin\alpha} \quad \text{Sum} = \frac{a}{1-r}$$

$$= \frac{\sin\alpha (\cos\alpha + i\sin\alpha)}{1 - \sin\alpha (\cos\alpha + i\sin\alpha)}$$

$$= \frac{\sin\alpha \cos\alpha + i\sin^2\alpha}{(1 - \sin\alpha \cos\alpha) - i\sin^2\alpha} \times \frac{(1 - \sin\alpha \cos\alpha) + i\sin^2\alpha}{(1 - \sin\alpha \cos\alpha) + i\sin^2\alpha}$$

$$C + iS = \frac{(\cos\alpha \sin\alpha + i\sin^2\alpha) ((1 - \sin\alpha \cos\alpha) + i\sin^2\alpha)}{(1 - \sin\alpha \cos\alpha)^2 + (\sin^2\alpha)^2}$$

Equating imaginary parts.

$$S = \frac{\cos\alpha \sin\alpha \sin^2\alpha + \sin^4\alpha (1 - \sin\alpha \cos\alpha)}{1 + \sin^2\alpha \cos^2\alpha - 2\sin\alpha \cos\alpha + \sin^4\alpha}$$

$$= \frac{\cos\alpha \sin\alpha \sin^2\alpha + \sin^4\alpha - \cos\alpha \sin\alpha \sin^2\alpha}{1 + \sin^2\alpha (\cos^2\alpha + \sin^2\alpha) - 2\cos\alpha \sin\alpha}$$

$$S = \frac{\sin^4\alpha}{1 + \sin^2\alpha - 2\cos\alpha \sin\alpha} \quad \text{Ans}$$

(76)

$$C + iS = (2\sin \theta/2)^{-n} \left[\cos n\left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i \sin \left(-n\left(\frac{\theta}{2} - \frac{\pi}{2}\right)\right) \right]$$

$$= (2\sin \theta/2)^{-n} \left[\cos n\left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i \sin n\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \right]$$

Equating imaginary part.

$$S = (2\sin \theta/2)^{-n} \sin n\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \quad \text{proved.}$$

$$11. \quad C = 1 + \frac{1}{2} \cos \theta + \frac{1 \cdot 3}{2 \cdot 4} \cos 2\theta + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cos 3\theta + \dots$$

$$\text{Let } S = \frac{1}{2} \sin \theta + \frac{1 \cdot 3}{2 \cdot 4} \sin 2\theta + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \sin 3\theta + \dots$$

$$C + iS = 1 + \frac{1}{2} (\cos \theta + i \sin \theta) + \frac{1 \cdot 3}{2 \cdot 4} (\cos 2\theta + i \sin 2\theta) + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} (\cos 3\theta + i \sin 3\theta) + \dots$$

$$= 1 + \frac{1}{2} e^{i\theta} + \frac{1 \cdot 3}{2 \cdot 4} e^{i2\theta} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} e^{i3\theta} + \dots$$

$$= (1 e^{i\theta})^{-1/2}$$

$$1 - \cos \theta = 2 \sin^2 \theta/2$$

$$= (1 - \cos \theta - i \sin \theta)^{-1/2}$$

$$= (2 \sin^2 \theta/2 - i 2 \sin \theta/2 \cos \theta/2)^{-1/2}$$

$$= (2 \sin \theta/2)^{-1/2} (\sin \theta/2 - i \cos \theta/2)^{-1/2}$$

$$= (2 \sin \theta/2)^{-1/2} \left(\cos \left(\frac{\pi}{2} - \frac{\theta}{2}\right) - i \sin \left(\frac{\pi}{2} - \frac{\theta}{2}\right) \right)^{-1/2}$$

$$= (2 \sin \theta/2)^{-1/2} \left(\cos \left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i \sin \left(-\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) \right)^{-1/2}$$

De-Moivre's Theorem

$$= (2 \sin \theta/2)^{-1/2} \left(\cos \left(-\frac{1}{2} \left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) + i \sin \left(-\frac{1}{2} \left(-\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)\right) \right)^{-1/2}$$

$$= (2 \sin \theta/2)^{-1/2} \left(\cos \left(\frac{\pi}{4} - \frac{\theta}{4}\right) + i \sin \left(\frac{\pi}{4} - \frac{\theta}{4}\right) \right)$$

equating real part.

$$C = (2 \sin \theta/2)^{-1/2} \cos \left(\frac{\pi}{4} - \frac{\theta}{4}\right) \quad \text{Ans}$$

$$12. \quad C = \cos \alpha - \frac{\cos(\alpha + 2\beta)}{3!} + \frac{\cos(\alpha + 4\beta)}{5!} - \dots$$

$$S = \sin \alpha - \frac{\sin(\alpha + 2\beta)}{3!} + \frac{\sin(\alpha + 4\beta)}{5!} - \dots$$

$$C + iS = (\cos \alpha + i \sin \alpha) - \frac{1}{3!} (\cos(\alpha + 2\beta) + i \sin(\alpha + 2\beta)) + \frac{1}{5!} (\cos(\alpha + 4\beta) + i \sin(\alpha + 4\beta)) + \dots$$

Chapter 6.

Plane Curves I

Topics of Exercise 6.1

Conic.

Def: The Conic is the set of points in a plane, the distance of each of which from a given point bears a constant ratio to its distance from a given straight line in the plane.

Focus.

The fixed point is called the Focus.

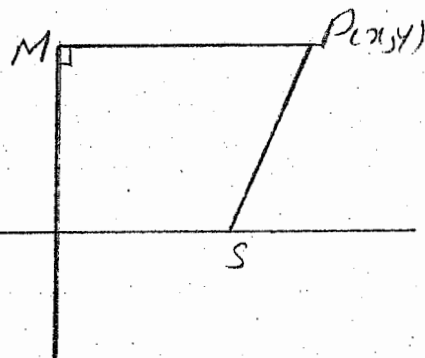
Directrix.

The fixed line is said to be the Directrix.

Eccentricity.

The constant ratio which is denoted by e , is known as the eccentricity of the Conic.

$$\text{i.e. } \frac{|PS|}{|PM|} = e$$



If $e = 1$

The Conic is a parabola.

If $e < 1$

The Conic is an ellipse.

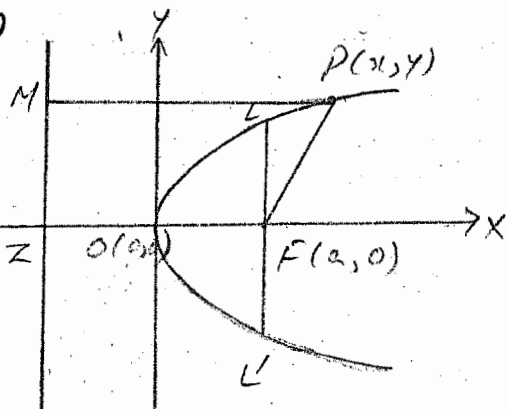
If $e > 1$

The Conic is a hyperbola.

The Parabola.

Def: "A Parabola is set of all points P in the plane that are equidistance from a fixed line and a fixed point in the plane. The fixed point does not lie on the fixed line."

The fixed line is called the Directrix of the parabola. and the fixed point is called its focus.



The straight line

through the focus and perpendicular to the directrix is called axis of the parabola. The point where the parabola meets its axis is called the vertex of the parabola.

The equation of a parabola with focus $F(a, 0)$, $a > 0$ and vertex at the origin is quite simple. In this case the equation of the directrix ZM is $x = -a$. If $P(x, y)$ is a point on the parabola, then by definition, P is equidistance from F and ZM .

$$\text{Hence } |PF| = |PM|$$

$$\text{i.e. } (x-a)^2 + y^2 = (x+a)^2$$

$$y^2 = 4ax$$

Which is standard equation of parabola.

Let the line through $F(a, 0)$ and perpendicular to the axis of the parabola meet the parabola at L and L' . If y' is ordinate of L , then $L(a, y')$ lies on $y^2 = 4ax$

$$\therefore y'^2 = 4a^2$$

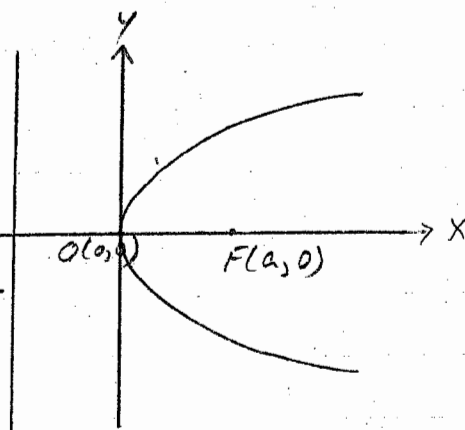
$$y' = \pm 2a$$

The length $LL' = 4a$ and is called latus rectum of the parabola.

Different Cases of Parabola.

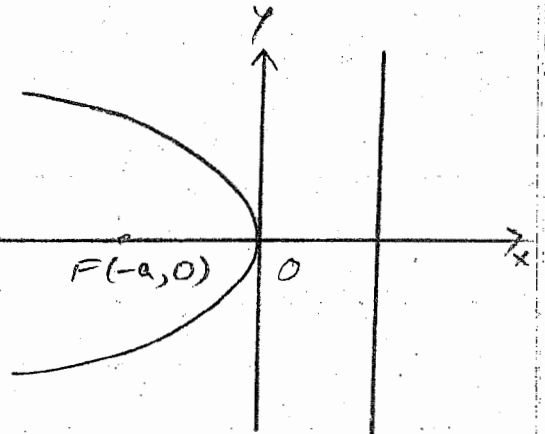
Case I)

If the equation of the parabola is $y^2 = 4ax$, $a > 0$ then the shape of the parabola is shown below.



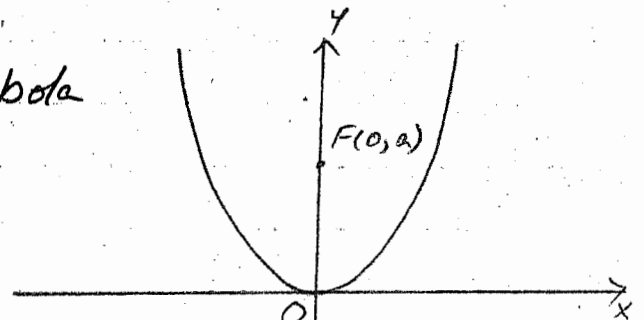
Case 2.

If the equation of the parabola is $y^2 = -4ax$ where $a > 0$, then the shape of this parabola is shown.



Case 3.

If the equation of the parabola is $x^2 = 4ay$ where $a > 0$ then the shape of the parabola is as shown.

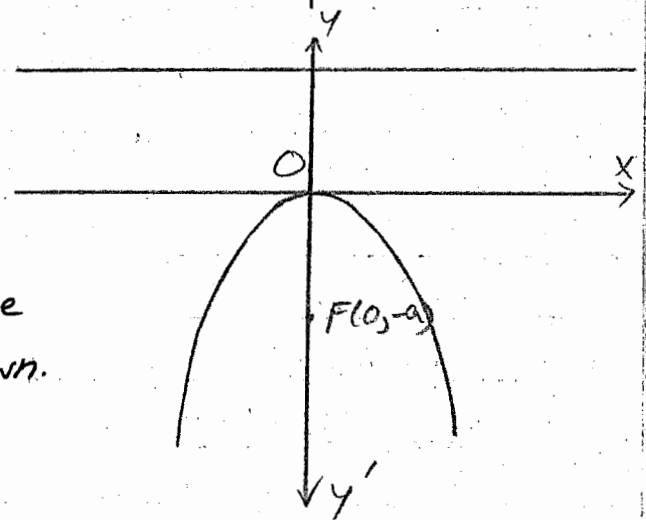


Case 4.

If the equation of the parabola is

$$x^2 = -4ay$$

where $a > 0$, then the shape of the parabola is as shown.



Case 5.

If the equation of the parabola is

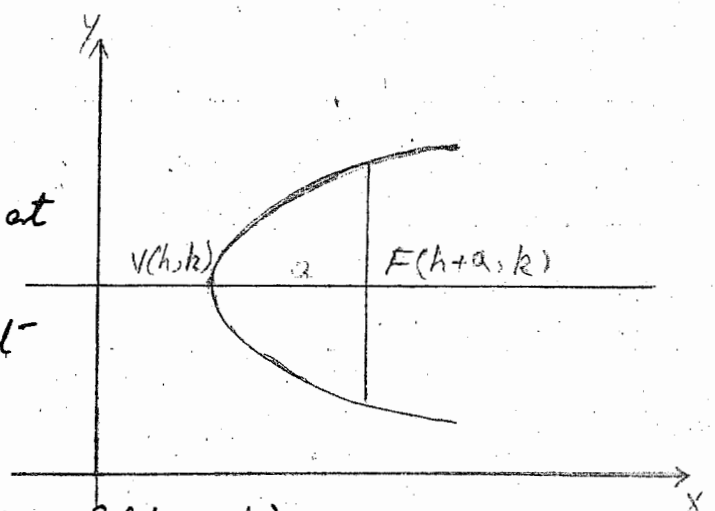
$$(y-k)^2 = 4a(x-h).$$

i) The vertex of parabola is at (h, k)

ii) Parabola will open in right side.

iii) The shape is shown.

iv) Focus of the parabola is $F(h+a, k)$.



Case 6

If the equation of the parabola is

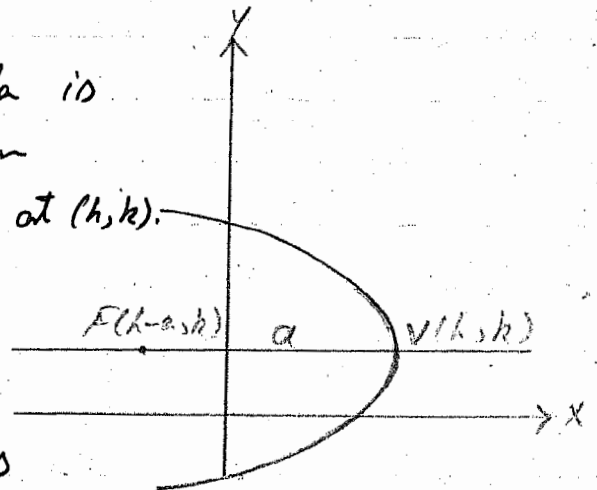
$$(y-k)^2 = -4a(x-h); a > 0 \text{ then}$$

i) The vertex of the parabola is at (h, k) .

ii) Parabola will open in left side.

iii) The shape is shown.

iv) The focus of the parabola is $(h-a, k)$.



Case 7

If the equation of the parabola

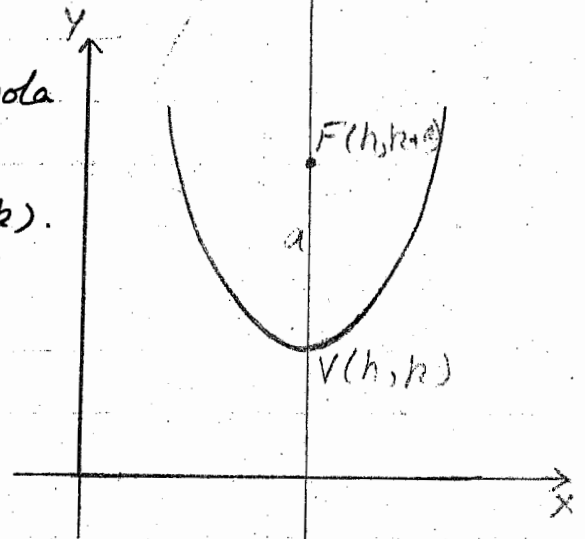
is $(x-h)^2 = 4a(y-k); a > 0$, then

i) The vertex of the parabola is (h, k) .

ii) Parabola will open in upward direction.

iii) The focus will be $(h, k+a)$

iv) The shape is as shown.



Case 8

If the equation of the parabola is

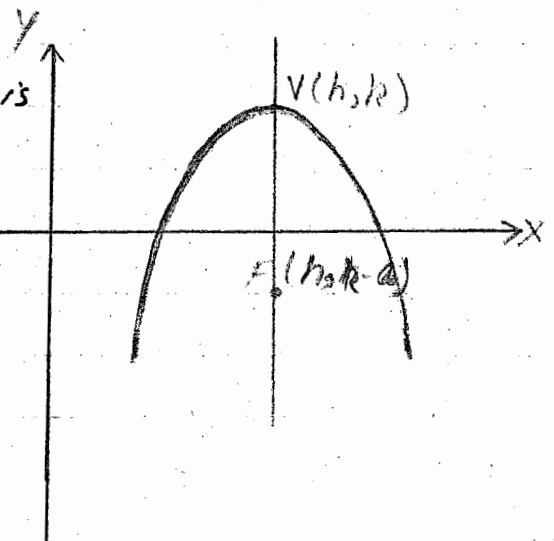
$$(x-h)^2 = -4a(y-k); a > 0 \text{ then}$$

i) The vertex of the parabola is $V(h, k)$.

ii) Parabola will open in downward direction.

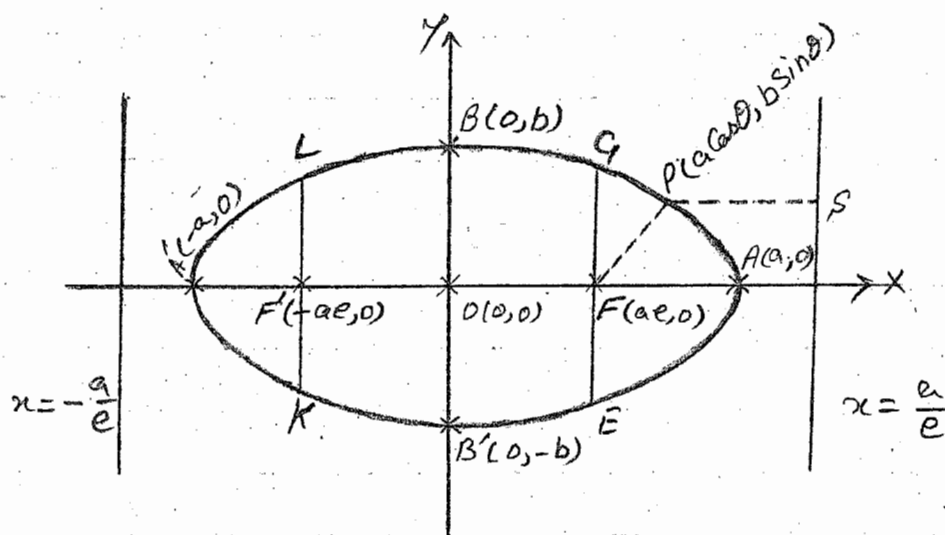
iii) The focus of the parabola is $(h, k-a)$.

iv) The shape is shown.



The Ellipse

- 1) The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $a^2 > b^2$ is called the equation of an ellipse.
- 2) The shape of ellipse is shown.

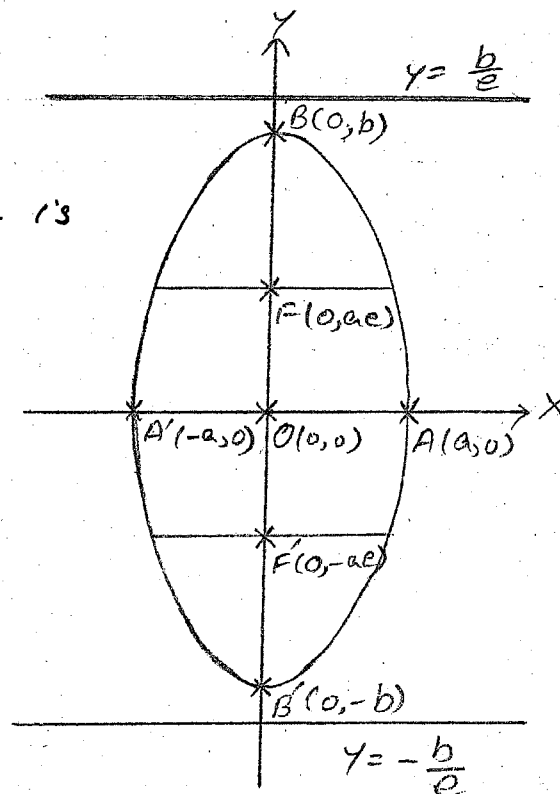


- 3) The pt. $O(0,0)$ is called centre of the ellipse.
- 4) The pts. $F(ae,0)$ and $F'(-ae,0)$ are called foci of the ellipse.
- 5) The pt $P(a \cos \theta, b \sin \theta)$ lies on the ellipse.
- 6) $\frac{|PS|}{|PF|}$ is called eccentricity and is denoted by e i.e. $\frac{|PS|}{|PF|} = e$.
For an ellipse $e < 1$.
- 7) $x = a \cos \theta$, $y = b \sin \theta$ are called parameter equation of the ellipse.
- 8) $A(a,0)$ and $A'(-a,0)$ are called vertices or end pts. of the major axis.
- 9) The segment $\overline{AA'}$ is called major axis of the ellipse. The length of the major axis of the ellipse is $2a$ i.e. $|\overline{AA'}| = 2a$.
- 10) The line segment $\overline{OA}$ and $\overline{OA'}$ are called semi major axis. The length of the semi major axis is ' a ' i.e. $|\overline{OA}| = |\overline{OA'}| = a$.
- 11) The pt. $B(0,b)$ and $B'(0,-b)$ are called end pt. of the minor axis.
- 12) The equation $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ are called directrices.
- 13) The lines $\overline{EC}$ and $\overline{KC}$ are called latus recta of the ellipse.

- 14) The line segment $\overline{BB'}$ is called minor axis of the ellipse. The length of the minor axis is $2b$ i.e. $\overline{BB'} = 2b$
- 15) The line segment $\overline{OB}$ and $\overline{OB'}$ are called semi minor axis. The length of the semi minor axis is b i.e. $|\overline{OB}| = |\overline{OB'}| = b$
- 16) Ellipse obeys the relation
- $$a^2 = b^2 + a^2 e^2$$

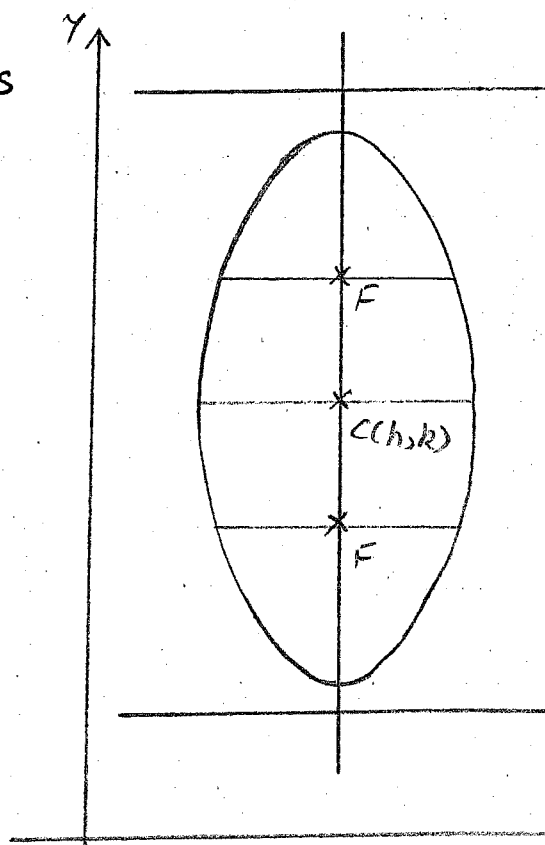
Case 2.

If the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $a^2 < b^2$ then the shape of the ellipse is shown.



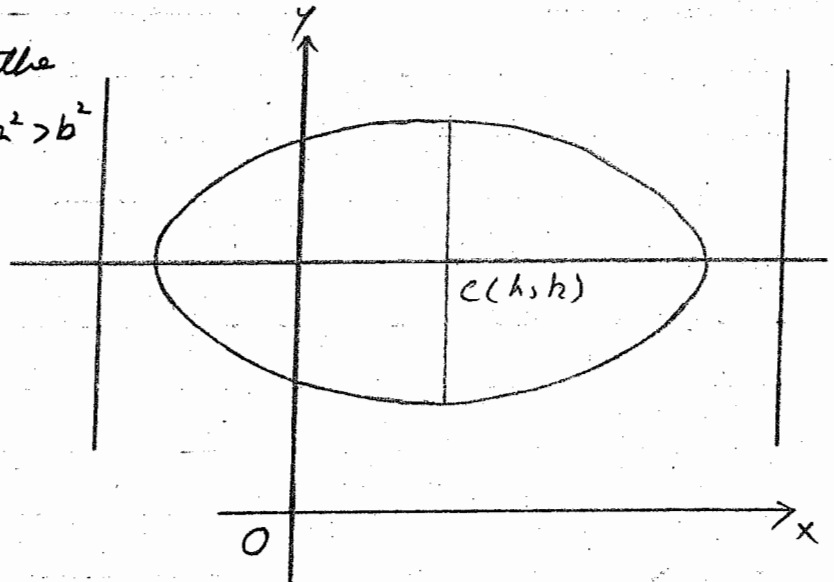
Case 3.

If the equation of the ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; $a^2 < b^2$ then centre of the ellipse is $C(h,k)$ and the shape is shown.



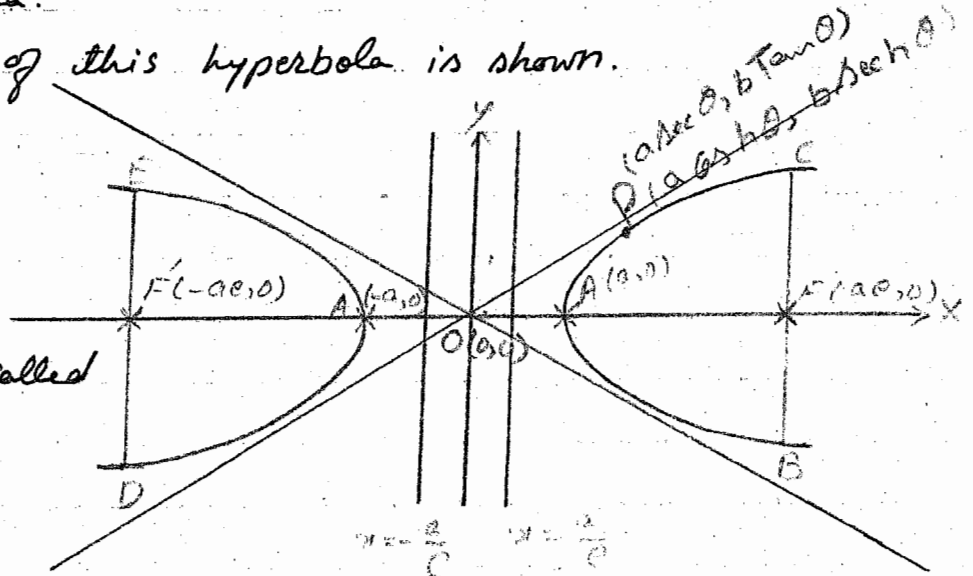
Case 4.

If the equation of the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; $a^2 > b^2$ then the centre of the ellipse is $C(h, k)$ and the shape is shown.



The Hyperbola.

- 1) The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is called the equation of the hyperbola.
- 2) The shape of this hyperbola is shown.
- 3) $O(0, 0)$ is called centre of the hyperbola.
- 4) $A(a, 0)$ and $A'(-a, 0)$ are called vertices of the hyperbola.
- 5) $F(ae, 0)$ and $F'(-ae, 0)$ (foci) are called foci of the hyperbola.
- 6) The line segments BC and DE are called latus recta of the hyperbola.
- 7) The $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola.
- 8) The equations $x = a \sec \theta$, $y = b \tan \theta$ are called parametric equations of the hyperbola.
- 9) Similarly the pt. $(a \cosh \theta, b \sinh \theta)$ lies on the hyperbola.
- 10) The equations $x = a \cosh \theta$, $y = b \sinh \theta$ are called parametric equations of the Hyperbola.
- 11) Hyperbola obeys the relation $a^2 e^2 = a^2 + b^2$



12) The ratio of the distance of any pt. P from the directrix to the distance of P from corresponding focus is called eccentricity of the hyperbola. and is denoted by e .

13) For hyperbola $e > 1$

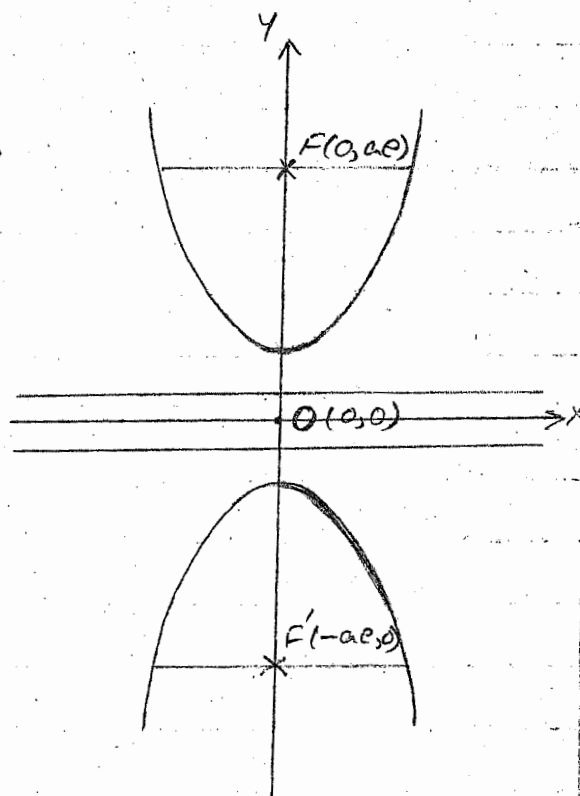
14) The lines $x = \pm \frac{a}{e}$ are called directrices of the hyperbola.

15) The lines $y = \pm \frac{b}{a} x$ are called asymptotes of the hyperbola.

Case 2.

If the equation of the hyperbola is $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ where $a^2 > b^2$.

Then the shape of the hyperbola is shown.

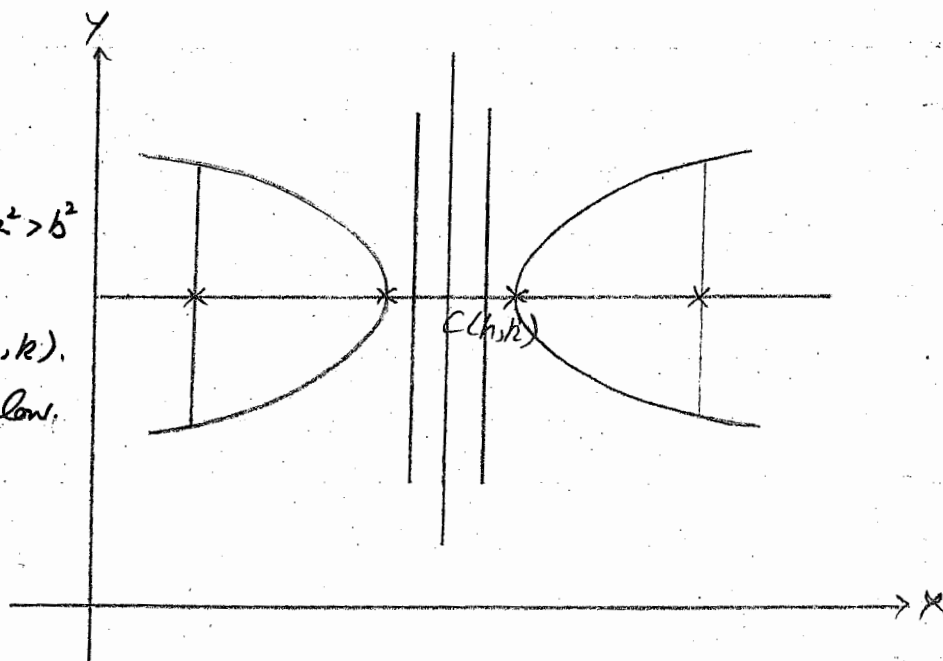


Case 3.

If the equation of the hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1; a^2 > b^2$$

Then centre of the hyperbola is at (h, k) .
shape is shown below.

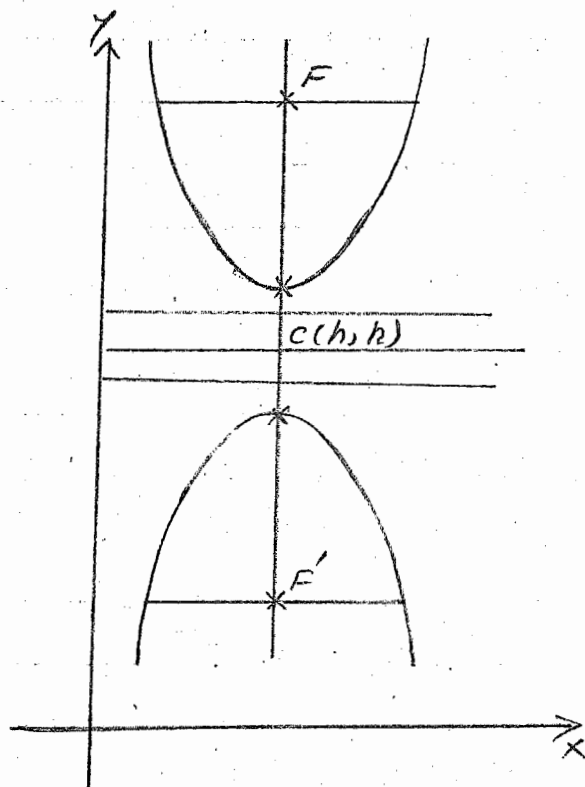


If the equation of the hyperbola is

$$\frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = 1 ; a^2 > b^2$$

Then the centre of the hyperbola is at (h, k) .

The shape is shown.



Some Useful Results to Recognize the Conic:

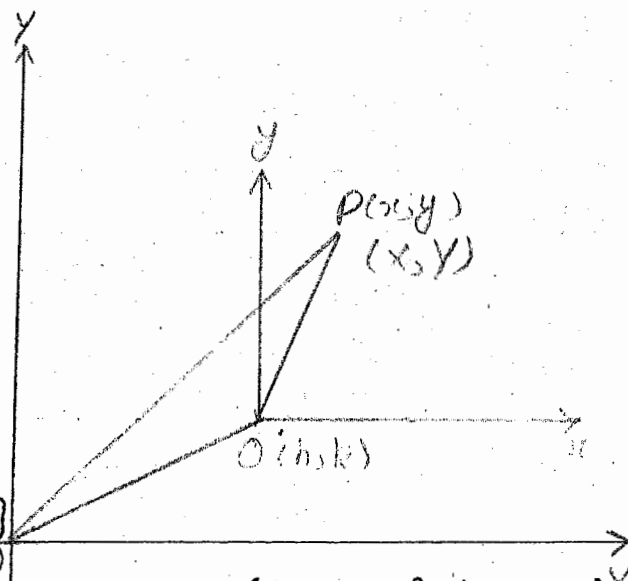
- i) The Conic is a parabola if $e = 1$
- ii) The Conic is an ellipse if $e < 1$
- iii) The Conic is a hyperbola if $e > 1$

Translation of axis.

Let (h, k) be the coordinates of O' w.r.t. ' O ' and let (x, y) and (X, Y) be the coordinates of P with respect to old and new axes.

By using vector addition.

$$\vec{OP} = \vec{OO'} + \vec{O'P}$$



$$\Rightarrow [x, y] = [h, k] + [X, Y]$$

$$\Rightarrow [x, y] = [h + X, k + Y]$$

$$\Rightarrow x = h + X, y = k + Y$$

$$\Rightarrow x = h + X, y = k + Y$$

$$\Rightarrow x = h + X, y = k + Y$$

$$\Rightarrow [x - h, y - k] = [X, Y]$$

$$\Rightarrow [x, y] = [h + X, k + Y]$$

$$\Rightarrow x = h + X, y = k + Y$$

$$\Rightarrow x = h + X, y = k + Y$$

Pair of Lines.

The most general equation of 2nd degree in x, y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- (1)}$$

where a, b and h are not simultaneously 0.

Test for 2nd degree equation.

To represent a pair of s.t lines is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Homogeneous 2nd Degree equation.

$$ax^2 + 2hxy + by^2 = 0 \quad \text{--- (1)}$$

This eq. is known as homogenous 2nd degree eq.

Dividing (1) by x^2

$$a + 2h \frac{y}{x} + \frac{by^2}{x^2} = 0$$

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0 \quad \text{--- (2)}$$

$\therefore$ It is Quadratic in $\frac{y}{x}$

$$\therefore \frac{y}{x} = \frac{-2h \pm \sqrt{(2h)^2 - 4ab}}{2b}$$

$$\frac{y}{x} = \frac{-2h \pm 2\sqrt{h^2 - ab}}{2b}$$

$$\frac{y}{x} = \frac{-h \pm \sqrt{h^2 - ab}}{b}$$

We observe that the eq. can be converted into two straight lines.

Angle between the Straight Lines.

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0 \quad \text{from eq. (2)}$$

$$\text{Let } \frac{y}{x} = m$$

$$\text{Then } bm^2 + 2hm + a = 0$$

$$\text{Let } m_1 \text{ and } m_2 \text{ be two values of } m \quad ax^2 + bx + c = 0$$

$$m_1 + m_2 = -\frac{2h}{b}$$

$$m_1 m_2 = \frac{a}{b}$$

$$\begin{array}{l} \text{Sum of roots, } S = -\frac{b}{a} \\ \text{Product of roots, } P = \frac{c}{a} \end{array}$$

Let ' θ ' be the angle between the lines then

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan \theta = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{(-2h/b)^2 - 4(a/b)}}{1 + a/b}$$

$$= \frac{\sqrt{4h^2/b^2 - 4a/b}}{\frac{b+a}{b}}$$

$$= \frac{\frac{b+a}{b} \sqrt{h^2 - ab}}{\frac{b+a}{b}}$$

$$= \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$(a+b)^2 - 4ab = (a-b)^2$$

Deduction.

For // lines $\theta = 0$

$$\frac{2\sqrt{h^2 - ab}}{a+b} = 0$$

$$\Rightarrow 2\sqrt{h^2 - ab} = 0$$

$$\Rightarrow h^2 - ab = 0$$

$$h^2 = ab$$

For perpendicular Lines $\theta = \frac{\pi}{2}$

$$\frac{2\sqrt{h^2 - ab}}{a+b} = \infty$$

$$\Rightarrow \frac{2\sqrt{h^2 - ab}}{\infty} = a+b$$

$$\Rightarrow 0 = a+b$$

e.g. $x^2 + 2xy - y^2 = 0$

Here $a = 1, b = -1$

$$a+b=0$$

Some Useful Results to Recognize the Conic.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

The Conic is hyperbola if $h^2 - ab > 0$

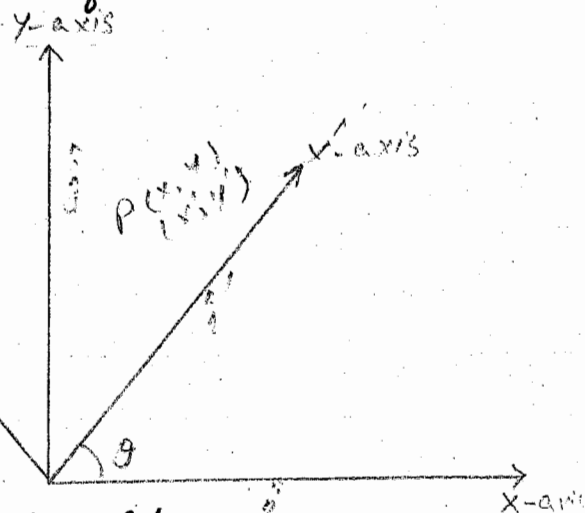
The Conic is an ellipse if $h^2 - ab < 0$

The Conic is a parabola if $h^2 - ab = 0$

Rotation of Axes.

Let the axes (x-axis, y-axis) be rotated through an angle θ , name the new axes as x' -axis and y' -axis.

Let (x, y) and (x', y') be the coordinates of P referred to old and new axes.



Then $\vec{OP} = x\hat{i} + y\hat{j}$ ——— ①

$\vec{OP}' = x'\hat{i}' + y'\hat{j}'$ ——— ②

$\therefore$ The unit vector $\hat{i}'$ makes an angle θ with the positive direction of x -axis.

$\therefore \hat{i}' = \cos\theta \hat{i} + \sin\theta \hat{j}$ ——— ③

Like wise

$\hat{j}' = \cos(90+\theta) \hat{i} + \sin(90+\theta) \hat{j}$

$\hat{j}' = -\sin\theta \hat{i} + \cos\theta \hat{j}$ ——— ④

③ and ④ in ②

$\vec{OP} = x'[\cos\theta \hat{i} + \sin\theta \hat{j}] + y'[-\sin\theta \hat{i} + \cos\theta \hat{j}]$

$\vec{OP} = x'\cos\theta \hat{i} + x'\sin\theta \hat{j} + (-y'\sin\theta \hat{i}) + y'\cos\theta \hat{j}$

$\vec{OP} = (x'\cos\theta - y'\sin\theta)\hat{i} + (x'\sin\theta + y'\cos\theta)\hat{j}$

$x\hat{i} + y\hat{j} = (x'\cos\theta - y'\sin\theta)\hat{i} + (x'\sin\theta + y'\cos\theta)\hat{j}$

Comparing Co-efficient of $\hat{i}$ and $\hat{j}$

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Where the matrix is called the Matrix of rotation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

In order to remove the product term (x, y) from the equation of Conic we make use of the substitution brought in due to rotation and apply the formula

$$\tan 2\theta = \frac{2h}{a-b}$$

Asymptotes of a Hyperbola.

$$x^2/a^2 - y^2/b^2 = 1$$

An asymptote for the curve $y = f(x)$ is a straight line $y = g(x)$ such that

$$\lim_{x \rightarrow \infty} (f(x) - g(x)) = 0$$

or $x \rightarrow -\infty$

Theorem.

The hyperbola $x^2/a^2 - y^2/b^2 = 1$ has asymptotes $y = \frac{b}{a}x$, $y = -\frac{b}{a}x$.

Proof:

$$x^2/a^2 - y^2/b^2 = 1$$

$$\Rightarrow y^2/b^2 = x^2/a^2 - 1$$

$$\Rightarrow y^2/b^2 = \frac{x^2 - a^2}{a^2}$$

$$\Rightarrow y^2 = \frac{b^2}{a^2} (x^2 - a^2)$$

$$\Rightarrow y = \pm \frac{b}{a} (\sqrt{x^2 - a^2}) \quad \text{--- ①}$$

from

$$\lim_{x \rightarrow \infty} (f(x) - g(x))$$

$$\lim_{x \rightarrow \infty} \left(\frac{b}{a} \sqrt{x^2 - a^2} - \frac{b}{a} x \right)$$

given line

$$y = \frac{b}{a} x$$

$$= \frac{b}{a} \left[\lim_{x \rightarrow \infty} \sqrt{x^2 - a^2} - x \right]$$

$$= \frac{b}{a} \left[\lim_{x \rightarrow \infty} \sqrt{x^2 - a^2} - x \right]$$

$$= \frac{b}{a} \left[\lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 - a^2} - x)(\sqrt{x^2 - a^2} + x)}{(\sqrt{x^2 - a^2} + x)} \right]$$

$$= \frac{b}{a} \left[\lim_{x \rightarrow \infty} -\frac{a^2}{\sqrt{x^2 - a^2} + x} \right]$$

$$= \frac{b}{a} \left[-\frac{a^2}{\infty} \right] = 0$$

in second quadrant from ①

$$y = -\frac{b}{a} \sqrt{x^2 - a^2}$$

$$\therefore y = -\frac{b}{a} x$$

$$y = -\frac{b}{a} x, \quad y = -\frac{b}{a} \sqrt{x^2 - a^2}$$

$$\text{Now } \lim_{x \rightarrow -\infty} \left(-\frac{b}{a} \sqrt{x^2 - a^2} + \frac{b}{a} x \right)$$

$$= \frac{b}{a} \lim_{x \rightarrow -\infty} (x - \sqrt{x^2 - a^2})$$

$$= \frac{b}{a} \lim_{x \rightarrow -\infty} \left[\frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} \right]$$

$$= \frac{b}{a} \lim_{x \rightarrow -\infty} \frac{a^2}{x + \sqrt{x^2 - a^2}}$$

$$= \frac{b}{a} \frac{a^2}{\infty}$$

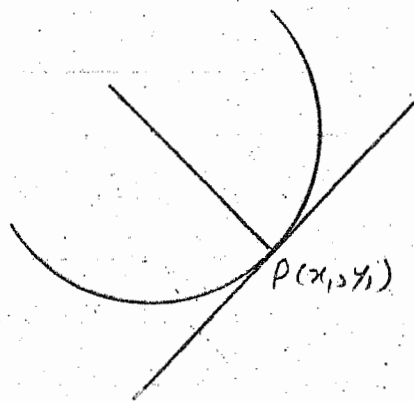
$$= 0$$

Hence the hyperbola $x^2/a^2 - y^2/b^2 = 1$ has the asymptotes
 $y = \frac{b}{a} x$, $y = -\frac{b}{a} x$.

Tangent and Normal

Let $y = f(x)$ be the curve
and $P(x_1, y_1)$ be any point on it.

$$\begin{aligned} \therefore y &= f(x) \\ \therefore \frac{dy}{dx} &= f'(x) \\ \left(\frac{dy}{dx} \right)_{P(x_1, y_1)} &= f'(x_1) \end{aligned}$$



Tangent. If a line touches the curve at the pt. $P(x_1, y_1)$.
Then this line is called Tangent to the curve at the
pt. $P(x_1, y_1)$. The equation of the Tangent at this point
 $P(x_1, y_1)$ to the curve $y = f(x)$ is given by

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= f'(x_1)(x - x_1) \end{aligned}$$

$$\text{Generally } y - y_1 = f'(x) \big|_P (x - x_1)$$

Normal. A line passing through the pt. P and $\perp$ to the
Tangent at the pt. P to the curve is called the
Normal at that pt. P .

The equation of the normal at the pt. $P(x_1, y_1)$
to the curve $y = f(x)$ is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$$

Generally

$$y - y_1 = -\frac{1}{f'(x)} \big|_P (x - x_1)$$

Example #7

Find the eq. of the normal to the parabola

$$y^2 = 4ax \text{ in the form } y = mx - 2a\frac{m}{2} - am^3$$

Soln.

$$y^2 = 4ax$$

$$2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\left. \frac{dy}{dx} \right|_{P(x_1, y_1)} = \frac{2a}{y_1}$$

$$\text{Slope of the normal} = -\frac{y_1}{2a} = m \text{ (say)}$$

$$-y_1 = 2am \Rightarrow y_1 = -2am \quad \text{--- (2)}$$

$\therefore (x_1, y_1)$ lies on $y^2 = 4ax$

$$\therefore y_1^2 = 4ax_1 \quad \text{--- (3)}$$

Put $y_1 = -2am$ in (3)

$$4a^2m^2 = 4ax_1$$

$$\Rightarrow x_1 = am^2 \quad \text{--- (4)}$$

$$P(x_1, y_1) = P(am^2, -2am)$$

eq. of the normal

$$y - (-2am) = m(x - am^2)$$

$$y + 2am = mx - am^3$$

$$y = mx - am^3 - 2am$$

is as required.

Parametric form of Parabola:-

$$y^2 = 4ax$$

$$x = at^2, y = 2at$$

$x = at^2, y = 2at$ are the equations representing the parabola in parametric form. where 't' is called the parameter.

Parametric form of Ellipse:-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x = a \cos \theta, y = b \sin \theta$ are the parametric equations of the ellipse.

Parametric form of Hyperbola:-

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

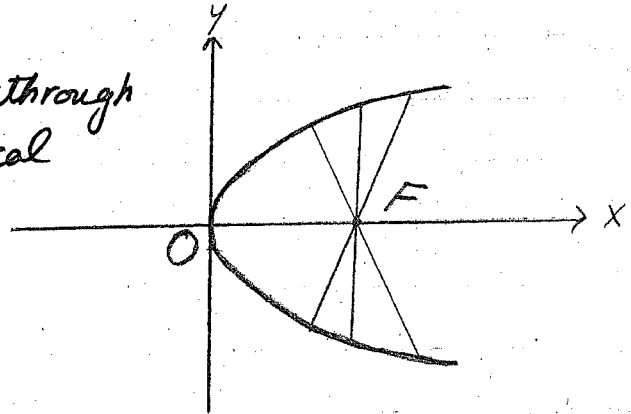
$$x = a \sec \theta; y = b \tan \theta$$

$$\text{or } x = a \cosh \theta; y = b \sinh \theta$$

are the parametric equations of the hyperbola.

Focal Chord

The chord passes through the focus is called focal chord.



Example # 8

Show that the points $(at_1^2, 2at_1)$ always lies on the parabola $y^2 = 4ax$. Find the condition that the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ may be a focal chord. Find an equation of the tangent to the parabola at $(at_1^2, 2at_1)$.

Soln. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$

Using two points formula.

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Putting $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$

$$\frac{y - 2at_1}{2at_2 - 2at_1} = \frac{x - at_1^2}{at_2^2 - at_1^2}$$

$$\frac{y - 2at_1}{2(t_2 - t_1)} = \frac{x - at_1^2}{t_2^2 - t_1^2}$$

$$\frac{y - 2at_1}{2(t_2 - t_1)} = \frac{x - at_1^2}{(t_2 - t_1)(t_2 + t_1)}$$

$$\frac{y - 2at_1}{2} = \frac{x - at_1^2}{t_2 + t_1} \quad \text{--- (1)}$$

Focus: $(a, 0)$, put $x = a$, $y = 0$ in (1)

$$\frac{0 - 2at_1}{2} = \frac{a - at_1^2}{t_2 + t_1}$$

$$-at_1 = \frac{a(1 - t_1^2)}{t_2 + t_1}$$

$$-t_1 = \frac{1-t_1^2}{t_1+t_2}$$

$$-t_1^2 - t_1 t_2 = 1 - t_1^2$$

$$-t_1 t_2 = 1$$

$$t_1 t_2 = -1$$

Is the required condition for the chord PQ to be focal chord.

Now equation of the Tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$y - y_1 = m(x - x_1) \quad \text{--- (2)}$$

$m =$ Slope of tangent at $P(at^2, 2at)$ is

$$m = \frac{dy}{dx} = \frac{d(2at)}{d(at^2)} = \frac{2a}{2at} = \frac{1}{t}$$

$\therefore$ We have to find the equation of Tangent at $P(at^2, 2at)$.

$\therefore$ Put $x_1 = at^2$, $y_1 = 2at$ and $m = \frac{1}{t}$ in (2) we have

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at^2 = x - at^2$$

$$yt = x - at^2 + 2at^2$$

i.e $yt = x + at^2$

Note. $x = at^2$, $y = 2at$ are called the parametric equations of the parabola $y^2 = 4ax$. The point $(at^2, 2at)$ is also referred to as point "t" on the parabola.

Pedal Equation:-

The pedal equation is an equation in 'p' and 'r' where 'r' is the distance of any point 'P' on the curve from O and p is the $\perp$ distance of O from the tangent at P.

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$.

Then $r = |OP| = \sqrt{x_1^2 + y_1^2}$

by distance formula.

$$r^2 = x_1^2 + y_1^2 \quad \text{--- (1)}$$

$\therefore P$ lies on the curve

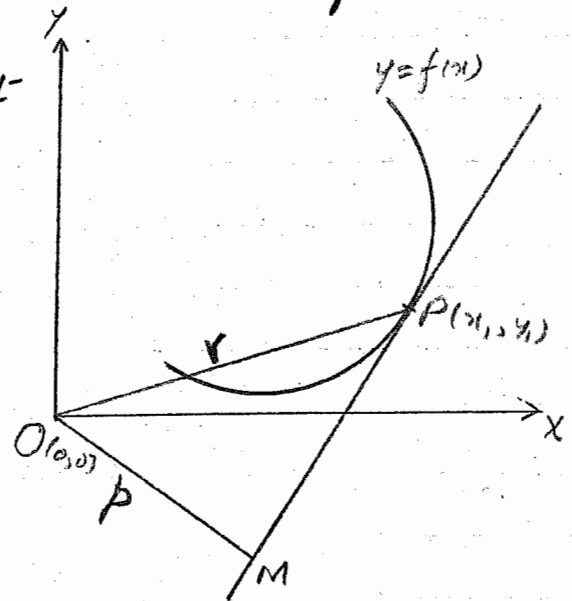
$$\therefore y_1 = f(x_1) \quad \text{--- (2)}$$

equation of the Tangent at $P(x_1, y_1)$.

$$(y - y_1) = f'(x_1)(x - x_1)$$

$$(y - y_1) = x f'(x_1) - x_1 f'(x_1)$$

$$x f'(x_1) - y + y_1 = x_1 f'(x_1)$$



Now p = Distance of $O(0,0)$ from tangent line.

$$p = \frac{|f'(x_1)(0) - 0 + y_1 - x_1 f'(x_1)|}{\sqrt{[f'(x_1)]^2 + 1^2}}$$

$$p = \frac{|y_1 - x_1 f'(x_1)|}{\sqrt{1 + f'(x_1)^2}} \quad \text{--- (3)}$$

Elimination of x_1 and y_1

from eq (1) to (3) will give

us the equation in p and r called the pedal equation.

The distance of Point $P(x_1, y_1)$ from line $ax + by + c = 0$ is given by $p = d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Example #9.

IF the tangent at any point of the parabola meets y-axis at A, then prove that

$$\hat{PAF} = 90^\circ$$

Where P is any point on the parabola, A is point on y-axis and F is focus.

Proof.

Consider that the tangent at pt. $P(at^2, 2at)$ of the parabola $y^2 = 4ax$ meets y-axis at the pt. A.

$F(a, 0)$ is the focus of this parabola.

We know that the equation of the tangent at the pt. $(at^2, 2at)$ is

$$x - ty + at^2 = 0$$

for the coordinate of A put $x=0$

$$\text{then } 0 - ty + at^2 = 0$$

$$at^2 = ty$$

$$y = at$$

the coordinate of $A(0, at)$

$$m_1 = \text{slope of } (\overline{PA}) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{at - 2at}{0 - at^2} = \frac{-at}{-at^2} = \frac{1}{t}$$

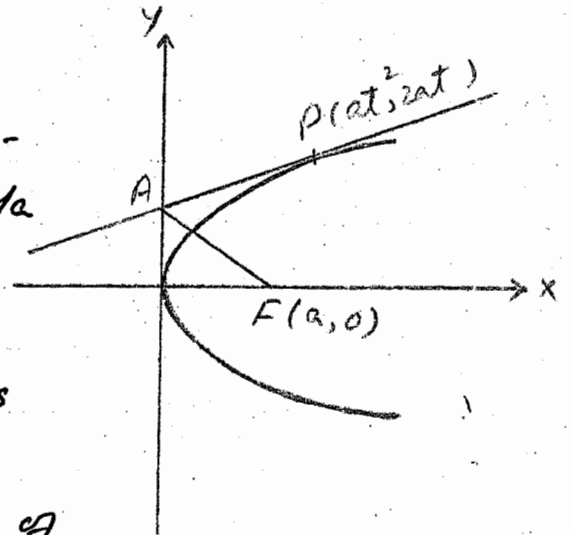
$$m_2 = \text{slope of } (\overline{FA}) = \frac{0 - at}{a - 0} = -\frac{at}{a} = -t$$

$$\text{Now } m_1 m_2 = \frac{1}{t} \cdot -t$$

$$= -1$$

$$\Rightarrow \overline{PA} \perp \overline{FA}$$

$$\Rightarrow \hat{PAF} = 90^\circ$$



Example #10

Find the locus of the middle points of a system of parallel chords of the parabola of the parabola $y^2 = 4ax$.

Soln. given $y^2 = 4ax$ — (1)

Consider a system of // chords of the parabola.

Let m be their slopes

and $y = mx + c$ — (2)

be the equation of the chord PQ representative of the // chords.

Further let $R = (h, k)$

be the middle point of the chord PQ of the parabola.

Now put (2) in (1)

$$(mx + c)^2 = 4ax$$

$$m^2x^2 + 2mcx + c^2 = 4ax$$

$$m^2x^2 + 2(mc - 2a)x + c^2 = 0 \quad \text{--- (3)}$$

$\therefore$ It is quadratic in x

$\therefore$ The values of x obtained from (3) will serve as the x -coordinates of P and Q .

$\therefore R$ is the mid pt. of PQ

$$\therefore h = \frac{x_1 + x_2}{2}$$

$$h = \frac{-2(mc - 2a)}{\frac{m^2}{2}}$$

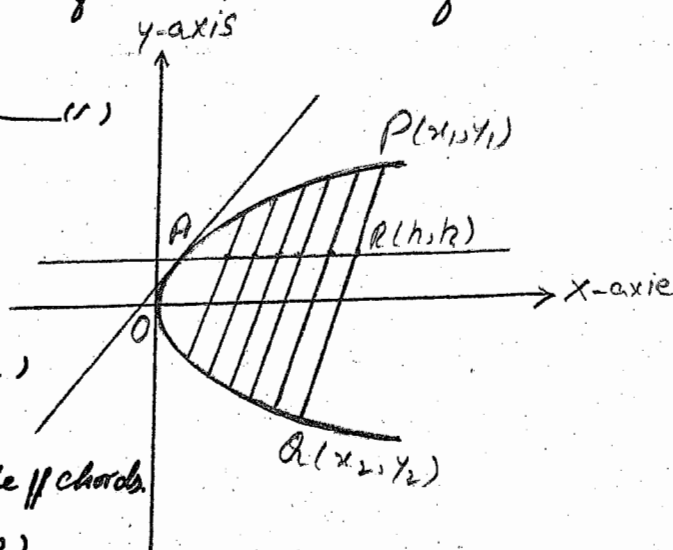
$$x_1 + x_2 = -\frac{b}{a}$$

$$h = \frac{-(mc - 2a)}{m^2} \quad \text{--- (4)}$$

$\therefore R = (h, k)$ lies on (2)

$$\therefore k = mh + c$$

$$\Rightarrow c = k - mh \quad \text{--- (5)}$$



$$(5) \text{ in (4) } \Rightarrow h = \frac{-(mk - mh) - 2a}{m^2}$$

$$hm^2 = -(mk - m^2h) - 2a$$

$$m^2h = -mk + m^2h + 2a$$

$$m^2h - m^2h = -mk + 2a$$

$$0 = -mk + 2a$$

$$mk = 2a$$

$$k = \frac{2a}{m}$$

i.e. the pt. $R(h, k)$ lies on the locus $y = \frac{2a}{m}$
 x^n of $y = \frac{2a}{m}$ and $y^2 = 4ax$

Put $y = \frac{2a}{m}$ in $y^2 = 4ax$

$$\frac{4a^2}{m^2} = 4ax$$

$$\frac{a}{m^2} = x$$

$$x = \frac{a}{m^2}$$

Point $(\frac{a}{m^2}, \frac{2a}{m})$ is the Locus.

Note. The locus of the middle points of parallel chords of a parabola is called a diameter of the parabola.

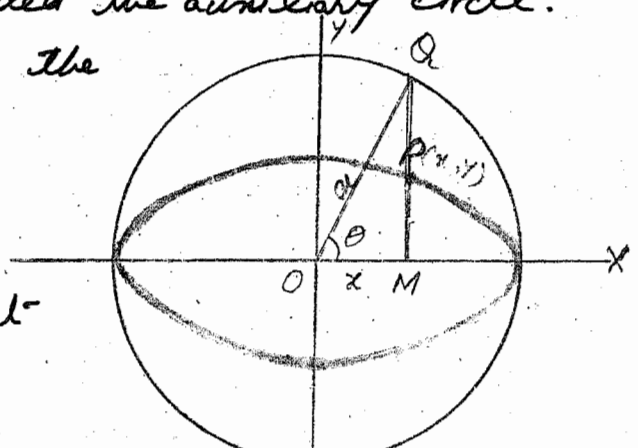
Auxiliary Circle.

Def. "The circle constructed on the Major axis of the ellipse as a diameter is called the auxiliary circle".

Let $P(x, y)$ be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Draw $PM \perp$ to x -axis.

Produce MP so that it meets the circle at Q . Join Q to O .



Let $\angle OMQ = \theta$

Then we call θ as the eccentric angle of P.
from r.t ΔOMQ

$$\frac{x}{a} = \cos \theta$$

$$x = a \cos \theta$$

Put $x = a \cos \theta$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \frac{a^2 \cos^2 \theta}{a^2} + \frac{y^2}{b^2} = 1$$

$$\cos^2 \theta + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \cos^2 \theta$$

$$\frac{y^2}{b^2} = \sin^2 \theta$$

$$y^2 = b^2 \sin^2 \theta$$

$$y = \pm b \sin \theta$$

$$y = b \sin \theta$$

i.e $P(x, y) = (a \cos \theta, b \sin \theta)$

Theorem

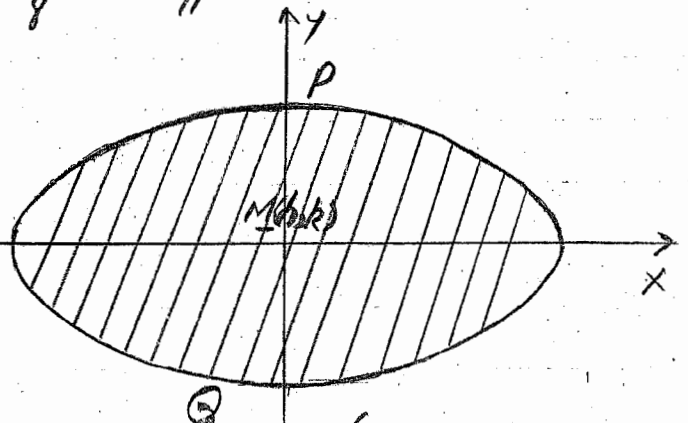
Show that the locus of the middle points of a system of // chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = -\frac{b^2}{a^2 m} x$$

where m is the slope of the // chords.

Proof: The equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

* The parallel chords



have slope m so that the equation to any one of them, say PQ is

$$y = mx + C \quad \text{--- (1)}$$

The straight line (1) meets the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at points whose abscissae are given by

$$\frac{x^2}{a^2} + \frac{(mx+C)^2}{b^2} = 1$$

$$\Rightarrow b^2 x^2 + a^2 (mx+C)^2 = a^2 b^2$$

$$\Rightarrow b^2 x^2 + a^2 (m^2 x^2 + 2mcx + C^2) = a^2 b^2$$

$$\Rightarrow b^2 x^2 + a^2 m^2 x^2 + 2a^2 mcx + a^2 C^2 = a^2 b^2$$

$$\Rightarrow x^2 (b^2 + a^2 m^2) + 2a^2 mcx + a^2 C^2 - a^2 b^2 = 0$$

$$\text{i.e. } x^2 (a^2 m^2 + b^2) + 2a^2 mcx + a^2 (C^2 - b^2) = 0$$

Let the roots of this equation be x_1, x_2 . Then x_1, x_2 are the abscissae of P and Q. Let $M(h, k)$ be the middle point of PQ. Then by using sum of the roots we have

$$h = \frac{x_1 + x_2}{2} = \frac{-2a^2 mc}{\frac{a^2 m^2 + b^2}{2}} = \frac{-a^2 mc}{a^2 m^2 + b^2} \quad \text{--- (2)}$$

$\therefore M(h, k)$ lies on (1).

$$\therefore k = mh + C$$

$$C = k - mh \quad \text{--- (3)}$$

(3) in (2)

$$\Rightarrow h = - \frac{a^2 m (k - mh)}{a^2 m^2 + b^2}$$

$$a^2 m^2 h + b^2 h = -a^2 mk + a^2 m^2 h$$

$$b^2 h = -a^2 mk$$

$$k = - \frac{b^2 h}{a^2 m}$$

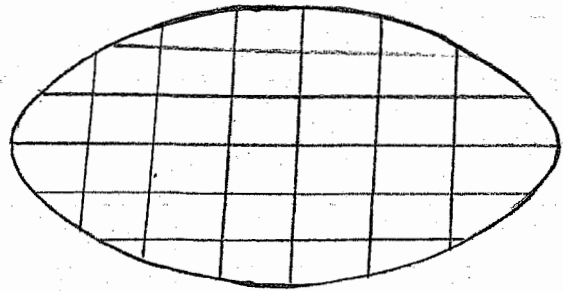
i.e. the pt. $M(h, k)$ lies on the locus, $y = - \frac{b^2 x}{a^2 m}$.

Diameter of an Ellipse.

Def. The locus of the middle pts. of a system of // chords of an ellipse is called a diameter of an ellipse.

Conjugate Diameter:-

Def. Two diameters of an ellipse are called conjugate if each bisects chord // to other.



Result for Conjugate Diameters.

For Conjugate diameters The product of their slopes = $-\frac{b^2}{a^2}$

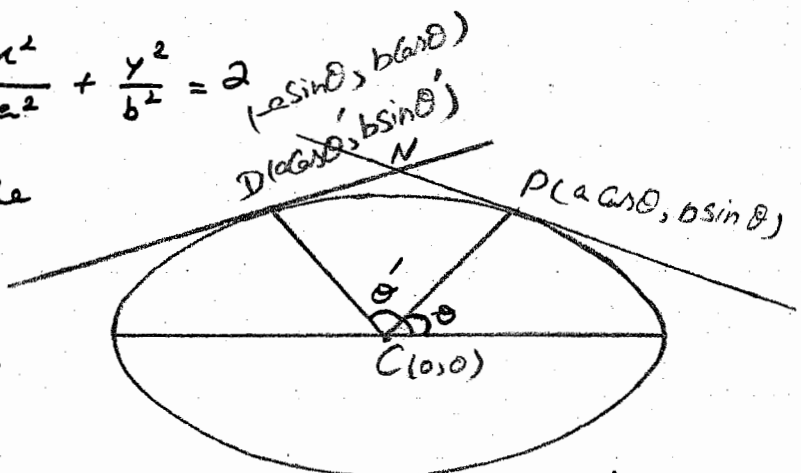
Theorem.

If CP and CD are semiconjugate diameters of an ellipse with centre C, show that

- The eccentric angles of P and D differ by a right-angle.
- $CP^2 + CD^2 = a^2 + b^2$, a constant.
- The locus of the point of intersection of tangents at P and D is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

Proof:- Let θ and θ' be the eccentric angles of P and D where CP and CD are the semi conjugate diameters.



Then $P = (a \cos \theta, b \sin \theta)$, $D = (a \cos \theta', b \sin \theta')$

$$\begin{aligned}\text{Slope of CP} &= \frac{b \sin \theta - 0}{a \cos \theta - 0} \\ &= \frac{b \sin \theta}{a \cos \theta}\end{aligned}$$

$$\text{Similarly slope of CD} = \frac{b \sin \theta'}{a \cos \theta'}$$

We use here the slope formula when the points are given i.e. $\frac{y_2 - y_1}{x_2 - x_1}$

$\therefore$ CP and CD are semi conjugate diameters

$$\therefore \frac{b \sin \theta}{a \cos \theta} \cdot \frac{b \sin \theta'}{a \cos \theta'} = -\frac{b^2}{a^2}$$

$$\Rightarrow \frac{\sin \theta \sin \theta'}{\cos \theta \cos \theta'} = -1$$

$$\Rightarrow \sin \theta \sin \theta' + \cos \theta \cos \theta' = 0$$

$$\cos(\theta' - \theta) = 0$$

$$\theta' - \theta = 90^\circ$$

$$\therefore \theta' - \theta = \pi/2$$

i.e. the eccentric angles θ' and θ of D and P differ by right angle.

Deduction.

$$\therefore \theta' = \pi/2 + \theta$$

$$\begin{aligned}\therefore D &= (a \cos(\pi/2 + \theta), b \sin(\pi/2 + \theta)) \\ &= (-a \sin \theta, b \cos \theta)\end{aligned}$$

ii)

$$\text{Target: } CP^2 + CD^2 = a^2 + b^2$$

$$\begin{aligned}|CP|^2 &= (a \cos \theta - 0)^2 + (b \sin \theta - 0)^2 \\ &= a^2 \cos^2 \theta + b^2 \sin^2 \theta\end{aligned}$$

$$\begin{aligned}\text{Now } |CD|^2 &= (-a \sin \theta - 0)^2 + (b \cos \theta - 0)^2 \\ &= a^2 \sin^2 \theta + b^2 \cos^2 \theta\end{aligned}$$

Now

$$\begin{aligned}|CP|^2 + |CD|^2 &= a^2 \cos^2 \theta + b^2 \sin^2 \theta + a^2 \sin^2 \theta + b^2 \cos^2 \theta \\ CP^2 + CD^2 &= a^2 (\cos^2 \theta + \sin^2 \theta) + b^2 (\cos^2 \theta + \sin^2 \theta) \\ CP^2 + CD^2 &= a^2 + b^2\end{aligned}$$

Proved.

iii) We know that the Eq. of Ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It can be written as

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Now Tangent at P is

$$\frac{x \cdot a \cos \theta}{a^2} + \frac{y \cdot b \sin \theta}{b^2} = 1$$

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \quad \text{--- (i)}$$

Tangent at D is

$$\frac{x(-a \sin \theta)}{a^2} + \frac{y(b \cos \theta)}{b^2} = 1$$

$$-\frac{x \sin \theta}{a} + \frac{y \cos \theta}{b} = 1 \quad \text{--- (ii)}$$

Squaring and adding (i) and (ii)

$$\frac{x^2 \cos^2 \theta}{a^2} + \frac{y^2 \sin^2 \theta}{b^2} + \frac{2xy \sin \theta \cos \theta}{ab} + \frac{x^2 \sin^2 \theta}{a^2} + \frac{y^2 \cos^2 \theta}{b^2} - 2 \frac{xy \sin \theta \cos \theta}{ab} = 1 + 1$$

$$\frac{x^2}{a^2} [\cos^2 \theta + \sin^2 \theta] + \frac{y^2}{b^2} [\cos^2 \theta + \sin^2 \theta] = 2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

Proved.

Exercise 6.1

Examine whether each of the given equations represents two straight lines. If so, find an equation of each straight line.

Q.1

$$10xy + 8x - 15y - 12 = 0 \quad \text{--- ①}$$

General equation of 2nd degree is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- ②}$$

From ① and ②

$$a=0, h=5, b=0, g=4, f=-\frac{15}{2}, c=-12$$

Now

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 0 & 5 & 4 \\ 5 & 0 & -\frac{15}{2} \\ 4 & -\frac{15}{2} & -12 \end{vmatrix}$$

$$= 0 - 5(-60 - (-30)) + 4(-\frac{75}{2} - 0)$$

$$= -5(-60 + 30) - 2(75)$$

$$= -5(-30) - 150$$

$$= 150 - 150$$

$$= 0$$

i.e ① represents two straight lines.

1) $\Rightarrow$

$$2x(5y+4) - 3(5y+4) = 0$$

$$(2x-3)(5y+4) = 0$$

$$2x-3=0, \quad 5y+4=0$$

are the two straight lines.

② # 2

$$2x^2 - xy + 5x - 2y + 2 = 0 \quad \text{--- ①}$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- ②}$$

Here

$$a=2, h=-\frac{1}{2}, b=0, g=\frac{5}{2}, f=-1, c=2$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & 0 & -1 \\ \frac{5}{2} & -1 & 2 \end{vmatrix}$$

$$= 2(0-1) + \frac{1}{2}(-1 + \frac{5}{2}) + \frac{5}{2}(\frac{1}{2} - 0)$$

$$= -2 + \frac{1}{2} \left(\frac{-2+5}{2} \right) + \frac{5}{4}$$

$$= -2 + \frac{3}{4} + \frac{5}{4} = \frac{-8+3+5}{4} = 0$$

Hence the given equation represents two straight lines.

$$1) \Rightarrow 2x^2 + x(5-y) + 2(1-y) = 0$$

$$x = \frac{-(5-y) \pm \sqrt{(5-y)^2 - 4(2)(2-2y)}}{2(2)}$$

$$x = \frac{-(5-y) \pm \sqrt{(5-y)^2 - 8(2-2y)}}{4}$$

$$x = \frac{-(5-y) \pm \sqrt{25+y^2-10y-16+16y}}{4}$$

$$x = \frac{-(5-y) \pm \sqrt{9+y^2+6y}}{4}$$

$$x = \frac{-(5-y) \pm \sqrt{(3+y)^2}}{4}$$

$$x = \frac{-(5-y) \pm (3+y)}{4}$$

$$x = \frac{-5+y+3+y}{4} \text{ and } \frac{-5+y-3-y}{4}$$

$$x = \frac{-2+2y}{4} \text{ and } \frac{-8}{4}$$

$$x = \frac{y-1}{2} \text{ and } -\frac{8}{4}$$

$$\Rightarrow x = \frac{y-1}{2} \text{ and } x = -\frac{8}{4}$$

$$2x - y + 1 = 0 \text{ and } 4x + 8 = 0$$

are the two straight lines.

② #3

$$6x^2 - 17xy - 3y^2 + 22x + 10y - 8 = 0 \quad \text{--- ①}$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here $a = 6, h = -\frac{17}{2}, b = -3, g = 11, f = 5, c = -8$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 6 & -\frac{17}{2} & 11 \\ -\frac{17}{2} & -3 & 5 \\ 11 & 5 & -8 \end{vmatrix}$$

$$= 6(24 - 25) + \frac{17}{2}(68 - 55) + 11(-\frac{85}{2} + 33)$$

$$= -6 + \frac{221}{2} - \frac{935}{2} + 363$$

$$= \frac{221}{2} - \frac{935}{2} + 357$$

$$= \frac{221 - 935 + 714}{2} = \frac{0}{2} = 0$$

Hence the given equation represents the two straight lines.

1) $\Rightarrow 6x^2 + x(22 - 17y) + (-3y^2 + 10y - 8) = 0$

$$x = \frac{-(22 - 17y) \pm \sqrt{(22 - 17y)^2 - 4(6)(-3y^2 + 10y - 8)}}{2(6)}$$

$$x = \frac{-22 + 17y \pm \sqrt{484 + 289y^2 - 748y + 72y^2 - 240y + 192}}{12}$$

$$x = \frac{-22 + 17y \pm \sqrt{676 + 361y^2 - 988y}}{12}$$

$$x = \frac{-22 + 17y \pm \sqrt{(26 - 19y)^2}}{12}$$

$$x = \frac{-22 + 17y \pm (26 - 19y)}{12}$$

$$\Rightarrow x = \frac{-22 + 17y + 26 - 19y}{12}, \quad x = \frac{-22 + 17y - 26 + 19y}{12}$$

$$x = \frac{4 - 2y}{12}, \quad x = \frac{-48 + 36y}{12}$$

$$x = \frac{2-y}{6}, \quad x = -4 + 3y$$

$6x + y - 2 = 0$, $x - 3y + 4 = 0$
are the two straight lines.

② #4 $10x^2 - 23xy - 5y^2 - 29x + 32y + 21 = 0$ ——— ①

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here $a = 10, h = -\frac{23}{2}, b = -5, g = -\frac{29}{2}, f = 16, c = 21$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 10 & -\frac{23}{2} & -\frac{29}{2} \\ -\frac{23}{2} & -5 & 16 \\ -\frac{29}{2} & 16 & 21 \end{vmatrix}$$

$$= 10(-105 - 256) + \frac{23}{2}(-\frac{483}{2} + 232) - \frac{29}{2}(-184 + \frac{145}{2})$$

$$= -3610 + \frac{23}{2}(\frac{-483 + 464}{2}) - \frac{29}{2}(\frac{-368 + 145}{2})$$

$$= -3610 + \frac{23}{2}(-\frac{19}{2}) - \frac{29}{2}(\frac{-223}{2})(-\frac{513}{2})$$

$$= -3610 - \frac{437}{4} + \frac{14877}{4}$$

$$= \frac{-14440 - 437 + 14877}{4} = 0$$

The given equation represents the pair of lines.

$\Rightarrow$

$$10x^2 + x(-29 - 23y) + (-5y^2 + 32y + 21) = 0$$

$$10x^2 - x(23y + 29) + (-5y^2 + 32y + 21) = 0$$

$$x = \frac{(23y + 29) \pm \sqrt{(23y + 29)^2 - 40(-5y^2 + 32y + 21)}}{20}$$

$$= \frac{23y + 29 \pm \sqrt{529y^2 + 841 + 1334y + 200y^2 - 1280y - 840}}{20}$$

$$= \frac{23y + 29 \pm \sqrt{729y^2 + 54y + 1}}{20}$$

$$\begin{aligned}
 &= \frac{23y+29 \pm (27y+1)}{20} \\
 &= \frac{23y+29+27y+1}{20}, \frac{23y+29-27y-1}{20} \\
 &= \frac{50y+30}{20}, \frac{-4y+28}{20} \\
 &= \frac{5y+3}{2}, \frac{-y+7}{5}
 \end{aligned}$$

Hence the lines are

$$2x - 5y - 3 = 0, \quad 5x + y - 7 = 0$$

③ #5.

$$6x^2 - 15y^2 - xy + 16x + 24y = 0 \quad \text{--- ①}$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0$$

Here $a = 6, h = -\frac{1}{2}, b = -15, g = 8, f = 12, C = 0$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & C \end{vmatrix} = \begin{vmatrix} 6 & -\frac{1}{2} & 8 \\ -\frac{1}{2} & -15 & 12 \\ 8 & 12 & 0 \end{vmatrix}$$

$$\begin{aligned}
 &= 6(-144) + \frac{1}{2}(-96) + 8(-6 + 120) \\
 &= -864 - 48 + 912 \\
 &= 0
 \end{aligned}$$

i.e ① Represents the two lines.

$$\Rightarrow 6x^2 + x(-y+16) + (-15y^2+24y) = 0$$

$$x = \frac{-(-y+16) \pm \sqrt{(-y+16)^2 - 4(6)(-15y^2+24y)}}{2(6)}$$

$$x = \frac{(y-16) \pm \sqrt{y^2 + 256 - 32y + 360y^2 - 576y}}{12}$$

$$x = \frac{y-16 \pm \sqrt{361y^2 - 608y + 256}}{12}$$

$$x = \frac{y-16 \pm (19y-16)}{12}$$

$$\Rightarrow x = \frac{y-16+19y-16}{12}, \quad x = \frac{y-16-19y+16}{12}$$

$$12x - 20y + 8 = 0, \quad 12x + 18y = 0$$

$$3x - 5y + 8 = 0, \quad 2x + 3y = 0$$

Are the two straight lines.

For what value of λ will each of the following equations represent a pair of straight lines.

① # 6.

$$\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0$$

Here $a = \lambda, h = -5, b = 12, g = \frac{5}{2}, f = -8, C = -3$

The given equation represents two straight lines if,

$$\begin{vmatrix} \lambda & -5 & \frac{5}{2} \\ -5 & 12 & -8 \\ \frac{5}{2} & -8 & -3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda(-36 - 64) + 5(15 + 20) + \frac{5}{2}(40 - 30) = 0$$

$$-100\lambda + 175 + 25 = 0$$

$$-100\lambda = -200$$

$$-\lambda = -2$$

$$\lambda = 2$$

Hence to represent a pair of straight lines λ must be equal to 2.

② # 7

$$\lambda xy + 5x + 3y + 2 = 0 \quad \text{----- ①}$$

Here $a = 0, h = \frac{\lambda}{2}, b = 0, g = \frac{5}{2}, f = \frac{3}{2}, C = 2$

The given equation represents two straight lines if

$$\begin{vmatrix} 0 & \lambda/2 & 5/2 \\ \lambda/2 & 0 & 3/2 \\ 5/2 & 3/2 & 2 \end{vmatrix} = 0$$

$$-\lambda/2(\lambda - \frac{15}{4}) + 5/2(\frac{3\lambda}{4}) = 0$$

$$-\lambda^2/2 + 15\lambda/8 + 15\lambda/8 = 0$$

$$-4\lambda^2 + 15\lambda + 15\lambda = 0$$

$$-4\lambda^2 + 30\lambda = 0$$

$$-2\lambda^2 + 15\lambda = 0$$

$$\lambda(-2\lambda + 15) = 0 \Rightarrow \lambda = 0 \text{ or } \lambda = +\frac{15}{2}$$

If $\lambda=0$, the given equation is linear and represent a straight line. Hence a point of lines $\lambda = \frac{15}{2}$ for (1) represent.

Q #8

$$4x^2 - 9y^2 - 2(8+\lambda)x - 18y = 29 + 2\lambda \quad \text{--- (1)}$$

$$4x^2 - 9y^2 - 16x - 2\lambda x - 18y - 29 - 2\lambda = 0$$

$$4x^2 - 9y^2 - 2x(8+\lambda) - 18y - 29 - 2\lambda = 0$$

Here $a = 4, h = 0, b = -9, g = -(8+\lambda), f = -9, c = -(29+2\lambda)$

$\therefore$ (1) Represents two straight lines if

$$\begin{vmatrix} 4 & 0 & -8-\lambda \\ 0 & -9 & -9 \\ -8-\lambda & -9 & -29-2\lambda \end{vmatrix} = 0$$

$$4(-9(-29-2\lambda) - 81) + (-8-\lambda)(-(-9(-8-\lambda))) = 0$$

$$4(261 + 18\lambda - 81) + (-8-\lambda)(-72 - 9\lambda) = 0$$

$$1044 + 72\lambda - 324 + 576 + 72\lambda + 72\lambda + 9\lambda^2 = 0$$

$$9\lambda^2 + 216\lambda + 1296 = 0$$

$$\lambda^2 + 24\lambda + 144 = 0$$

$$(\lambda + 12)^2 = 0$$

$$\lambda + 12 = 0$$

$$\lambda = -12$$

Hence $\lambda = -12$

Find the angle b/w each of the following pairs of lines.

Q #9

$$x^2 - 2xy \tan \theta - y^2 = 0 \quad \text{--- (1)}$$

$$ax^2 + 2hxy + by^2 = 0$$

Here $a = 1, h = -\tan \theta, b = -1$

$$\therefore a = 1, b = -1$$

$$\therefore a + b = 0$$

Lines generated by (1) are $\perp$ i.e the angle b/w the lines = 90°

Q #10.

$$3x^2 + 7xy + 2y^2 = 0 \quad \text{--- (1)}$$

$$ax^2 + 2hxy + by^2 = 0$$

Here

$$a = 3, \quad h = 7/2, \quad b = 2$$

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{49/4 - 6}}{5}$$

$$= \frac{2\sqrt{\frac{49-24}{4}}}{5} = \frac{2\sqrt{25/4}}{5} = \frac{2 \cdot 5/2}{5} = 1$$

$$\theta = \tan^{-1}(1)$$

$$\theta = \pi/4$$

i.e. the angle b/w the lines is $\frac{\pi}{4}$.

Q #11

$$11x^2 + 16xy - y^2 = 0 \quad \text{--- (1)}$$

$$a = 11, \quad h = 8, \quad b = -1$$

$$\tan \theta = \frac{2\sqrt{64+11}}{11-1} = \frac{2\sqrt{75}}{10} = \frac{2 \times 5\sqrt{3}}{10}$$

$$\tan \theta = \sqrt{3}$$

$$\theta = \tan^{-1}(\sqrt{3})$$

$$\theta = 60^\circ$$

i.e. the angle b/w the curves is 60° .

Q #12

$$x^2 + 4xy + y^2 - 6x - 3 = 0$$

$$a = 1, \quad h = 2, \quad b = 1$$

$$\tan \theta = \frac{2\sqrt{4-1}}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$\tan \theta = \sqrt{3}$$

$$\theta = 60^\circ$$

i.e. the angle b/w the lines = 60°

Q #13

$$6x^2 + xy - y^2 - 21x - 8y + 9 = 0$$

Here $a = 6$, $b = -1$, $h = \frac{1}{2}$

$$\begin{aligned} \tan \theta &= \frac{2\sqrt{\frac{1}{4} + 6}}{6-1} = \frac{2\sqrt{\frac{4+24}{4}}}{5} \\ &= \frac{\sqrt{28}}{5} = \frac{5}{5} = 1 \end{aligned}$$

$$\theta = \tan^{-1}(1)$$

$$\theta = 45^\circ$$

i.e the angle b/w the lines is 45° .

Q #14 Show that $16xy - 6x + 8y - 3 = 0$ represents a pair of straight lines. Also prove that this together with the coordinate axes form a rectangle and find the area enclosed by the rectangle.

Soln.

$$16xy - 6x + 8y - 3 = 0 \quad \text{--- (1)}$$

$$2x(8y - 3) + (8y - 3) = 0$$

$$(2x+1)(8y-3) = 0$$

$$\Rightarrow 2x+1=0, \quad 8y-3=0$$

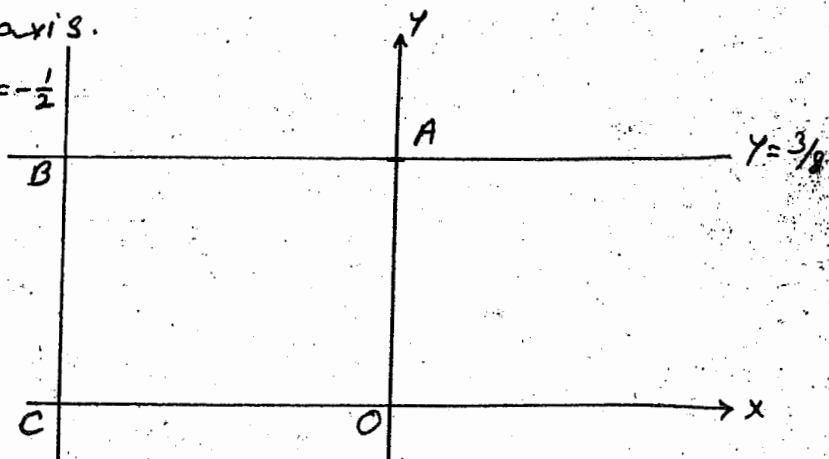
$\therefore$ (1) has been factorized into two ^{linear} factors

$\therefore$ The resulting equations represents the two straight lines.

We can write $2x+1=0$ and $8y-3=0$ in the form $x = -\frac{1}{2}$, slope $\parallel$ to y-axis and $y = \frac{3}{8}$ $\parallel$ line $\parallel$ to x-axis.

Area of the Triangle

$$\begin{aligned} &= 10 C |OA| \\ &= |-\frac{1}{2}| | \frac{3}{8} | \\ &= \frac{3}{16} \text{ Sq. Unit} \end{aligned}$$



☺ #15. Show that an equation of rectangular hyperbola $x^2 - y^2 = 1$ referred to its asymptotes as axis is $x'y' = -\frac{1}{2}$

Soln

The given rectangular hyperbola is

$$x^2 - y^2 = 1 \quad \text{--- (1)}$$

We know that the equations of the transformations are

$$x = x' \cos \theta - y' \sin \theta \quad \text{--- (2)}$$

$$y = x' \sin \theta + y' \cos \theta \quad \text{--- (3)}$$

For the eq. referred to its asymptotes, we know that $\theta = \pi/4$

$$2) \Rightarrow x = x' \cos(\pi/4) - y' \sin(\pi/4)$$

$$x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}$$

$$x = \frac{1}{\sqrt{2}} (x' - y') \quad \text{--- (4)}$$

$$3) \Rightarrow y = x' \sin(\pi/4) + y' \cos(\pi/4)$$

$$y = x' \frac{1}{\sqrt{2}} + y' \frac{1}{\sqrt{2}}$$

$$y = \frac{1}{\sqrt{2}} (x' + y') \quad \text{--- (5)}$$

Putting these values in (1), we have

$$\frac{1}{2} (x' - y')^2 - \frac{1}{2} (x' + y')^2 = 1$$

$$x'^2 + y'^2 - 2x'y' - x'^2 - y'^2 - 2x'y' = 2$$

$$-4x'y' = 2$$

$$x'y' = -\frac{1}{2}$$

Angle b/w the asymptotes and the axes is $\pi/4$

Note The asymptotes $y = \pm x$ or regarded as axes. Keeping these axes in view the equation $x^2 - y^2 = 1$ is reduced to

$$x'y' = -\frac{1}{2} \text{ by proceeding as above.}$$

Analyze and graph the conic represented by each of the following equations.

☺ #16 - ☺ #25

For ☺.16 - ☺.25

See the graph Book.

Written by
Shahid Javed

Some Useful results:-

Eq. of the Tangent at any point $P(x_1, y_1)$ to the parabola is

$$yy_1 = 2a(x + x_1)$$

Eq. of the Tangent at any point $P(x_1, y_1)$ to the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

In Parabola $x = at^2, y = 2at$ is the parametric form and $(at^2, 2at) = T$

Exercise 6.2

Find equations of tangent and normal to each of the following curves at the indicated point (P-1 to 4)

① #1.

$$y^2 = 4ax \text{ at } (a, -2a)$$

The given eq. is

$$y^2 = 4ax \text{ — (1)}$$

Diff. w.r.t. 'x'

$$2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Now $\frac{dy}{dx} \Big|_{(a, -2a)} = \frac{2a}{-2a} = -1$

Eq. of tangent at any point $P(x_1, y_1)$ is

$$y - y_1 = \frac{dy}{dx} \Big|_P (x - x_1)$$

Now

Eq. of tangent at $(a, -2a)$ is

$$(y + 2a) = -1(x - a)$$

$$y + 2a = -x + a$$

$$x + y + a = 0$$

Is the required eq. of Tangent.

Now Eq. of the normal at any point $P(x_1, y_1)$ is

$$y - y_1 = -\frac{dx}{dy} (x - x_1)$$

Eq. of the normal at $(a, -2a)$ is

$$(y+2a) = -\frac{1}{-1} (x-a)$$

$$x-y-3a=0$$

Is the required eq. of the normal.

Q#2

$$xy=c^2 \text{ at } (ct, \frac{c}{t})$$

The given eq. is

$$xy=c^2$$

Diff. w.r.t. 'x'

$$y + x \frac{dy}{dx} = 0$$

$$\Rightarrow x \frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = -y/x$$

Now

$$\left. \frac{dy}{dx} \right|_{(ct, \frac{c}{t})} = -\frac{c/t}{ct} = -\frac{c}{t} \cdot \frac{1}{ct} = -\frac{1}{t^2}$$

Eq. of the tangent at $(ct, \frac{c}{t})$ is

$$y - \frac{c}{t} = -\frac{1}{t^2} (x - ct)$$

$$t^2 y - tc = -x + ct$$

$$x + yt^2 - 2ct = 0$$

$$x + yt^2 = 2ct$$

Eq. of Normal at $(ct, \frac{c}{t})$ is

$$y - \frac{c}{t} = t^2 (x - ct)$$

$$yt - c = t^3 x - ct^4$$

$$xt^3 - yt - ct^4 + c = 0$$

$$xt^3 - yt = \frac{ct^4 - c}{t}$$

$$xt^3 - yt = c(t^3 - 1)$$

Q.3. $x(x^2 + y^2) - ay^2 = 0$ at $x = \frac{a}{2}$

The eq. is

$$x(x^2 + y^2) - ay^2 = 0 \quad \text{--- (i)}$$

$$x^3 + xy^2 - ay^2 = 0 \quad \text{--- (ii)}$$

Diff. w.r.t. x

$$3x^2 + y^2 + 2xy \frac{dy}{dx} - 2ay \frac{dy}{dx} = 0$$

$$3x^2 + y^2 + (2xy - 2ay) \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = - \frac{3x^2 + y^2}{2xy - 2ay} \quad \text{--- (iii)}$$

For value of y we put the value of x in eq. (ii)

$$\left(\frac{a}{2}\right)^3 + \left(\frac{a}{2}\right)y^2 - ay^2 = 0$$

$$\frac{a^3}{8} + \frac{a}{2}y^2 - ay^2 = 0$$

$$\frac{a^3}{8} - ay^2/2 = 0$$

$$\frac{a^3}{8} = ay^2/2$$

$$y^2 = \frac{a^2}{4}$$

$$\Rightarrow y = \pm \frac{a}{2}$$

$\Rightarrow$ We have to find the eq. of tangent and normal at $(\frac{a}{2}, \frac{a}{2})$ and $(\frac{a}{2}, -\frac{a}{2})$.

Now we find the eq. of Tangent and Normal at $(\frac{a}{2}, \frac{a}{2})$

$$\begin{aligned} \text{iii) } \Rightarrow \quad \frac{dy}{dx} \bigg|_{(\frac{a}{2}, \frac{a}{2})} &= - \frac{3(\frac{a}{2})^2 + (\frac{a}{2})^2}{2(\frac{a}{2})(\frac{a}{2}) - 2a(\frac{a}{2})} \\ &= - \frac{3\frac{a^2}{4} + \frac{a^2}{4}}{\frac{a^2}{2} - a^2} \\ &= - \frac{\frac{4a^2}{4}}{\frac{a^2 - 4a^2}{4}} = - \frac{4a^2/4}{-3a^2/4} \\ &= \frac{4a^2/4}{3a^2/4} = \frac{4}{3} \end{aligned}$$

$$= \frac{a^2}{a^2} \times \frac{2}{a^2} = 2$$

Eq. of Tangent :- $y - y_1 = \left(\frac{dy}{dx} \right)_p (x - x_1)$

$$(y - a/2) = 2(x - a/2)$$

$$y - a/2 = 2x - a$$

$$\Rightarrow 2x - y + a/2 - a = 0$$

$$\Rightarrow 2x - y - a/2 = 0$$

$$\Rightarrow 4x - 2y - a = 0$$

Is the required Eq. of Tangent.

Eq. of Normal :- $y - y_1 = -\left(\frac{dx}{dy} \right)_p (x - x_1)$

$$y - a/2 = -\frac{1}{2}(x - a/2)$$

$$2y - a = -x + a/2$$

$$x + 2y - \frac{3a}{2} = 0$$

$$2x + 4y - 3a = 0$$

Is the required Eq. of Normal.

Eq. of Tangent and Normal at $(a/2, -a/2)$

$$\begin{aligned} \text{iii) } \Rightarrow \left(\frac{dy}{dx} \right)_{(a/2, -a/2)} &= - \frac{3(a/2)^2 + (-a/2)^2}{2(a/2)(-a/2) - 2a(-a/2)} \\ &= - \frac{3a^2 + a^2}{-a^2 + a^2} = - \frac{4a^2}{-a^2 + 2a^2} \\ &= - a^2 \times \frac{2}{a^2} \\ &= -2 \end{aligned}$$

Eq. of Tangent: $y - y_1 = \left(\frac{dy}{dx} \right)_p (x - x_1)$

$$(y + a/2) = -2(x - a/2)$$

x by (2)

$$2y + a = -4x + 2a$$

$$4x + 2y - a = 0$$

Hence the Eq. of Tangent at $(a/2, -a/2)$ is

$$4x + 2y - a = 0$$

Eq. of Normal: $y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_P} (x - x_1)$

$$(y + a/2) = 2(x - a/2)$$

$$y + a/2 = 2x - a$$

$$2x - y - a - a/2 = 0$$

$$2x - y - 3a/2 = 0$$

$$4x - 2y = 3a$$

Is the required Eq. of Normal.

Q#4. $c^2(x^2 + y^2) = x^2 y^2$ at $\left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right)$

$$c^2 x^2 + c^2 y^2 = x^2 y^2$$

Dividing both sides by $x^2 y^2 c^2$, we have

$$\frac{1}{y^2} + \frac{1}{x^2} = \frac{1}{c^2}$$

$$x^{-2} + y^{-2} = c^{-2} \quad \text{at } \left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right) \quad (1)$$

Diff. w.r.t. x

$$-2x^{-3} - 2y^{-3} \frac{dy}{dx} = 0 \Rightarrow -x^{-3} = y^{-3} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-3}}{y^{-3}} = -\frac{y^3}{x^3}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right)} = -\frac{\left(\frac{c}{\sin\theta}\right)^3}{\left(\frac{c}{\cos\theta}\right)^3} = -\frac{c^3}{\sin^3\theta} \times \frac{\cos^3\theta}{c^3}$$

(say) $m = -\frac{\cos^3\theta}{\sin^3\theta}$

Eq. of the tangent: $y - y_1 = m(x - x_1)$

$$y - \frac{c}{\sin\theta} = -\frac{\cos^3\theta}{\sin^3\theta} \left(x - \frac{c}{\cos\theta}\right)$$

$$y \sin^3\theta - c \sin^2\theta = -\cos^3\theta \left(x - \frac{c}{\cos\theta}\right)$$

$$y \sin^3 \theta - c \sin^2 \theta = c \cos^2 \theta - x \cos^3 \theta$$

$$x \cos^3 \theta + y \sin^3 \theta = c (\cos^2 \theta + \sin^2 \theta)$$

$$x \cos^3 \theta + y \sin^3 \theta = c$$

Is the required eq. of Tangent.

Eq. of Normal: $y - y_1 = -\frac{1}{m} (x - x_1)$

$$\left(y - \frac{c}{\sin \theta}\right) = \frac{\sin^3 \theta}{\cos^3 \theta} \left(x - \frac{c}{\cos \theta}\right)$$

$$y \cos^3 \theta - \frac{c \cos^3 \theta}{\sin \theta} = \sin^3 \theta \left(x - \frac{c}{\cos \theta}\right)$$

$$y \cos^3 \theta - \frac{c \cos^3 \theta}{\sin \theta} = x \sin^3 \theta - \frac{c \sin^3 \theta}{\cos \theta}$$

$$x \sin^3 \theta - y \cos^3 \theta = \frac{c \sin^3 \theta}{\cos \theta} - \frac{c \cos^3 \theta}{\sin \theta}$$

$$x \sin^3 \theta - y \cos^3 \theta = \frac{c \sin^4 \theta - c \cos^4 \theta}{\cos \theta \sin \theta} *$$

$$x \sin^3 \theta - y \cos^3 \theta = -\frac{2c \cos 2\theta}{2 \sin \theta \cos \theta}$$

$$x \sin^3 \theta - y \cos^3 \theta = -\frac{2c \cos 2\theta}{\sin 2\theta} = -2c \cot 2\theta$$

$$x \sin^3 \theta - y \cos^3 \theta + 2c \cot 2\theta = 0$$

Is the required eq. of Normal.

*

$$\frac{c \sin^4 \theta - c \cos^4 \theta}{\cos \theta \sin \theta} = \frac{c (\sin^4 \theta - \cos^4 \theta)}{\cos \theta \sin \theta}$$

$$= \frac{c [(\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta)]}{\cos \theta \sin \theta}$$

$$= \frac{c (\sin^2 \theta - \cos^2 \theta)}{\cos \theta \sin \theta}$$

$$= -\frac{c (\cos \theta \cos \theta - \sin \theta \sin \theta)}{\cos \theta \sin \theta} = \frac{c \cos(\theta + \theta)}{\cos \theta \sin \theta}$$

$$= -\frac{2c \cos 2\theta}{2 \cos \theta \sin \theta} = -\frac{2c \cos 2\theta}{\sin 2\theta} = -2c \cot 2\theta$$

Find the points where the tangent is // to the x-axis and where it is // to the y-axis for each of the given curves. (PS-7):

Q#5.

$$x^3 + y^3 = a^3 \quad \text{--- (i)}$$

Diff. w.r.t. 'x'

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \quad \text{--- (ii)}$$

If the Tangent is // to x-axis then $\frac{dy}{dx} = 0$

$$\Rightarrow -\frac{x^2}{y^2} = 0$$

$$\Rightarrow x = 0$$

Put in (i)

$$y^3 = a^3$$

$$\Rightarrow y = a$$

Hence the tangent is // to x-axis at $(0, a)$

Now If the tangent is // to y-axis then $\frac{dy}{dx} = \infty$

$$\Rightarrow -\frac{x^2}{y^2} = \infty$$

$$\Rightarrow y^2 = 0$$

$$\Rightarrow y = 0$$

Put in (i)

$$x^3 = a^3$$

$$\Rightarrow x = a$$

Hence the tangent will // to y-axis at $(a, 0)$

Q#6.

$$x^3 + y^3 = 3axy \quad \text{--- (i)}$$

Diff. w.r.t. 'x'

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a \left[y + x \frac{dy}{dx} \right]$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3ay + 3ax \frac{dy}{dx}$$

$$(3y^2 - 3ax) \frac{dy}{dx} = 3ay - 3x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

For Tangent Parallel to x-axis

$$\frac{dy}{dx} = 0$$

$$\frac{ay - x^2}{y^2 - ax} = 0$$

$$ay - x^2 = 0$$

$$ay = x^2$$

$$y = \frac{x^2}{a} \quad \text{--- (ii)}$$

Put in (i)

$$x^3 + \left(\frac{x^2}{a}\right)^3 = 3ax \cdot \frac{x^2}{a}$$

$$x^3 + \frac{x^6}{a^3} = 3x^3$$

$$\frac{x^6}{a^3} = 2x^3$$

$$x^3 = 2a^3$$

$$x = 2^{1/3} a$$

Put in (ii)

$$y = \frac{(2^{1/3} a)^2}{a}$$

$$y = (2)^{2/3} a$$

∴ The tangent is parallel to x-axis at $(2^{1/3} a, 2^{2/3} a)$

Now for tangent parallel to y-axis.

$$\frac{dy}{dx} = \infty$$

$$\frac{ay - x^2}{y^2 - ax} = \infty$$

$$\Rightarrow y^2 - ax = 0$$

$$\Rightarrow y^2 = ax$$

$$\Rightarrow x = \frac{y^2}{a} \text{ --- (iii)}$$

Put in (i)

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a \frac{y^2}{a} \cdot y$$

$$\frac{y^6}{a^3} + y^3 = 3y^3 \Rightarrow \frac{y^6}{a^3} = 2y^3$$

$$\Rightarrow y^3 = 2a^3$$

$$\Rightarrow y = 2^{1/3} \cdot a \text{ --- (iv)}$$

put in (iii)

$$\Rightarrow x = \frac{(2^{1/3} \cdot a)^2}{a}$$

$$x = 4^{1/3} \cdot a$$

Hence tangent will be parallel to y-axis at $(4^{1/3}a, 2^{1/3}a)$

Q #7.

$$25x^2 + 12xy + 4y^2 = 1 \text{ --- (i)}$$

Diff. w.r.t 'x'

$$50x + 12y + 12x \frac{dy}{dx} + 8y \frac{dy}{dx} = 0$$

$$(12x + 8y) \frac{dy}{dx} = - \frac{50x + 12y}{1}$$

$$\frac{dy}{dx} = - \frac{50x + 12y}{12x + 8y}$$

The tangent will be parallel to x-axis if

$$\frac{dy}{dx} = 0$$

$$\Rightarrow - \frac{50x + 12y}{12x + 8y} = 0 \Rightarrow 50x + 12y = 0$$

$$\Rightarrow 25x + 6y = 0$$

$$x = -\frac{6}{25}y \quad \text{--- (ii')}$$

Put in (i)

$$25\left(-\frac{6}{25}y\right)^2 + 12\left(-\frac{6}{25}y\right)y + 4y^2 = 1$$

$$\frac{36}{25}y^2 - \frac{72}{25}y^2 + 4y^2 = 1$$

$$y^2 \left(\frac{36 - 72 + 100}{25} \right) = 1$$

$$y^2 \left(\frac{64}{25} \right) = 1$$

$$y^2 = \frac{25}{64}$$

$$y = \pm \frac{5}{8} \quad \text{--- (iii')}$$

Put in (ii')

$$x = -\frac{6}{25}\left(\frac{5}{8}\right), \quad x = -\frac{6}{25}\left(-\frac{5}{8}\right)$$

$$x = -\frac{3}{20}, \quad x = \frac{3}{20}$$

Hence the tangent is parallel to x-axis at $\left(-\frac{3}{20}, \frac{5}{8}\right)$ and $\left(\frac{3}{20}, -\frac{5}{8}\right)$.

Now the tangent will parallel to y-axis if.

$$\frac{dy}{dx} = \infty$$

$$-\frac{50x + 12y}{12x + 8y} = \infty$$

$$\Rightarrow 12x + 8y = 0$$

$$3x + 2y = 0$$

$$y = -\frac{3x}{2} \quad \text{--- (iv)}$$

Put in (i)

$$25x^2 + 12x\left(-\frac{3x}{2}\right) + 4\left(-\frac{3}{2}x\right)^2 = 1$$

$$25x^2 - 18x^2 + 9x^2 = 1$$

$$16x^2 = 1$$

$$x^2 = \frac{1}{16}$$

$$x = \pm \frac{1}{4} \quad \text{--- (v)}$$

Put in (iv)

$$y = -\frac{3}{2} \left(\frac{1}{4} \right) = -\frac{3}{8}$$

$$y = -\frac{3}{2} \left(-\frac{1}{4} \right) = \frac{3}{8}$$

Hence the tangent is parallel to y-axis at
 $\left(\frac{1}{4}, -\frac{3}{8} \right), \left(-\frac{1}{4}, \frac{3}{8} \right)$

Q#8. If $P = x \cos \theta + y \sin \theta$ touches the curve

$$\left(\frac{x}{a} \right)^{\frac{n}{n-1}} + \left(\frac{y}{b} \right)^{\frac{n}{n-1}} = 1$$

Prove that $P^n = (a \cos \theta)^n + (b \sin \theta)^n$.

Proof:-

The given eq. is $\left(\frac{x}{a} \right)^{\frac{n}{n-1}} + \left(\frac{y}{b} \right)^{\frac{n}{n-1}} = 1$ --- (i)

Diff. w.r.t. 'x'

$$\frac{n}{n-1} \left(\frac{x}{a} \right)^{\frac{n}{n-1}-1} \cdot \frac{1}{a} + \frac{n}{n-1} \left(\frac{y}{b} \right)^{\frac{n}{n-1}-1} \cdot \frac{1}{b} \frac{dy}{dx} = 0$$

$$\frac{n}{n-1} \left[\frac{1}{a} \left(\frac{x}{a} \right)^{\frac{n-n+1}{n-1}} + \frac{1}{b} \left(\frac{y}{b} \right)^{\frac{n-n+1}{n-1}} \frac{dy}{dx} \right] = 0$$

$$\frac{1}{a} \left(\frac{x}{a} \right)^{\frac{1}{n-1}} + \frac{1}{b} \left(\frac{y}{b} \right)^{\frac{1}{n-1}} \frac{dy}{dx} = 0$$

$$\frac{1}{b} \left(\frac{y}{b} \right)^{\frac{1}{n-1}} \frac{dy}{dx} = -\frac{1}{a} \left(\frac{x}{a} \right)^{\frac{1}{n-1}}$$

$$\frac{dy}{dx} = -\frac{\frac{1}{a} \left(\frac{x}{a} \right)^{\frac{1}{n-1}}}{\frac{1}{b} \left(\frac{y}{b} \right)^{\frac{1}{n-1}}}$$

Let $P(x_1, y_1)$ be any point on the curve.

then

$$\left(\frac{dy}{dx} \right)_{(x_1, y_1)} = - \frac{\frac{1}{a} \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}}}{\frac{1}{b} \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}}}$$

Thus the eq. of the tangent at $P(x_1, y_1)$ is given by

$$y - y_1 = \left. \frac{dy}{dx} \right|_P (x - x_1)$$

$$\Rightarrow y - y_1 = - \frac{\frac{1}{a} \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}}}{\frac{1}{b} \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}}} (x - x_1)$$

Multiplying by $\frac{1}{b} \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}}$

$$\Rightarrow \left(\frac{y}{b} \right) \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}} - \left(\frac{y_1}{b} \right) \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}} = - \left(\frac{x}{a} \right) \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}} + \left(\frac{x_1}{a} \right) \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}}$$

$$\left(\frac{y}{b} \right) \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}} - \left(\frac{y_1}{b} \right)^{\frac{n}{n-1}} = - \left(\frac{x}{a} \right) \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}} + \left(\frac{x_1}{a} \right)^{\frac{n}{n-1}}$$

$$\Rightarrow \frac{x}{a} \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}} + \frac{y}{b} \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}} = \left(\frac{x_1}{a} \right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b} \right)^{\frac{n}{n-1}}$$

$$\frac{x}{a} \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}} + \frac{y}{b} \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}} = 1 \quad \text{--- (2)} \quad \text{from (1)}$$

But $p = x \cos \theta + y \sin \theta$ --- (3)

Comparing (2) and (3)

$$\frac{\frac{1}{a} \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}}}{\cos \theta} = \frac{\frac{1}{b} \left(\frac{y_1}{b} \right)^{\frac{1}{n-1}}}{\sin \theta} = \frac{1}{p} \quad \text{--- (4)}$$

$$\Rightarrow \frac{\frac{1}{a} \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}}}{\cos \theta} = \frac{1}{p} \Rightarrow \frac{1}{a} \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}} = \frac{\cos \theta}{p}$$

$$\Rightarrow p \left(\frac{x_1}{a} \right)^{\frac{1}{n-1}} = a \cos \theta$$

$$p^n \left(\frac{x_1}{a} \right)^{\frac{n}{n-1}} = (a \cos \theta)^n \quad \text{--- (5)}$$

and similarly

$$p^n \left(\frac{y_1}{b} \right)^{\frac{n}{n-1}} = (b \sin \theta)^n \quad \text{--- (6)}$$

Comparing form
for $a_1 x_1 + b_1 y_1 + c_1 = 0$
 $\frac{a_1 x_1 + b_1 y_1 + c_1}{\sqrt{a_1^2 + b_1^2}} = 0$
i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$(5) + (6) \Rightarrow$$

$$p^n \left(\frac{x_1}{a} \right)^{\frac{n}{n-1}} + p^n \left(\frac{y_1}{b} \right)^{\frac{n}{n-1}} = (a \cos \theta)^n + (b \sin \theta)^n$$

$$p^n \left[\left(\frac{x_1}{a} \right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b} \right)^{\frac{n}{n-1}} \right] = (a \cos \theta)^n + (b \sin \theta)^n$$

$$p^n (1) = (a \cos \theta)^n + (b \sin \theta)^n$$

$$\Rightarrow p^n = (a \cos \theta)^n + (b \sin \theta)^n$$

Is as required.

⑨ #9. The tangent at any point on the curve $x^3 + y^3 = 2a^3$ makes intercepts p and q on the coordinate axes. Show that-

$$p^{-3/2} + q^{-3/2} = 2^{-1/2} a^{-3/2}$$

Soln.

The given eq. of the curve is

$$x^3 + y^3 = 2a^3 \quad \text{--- (A)}$$

Eq. of the tangent at any pt $P(x_1, y_1)$ will be

$$x_1^2 x + y_1^2 y = 2a^3$$

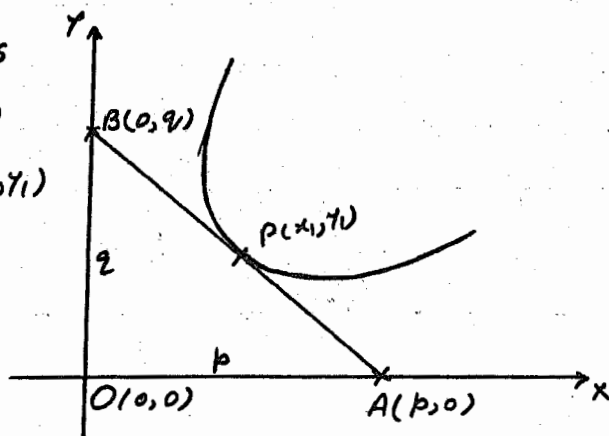
$\therefore A(p, 0)$ lies on the tangent

$$\therefore p x_1^2 + 0 = 2a^3$$

$$x_1^2 = \frac{2a^3}{p}$$

$$x_1 = \frac{2^{1/2} a^{3/2}}{p^{1/2}}$$

$$x_1^3 = \frac{2^{3/2} a^{9/2}}{p^{3/2}} \quad \text{--- (i)}$$



Also $B(0, q)$ lies on the tangent

$$\therefore 0 + y_1^2 q = 2a^3$$

$$y_1^2 = \frac{2a^3}{q} \Rightarrow y_1 = \frac{2^{1/2} a^{3/2}}{q^{1/2}}$$

$$\Rightarrow y_1^3 = \frac{2^{3/2} a^{9/2}}{q^{3/2}} \quad \text{--- (ii)}$$

(i) + (ii) $\Rightarrow$

$$x_1^3 + y_1^3 = \frac{2^{3/2} a^{9/2}}{p^{3/2}} + \frac{2^{3/2} a^{9/2}}{q^{3/2}}$$

$$2a^3 = 2^{3/2} a^{9/2} \left(\frac{1}{p^{3/2}} + \frac{1}{q^{3/2}} \right) \quad \text{from (A)}$$

$$\Rightarrow p^{-3/2} + q^{-3/2} = \frac{1}{2} a^{-3/2}$$

As required.

Alternative Method.

The given curve is

$$x^3 + y^3 = 2a^3 \quad \text{--- (i)}$$

Eq. of the tangent on the curve at any point $P(x_1, y_1)$ is

$$x \cdot x_1^2 + y \cdot y_1^2 = 2a^3 \quad \text{--- (ii)}$$

Interception with x-axis:

When the curve intercept with x-axis then $y=0$

$$\text{ii) } \Rightarrow x \cdot x_1^2 = 2a^3$$

$$\Rightarrow x = \frac{2a^3}{x_1^2}$$

$\therefore$ The curve makes intercept with x-axis at p

$$\therefore p = \frac{2a^3}{x_1^2} \Rightarrow x_1 = \frac{2^{1/2} a^{3/2}}{p^{1/2}}$$

Interception with y-axis.

When the curve intercept with y-axis then $x=0$

$$\text{ii) } \Rightarrow y \cdot y_1^2 = 2a^3$$

$$y = \frac{2a^3}{y_1^2}$$

$\therefore$ the curve makes intercept with y-axis at q

$$\therefore q = \frac{2a^3}{y_1^2} \Rightarrow y_1 = \frac{2^{1/2} a^{3/2}}{q^{1/2}}$$

$\therefore P(x_1, y_1)$ lies on the Curve

$$\therefore x_1^3 + y_1^3 = 2a^3$$

$$\left(\frac{2^{1/2} a^{3/2}}{p^{1/2}}\right)^3 + \left(\frac{2^{1/2} a^{3/2}}{q^{1/2}}\right)^3 = 2a^3$$

$$\frac{2^{3/2} a^{9/2}}{p^{3/2}} + \frac{2^{3/2} a^{9/2}}{q^{3/2}} = 2a^3$$

$$\Rightarrow p^{-3/2} + q^{-3/2} = \frac{2a^3}{2^{3/2} a^{9/2}}$$

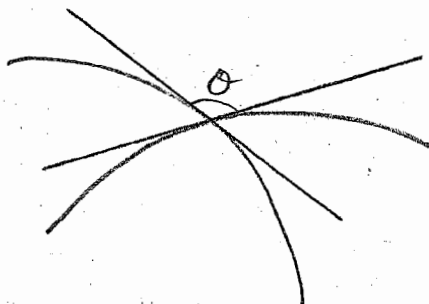
$$\Rightarrow p^{-3/2} + q^{-3/2} = 2^{-1/2} a^{-3/2}$$

As required.

Angle of intersection of two curves: OR

Angle between the two curves:-

Def:- Angle between two curves at their common point of x^n is defined as the angle between the tangents at this point.



Find the angle of intersection of the given curves (10-12)

Q#10 The parabolas $y^2 = 4ax$ and $x^2 = 4by$ at the point other than $(0,0)$.

The given curves are

$$y^2 = 4ax \quad \text{--- (i)}$$

$$x^2 = 4by \quad \text{--- (ii)}$$

$$\text{i)} \Rightarrow x = \frac{y^2}{4a} \quad \text{--- (iii)}$$

Put in (ii)

$$\left(\frac{y^2}{4a}\right)^2 = 4by$$

$$\frac{y^4}{16a^2} = 4by \Rightarrow y^4 = 64a^2by$$

$$y^4 - 64a^2by = 0 \Rightarrow y(y^3 - 64a^2b) = 0$$

$$\Rightarrow y=0 \quad \text{or} \quad y^3 - 64a^2b = 0 \Rightarrow y^3 = 64a^2b$$

$$y = 4a^{2/3}b^{1/3}$$

$$\Rightarrow y=0, \quad y = 4a^{2/3}b^{1/3}$$

Case I. if $y=0$ put in (iii)

$$\Rightarrow x=0$$

$\Rightarrow (0,0)$ is the point of intersection.

Case II. if $y = 4a^{2/3}b^{1/3}$

$$\begin{aligned} \Rightarrow x &= (4a^{2/3}b^{1/3})^2 / 4a \\ &= 16a^{4/3}b^{2/3} / 4a \\ &= 4a^{1/3}b^{2/3} \end{aligned}$$

$\Rightarrow (4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ is the point of intersection.

By the given condition we find the angle of intersection at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$.

Diff. (i) w.r.t. 'x'

$$2y \frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$m_1 = \left. \frac{dy}{dx} \right|_p = \frac{2a}{4a^{2/3}b^{1/3}}$$

$$m_1 = \frac{a^{1/3}}{2b^{1/3}}$$

Diff. (ii) w.r.t. 'x'

$$2x = 4b \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{2b}$$

$$m_2 = \left. \frac{dy}{dx} \right|_p = \frac{4a^{1/3}b^{2/3}}{2b}$$

$$m_2 = \frac{2a^{1/3}}{b^{1/3}}$$

Let θ be the angle between the tangents at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

Then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\tan \theta = \left| \frac{\frac{a^{1/3}}{2b^{1/3}} - \frac{2a^{1/3}}{b^{1/3}}}{1 + \frac{a^{1/3}}{2b^{1/3}} \cdot \frac{2a^{1/3}}{b^{1/3}}} \right|$$

$$\tan \theta = \left| \frac{\frac{a^{1/3} - 4a^{1/3}}{2b^{1/3}}}{1 + \frac{2a^{2/3}}{2b^{2/3}}} \right|$$

$$\tan \theta = \left| \frac{\frac{-3a^{1/3}}{2b^{1/3}}}{\frac{2b^{2/3} + 2a^{2/3}}{2b^{2/3}}} \right|$$

$$\tan \theta = \frac{3a^{1/3}}{2b^{1/3}} \cdot \frac{2b^{2/3}}{2b^{2/3} + 2a^{2/3}}$$

$$\tan \theta = \frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}$$

$$\Rightarrow \theta = \tan^{-1} \left\{ \frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})} \right\}$$

Q#11. $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2 \sqrt{2}$

We have to find the angle of intersection between the curves

$$x^2 - y^2 = a^2 \quad \text{--- (i)}$$

$$x^2 + y^2 = a^2 \sqrt{2} \quad \text{--- (ii)}$$

Let $P(x, y)$ be the pt. of intersection of (i) and (ii)
Diff. (i) w.r.t. 'x' Diff. (ii) w.r.t. 'x'

$$2x - 2y \frac{dy}{dx} = 0$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$m_1 = \frac{dy}{dx} = \frac{x}{y}$$

$$m_2 = \frac{dy}{dx} = -\frac{x}{y}$$

Let θ be the angle of intersection of (i) and (ii)
then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\tan \theta = \left| \frac{x/y + x/y}{1 - x^2/y^2} \right|$$

$$\tan \theta = \frac{2x/y}{\frac{y^2 - x^2}{y^2}} = \frac{2xy}{y^2 - x^2} \quad \text{--- (A)}$$

Now we find the coordinate (x, y) from (i) and (ii)

$$\Rightarrow x^2 - y^2 = a^2 \Rightarrow x^2 = y^2 + a^2 \quad \text{--- (iii)}$$

(iii) Put in (ii)

$$y^2 + a^2 + y^2 = a^2 \sqrt{2}$$

$$a^2 + 2y^2 = a^2 \sqrt{2}$$

$$2y^2 = a^2(\sqrt{2} - 1)$$

$$y^2 = \frac{a^2}{2}(\sqrt{2} - 1) \quad \text{--- (iv)}$$

Put (iv) in (iii)

$$x^2 = \frac{a^2}{2}(\sqrt{2} - 1) + a^2$$

$$x^2 = \frac{a^2}{2}(\sqrt{2} - 1 + 2)$$

$$x^2 = \frac{a^2}{2}(\sqrt{2} + 1) \quad \text{--- (v)}$$

$$(iv) \times (v) \Rightarrow x^2 y^2 = \frac{a^2}{2}(\sqrt{2} + 1) \cdot \frac{a^2}{2}(\sqrt{2} - 1)$$

$$x^2 y^2 = \frac{a^4}{4}(2 - 1) = \frac{a^4}{4}$$

$$xy = \pm \left(\frac{a^2}{2}\right) \quad \text{--- (vi)}$$

$$\text{Now } i) \Rightarrow x^2 - y^2 = a^2$$

$$y^2 - x^2 = -a^2 \quad \text{--- (vii)}$$

Put (vi) and (vii) in (A)

$$\Rightarrow \tan \theta = \frac{2(\pm \frac{a^2}{2})}{-a^2}$$

$$= \frac{\pm a^2}{-a^2}$$

$$= \frac{+a^2}{-a^2}, \quad -\frac{a^2}{-a^2}$$

$$\Rightarrow \tan \theta = -1 \text{ and } \tan \theta = 1$$

$$\Rightarrow \theta = \tan^{-1}(\pm 1)$$

$$\Rightarrow \theta = 45^\circ$$

Q#12.

$$y^2 = ax \text{ and } x^3 + y^3 = 3axy$$

Here

$$y^2 = ax \text{ — (i)}$$

$$x^3 + y^3 = 3axy \text{ — (ii)}$$

1) $\Rightarrow$

$$x = \frac{y^2}{a} \text{ — (iii)}$$

(iii) in (ii) $\Rightarrow$

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a\left(\frac{y^2}{a}\right)(y)$$

$$\frac{y^6}{a^3} + y^3 = 3y^3$$

$$\frac{y^6}{a^3} = 2y^3$$

$$y^6 = 2a^3y^3$$

$$y^6 = 2a^3y^3$$

$$y^6 - 2a^3y^3 = 0$$

$$y^3(y^3 - 2a^3) = 0$$

$$\Rightarrow y = 0, \quad y^3 = 2a^3$$

$$y = 0, \quad y = 2^{1/3}a$$

Two cases arise here.

Case I if $y = 0$ put in (iii)

$$x = 0$$

$\Rightarrow$ The point of intersection is $(0, 0)$.

Case II if $y = 2^{1/3}a$ put in (iii)

$$x = \frac{(2^{1/3}a)^2}{a}$$

$$x = \frac{2^{2/3}a^2}{a}$$

$$x = 2^{1/3}a$$

$\Rightarrow$ The point of intersection is $(2^{1/3}a, 2^{1/3}a)$

First we find the angle of intersection at $(0,0)$

Diff. (i) w.r.t. 'x'

$$2y \frac{dy}{dx} = a$$

$$\frac{dy}{dx} = \frac{a}{2y} \quad \text{--- (iv)}$$

$$m_1 = \left. \frac{dy}{dx} \right|_{(0,0)} = \infty$$

$$m_1 = \infty$$

Diff. (ii) w.r.t. 'x'

$$3x^2 + 3y^2 \frac{dy}{dx} = 3ay + 3ax \frac{dy}{dx}$$

$$(y^2 - ax) \frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \quad \text{--- (v)}$$

$$m_2 = \left. \frac{dy}{dx} \right|_{(0,0)} = \frac{0}{0}$$

i.e. The angle of intersection is undefined at $(0,0)$.
Now we find the angle of intersection at $(2^{1/3}a, 2^{1/3}a)$.

$$(iv) \Rightarrow m_1 = \left. \frac{dy}{dx} \right|_P = \frac{a}{2 \cdot 2^{1/3}a} = \frac{1}{2^{4/3}}$$

$$(v) \Rightarrow m_2 = \left. \frac{dy}{dx} \right|_P = \frac{a(2^{1/3}a) - (2^{2/3}a)^2}{(2^{4/3}a)^2 - a(2^{1/3}a)} = \infty$$

Let θ be the angle of intersection.

then

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan \theta = \frac{m_2 \left(\frac{m_1}{m_2} - 1 \right)}{m_2 \left(\frac{1}{m_2} + m_1 \right)}$$

$$= \frac{m_1/m_2 - 1}{2/20 + \frac{1}{2^{4/3}}}$$

$$\tan \theta = - \frac{1}{\frac{1}{2^{4/3}}} = -2^{4/3}$$

$$\theta = \tan^{-1}(-2^{4/3})$$

$$\theta = 68^\circ 21'$$

Is the required angle of intersection.

Q#13. Find the condition that the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ should intersect orthogonally.

Here $ax^2 + by^2 = 1$ _____ (i)

$a_1x^2 + b_1y^2 = 1$ _____ (ii)

Diff. (i) w.r.t. 'x'

$$2ax + 2by \frac{dy}{dx} = 0$$

Diff. (ii) w.r.t. 'x'

$$2a_1x + 2b_1y \frac{dy}{dx} = 0$$

$$m_1 = \frac{dy}{dx} = -\frac{ax}{by}$$

$$m_2 = \frac{dy}{dx} = -\frac{a_1x}{b_1y}$$

$\therefore$ the curves cut orthogonally

$$\therefore m_1 \cdot m_2 = -1$$

$$-\frac{ax}{by} \cdot -\frac{a_1x}{b_1y} = -1$$

$$\frac{aa_1x^2}{bb_1y^2} = -1 \quad \text{_____ (iii)}$$

$$(b - b_1) \Rightarrow (a - a_1)x^2 + (b - b_1)y^2 = 0$$

$$\frac{x^2}{y^2} = -\frac{(b - b_1)}{(a - a_1)} \quad \text{_____ (iv)}$$

Put in (iii)

$$+ \frac{aa_1}{bb_1} \cdot -\frac{(b - b_1)}{a - a_1} = -1$$

$$\frac{a a_1}{a - a_1} = \frac{b b_1}{b - b_1}$$

$$\Rightarrow \frac{a - a_1}{a a_1} = \frac{b - b_1}{b b_1}$$

$$\frac{a}{a a_1} - \frac{a_1}{a a_1} = \frac{b}{b b_1} - \frac{b_1}{b b_1}$$

$$\frac{1}{a_1} - \frac{1}{a} = \frac{1}{b} - \frac{1}{b}$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$$

Is the required condition.

Q# 14.

Show that the pedal eq. of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is
$$\frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}$$

Soln.

Here the eq. of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- (1)}$$

Diff. w.r.t. 'x'

$$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = - \frac{a^2 y}{b^2 x}$$

$$\frac{dy}{dx} = - \frac{b^2 x}{a^2 y}$$

Let $P(x_1, y_1)$ be the point where the tangent is drawn.
Then
$$\left. \frac{dy}{dx} \right|_P = - \frac{b^2 x_1}{a^2 y_1}$$

If p is the length of the $\perp$ (from) of the tangent from $O(0,0)$ then

$$p = \frac{|y_1 - x_1 y_1'|}{\sqrt{1 + y_1'^2}}$$

$$\Rightarrow p = \frac{y_1 - x_1 \left(-\frac{b^2 x_1}{a^2 y_1} \right)}{\sqrt{1 + \left(-\frac{b^2 x_1}{a^2 y_1} \right)^2}}$$

$$p = \frac{y_1 + \frac{b^2 x_1^2}{a^2 y_1}}{\sqrt{\frac{a^4 y_1^2 + b^4 x_1^2}{a^4 y_1^2}}} = \frac{\frac{a^2 y_1^2 + b^2 x_1^2}{a^2 y_1}}{\frac{\sqrt{a^4 y_1^2 + b^4 x_1^2}}{a^2 y_1}}$$

$$p = \frac{a^2 y_1^2 + b^2 x_1^2}{\sqrt{a^4 y_1^2 + b^4 x_1^2}} \quad (2)$$

from (1)

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\Rightarrow b^2 x_1^2 + a^2 y_1^2 = a^2 b^2 \quad (3)$$

Put in (2)

$$p = \frac{a^2 b^2}{a^2 b^2 \sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}}}$$

$$p^2 = \frac{1}{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}} \quad (A)$$

We know that $r^2 = x_1^2 + y_1^2$ (4)

$$\frac{r^2}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2}$$

$$\frac{r^2}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{y_1^2}{b^2}$$

$$\frac{r^2}{a^2} = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \right) + \left(\frac{y_1^2}{a^2} - \frac{y_1^2}{b^2} \right)$$

$$\frac{r^2}{a^2} = 1 + y_1^2 \left(\frac{1}{a^2} - \frac{1}{b^2} \right)$$

$$\frac{r^2}{a^2} - 1 = y_1^2 \left(\frac{1}{a^2} - \frac{1}{b^2} \right)$$

$$\frac{r^2 - a^2}{a^2} = y_1^2 \left(\frac{1}{a^2} - \frac{1}{b^2} \right)$$

$$\frac{r^2 - a^2}{a^2} = \gamma_1^2 \left(\frac{b^2 - a^2}{a^2 b^2} \right)$$

$$\frac{b^2(r^2 - a^2)}{b^2 - a^2} = \gamma_1^2 \quad \text{--- (5)}$$

Put in (4)

$$r^2 = x_1^2 + \frac{b^2(r^2 - a^2)}{b^2 - a^2}$$

$$x_1^2 = r^2 - \frac{b^2(r^2 - a^2)}{b^2 - a^2}$$

$$x_1^2 = \frac{r^2 b^2 - r^2 a^2 - b^2 r^2 + b^2 a^2}{b^2 - a^2}$$

$$x_1^2 = \frac{a^2(b^2 - r^2)}{b^2 - a^2} \quad \text{--- (6)}$$

$$\begin{aligned} A) \Rightarrow \quad \frac{1}{p^2} &= \frac{x_1^2}{a^4} + \frac{\gamma_1^2}{b^4} \\ \Rightarrow \quad \frac{1}{p^2} &= \frac{\frac{a^2(b^2 - r^2)}{b^2 - a^2}}{a^4} + \frac{\frac{b^2(r^2 - a^2)}{b^2 - a^2}}{b^4} \\ \frac{1}{p^2} &= \frac{b^2 - r^2}{a^2(b^2 - a^2)} + \frac{r^2 - a^2}{b^2(b^2 - a^2)} \\ \frac{1}{p^2} &= \frac{b^2(b^2 - r^2) + a^2(r^2 - a^2)}{a^2 b^2 (b^2 - a^2)} \\ \frac{1}{p^2} &= \frac{b^4 - b^2 r^2 + a^2 r^2 - a^4}{a^2 b^2 (b^2 - a^2)} \\ \frac{1}{p^2} &= \frac{b^4 - a^4 + r^2(a^2 - b^2)}{a^2 b^2 (b^2 - a^2)} \\ \frac{1}{p^2} &= \frac{(b^2 - a^2)(b^2 + a^2) - r^2(b^2 - a^2)}{a^2 b^2 (b^2 - a^2)} \\ \frac{1}{p^2} &= \frac{[b^2 + a^2 - r^2](b^2 - a^2)}{a^2 b^2 (b^2 - a^2)} \end{aligned}$$

$$\frac{1}{b^2} = \frac{b^2 + a^2 - r^2}{a^2 b^2}$$

$$\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}$$

Is as required.

Q#15.

Show that the pedal equation of the curve

$$c^2(x^2 + y^2) = x^2 y^2$$

is

$$\frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$$

Soln. Here

$$c^2(x^2 + y^2) = x^2 y^2 \quad \text{--- (i)}$$

$\div$ by $c^2 x^2 y^2$ we have

$$\frac{1}{y^2} + \frac{1}{x^2} = \frac{1}{c^2} \quad \text{--- (ii)}$$

Differentiating, we have

$$-2 \cdot \frac{1}{y^3} \frac{dy}{dx} + (-2 \frac{1}{x^3}) = 0$$

$$-2 \cdot \frac{1}{y^3} \frac{dy}{dx} = \frac{2}{x^3}$$

$$\Rightarrow \frac{dy}{dx} = - \frac{2y^3}{2x^3}$$

$$\frac{dy}{dx} = - \frac{y^3}{x^3}$$

Let $p(x_1, y_1)$ be any point on the curve where the tangent drawn. Then the slope of the tangent at that point is given by

$$\left. \frac{dy}{dx} \right|_p = - \frac{y_1^3}{x_1^3}$$

If p is the length of the $\perp$ on the tangent from $O(0,0)$. Then

$$p = \frac{|y_1 - x_1 y_1'|}{\sqrt{1 + y_1'^2}}$$

$$p = \frac{y_1 + x_1 \left(\frac{y_1^3}{x_1^3} \right)}{\sqrt{1 + \left(-\frac{y_1^3}{x_1^3} \right)^2}}$$

$$p = \frac{y_1 + \frac{y_1^3}{x_1^2}}{\sqrt{1 + \frac{y_1^6}{x_1^6}}}$$

$$p = \frac{\frac{x_1^2 y_1 + y_1^3}{x_1^2}}{\sqrt{\frac{x_1^6 + y_1^6}{x_1^6}}} = \frac{\frac{x_1^2 y_1 + y_1^3}{x_1^2}}{\frac{\sqrt{x_1^6 + y_1^6}}{x_1^3}}$$

$$p = \frac{\frac{x_1^2 y_1 + y_1^3}{x_1^2} \cdot \frac{x_1^3}{\sqrt{x_1^6 + y_1^6}}}{1}$$

$$p = \frac{x_1^3 y_1 + x_1 y_1^3}{\sqrt{x_1^6 + y_1^6}}$$

$$p = \frac{x_1^3 y_1^3 \left(\frac{1}{y_1^2} + \frac{1}{x_1^2} \right)}{\sqrt{x_1^6 y_1^6 \left(\frac{1}{y_1^6} + \frac{1}{x_1^6} \right)}}$$

$$p = \frac{\frac{1}{y_1^2} + \frac{1}{x_1^2}}{\sqrt{\frac{1}{y_1^6} + \frac{1}{x_1^6}}}$$

$$p = \frac{\frac{1}{y_1^2} + \frac{1}{x_1^2}}{\sqrt{\left(\frac{1}{y_1^2} \right)^3 + \left(\frac{1}{x_1^2} \right)^3}}$$

By using $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$$p = \frac{\frac{1}{y_1^2} + \frac{1}{x_1^2}}{\sqrt{\left(\frac{1}{y_1^2} + \frac{1}{x_1^2} \right)^3 - 3 \left(\frac{1}{x_1^2} \right) \left(\frac{1}{y_1^2} \right) \left(\frac{1}{x_1^2} + \frac{1}{y_1^2} \right)}}$$

$$p = \frac{\frac{1}{c^2}}{\sqrt{\left(\frac{1}{c^2} \right)^3 - \frac{3}{x_1^2 y_1^2} \cdot \frac{1}{c^2}}}$$

$$p^2 = \frac{\frac{1}{c^4}}{\frac{1}{c^6} - \frac{3}{c^2 x_1^2 y_1^2}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{\frac{1}{c^6} - \frac{3}{c^2 x_1^2 y_1^2}}{\frac{1}{c^4}}$$

$$\frac{1}{p^2} = c^4 \left[\frac{1}{c^6} - \frac{3}{c^2 x_1^2 y_1^2} \right]$$

$$\frac{1}{p^2} = \frac{1}{c^2} - \frac{3c^2}{x_1^2 y_1^2} \quad \text{--- (iii)}$$

We know that $r^2 = x_1^2 + y_1^2$

$$\frac{r^2}{x_1^2 y_1^2} = \frac{1}{y_1^2} + \frac{1}{x_1^2}$$

$$\frac{r^2}{x_1^2 y_1^2} = \frac{1}{c^2}$$

from (ii)

$$\frac{c^2}{x_1^2 y_1^2} = \frac{1}{r^2}$$

Put in (iii), we have

$$\frac{1}{p^2} = \frac{1}{c^2} - \frac{3}{r^2}$$

$$\frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$$

Is as required.

Q#16. Show that from any point three normals can be drawn to a parabola $y^2 = 4ax$ and the sum of slopes of the three normals is zero.

Soln.

We know that the eq. of the normal to the parabola

$$y^2 = 4ax$$

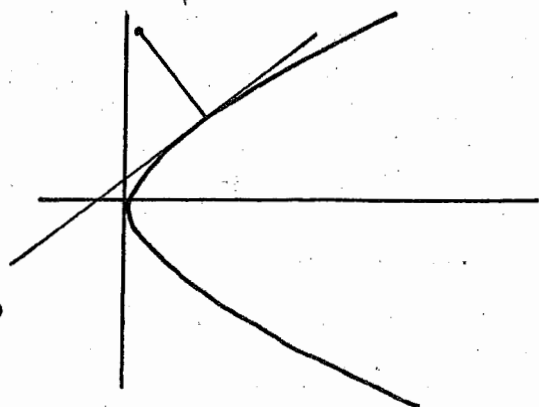
$$\text{is } y = mx - 2am - am^3$$

$$\Rightarrow am^3 + 2am - mx + y = 0$$

$$am^3 + 0m^2 + (2a - x)m + y = 0$$

$\therefore$ this eq. is cubic in m

$\therefore$ We have three values of m from this eq.



m being slope of the normal we can say that three normals can be drawn to this parabola.

If m_1, m_2 and m_3 are the roots of this eq. then we know that

$$\begin{aligned} m_1 + m_2 + m_3 &= - \frac{\text{Co-eff. of } m^2}{\text{Co-eff. of } m^3} \\ &= - \frac{0}{a} \\ &= 0 \end{aligned}$$

Hence the sum of the slopes of the three normals is zero.

$$ax^2 + bx + c = 0$$

If α & β are the roots

$$\text{then } \alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

Similarly for

$$ax^3 + bx^2 + cx + d = 0$$

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Q#17. Show that tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix. (To understand this question see Example 8).

Soln. Let the focal chord (AB)

of the parabola

$$y^2 = 4ax \quad \text{--- (i)}$$

having the focus at

$F(a, 0)$. Let $A(at_1^2, 2at_1)$

and $B(at_2^2, 2at_2)$ be the extremities of the focal chord.

Then we know that if $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$

be the extremities of a focal chord then

$$t_1 t_2 = -1 \quad \text{--- (ii)}$$

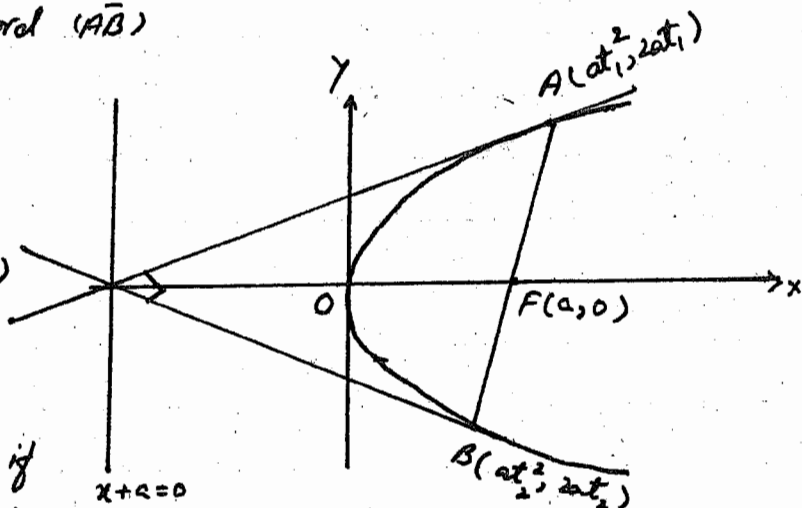
Also the equations of the tangents at A and B

$$\text{are } t_1 y = x + at_1^2 \Rightarrow x - t_1 y + at_1^2 = 0 \quad \text{--- (iii)}$$

$$t_2 y = x + at_2^2 \Rightarrow x - t_2 y + at_2^2 = 0 \quad \text{--- (iv)}$$

Now

$$m_1 = \text{slope of the tangent at } A = - \frac{\text{Co-eff. of } x}{\text{Co-eff. of } y}$$



$$y \cdot 2at_1 = 2a(x + at_1^2)$$

$$t_1 y = x + at_1^2$$

$$\text{Using } y^2 = 2a(x + x_1)$$

$$m_1 = -\frac{1}{-\frac{1}{t_1}} = \frac{1}{t_1}$$

$$m_2 = \text{slope of the tangent at } B = -\frac{1}{-\frac{1}{t_2}} = \frac{1}{t_2}$$

$$\text{Now } m_1 \cdot m_2 = \frac{1}{t_1} \cdot \frac{1}{t_2} = \frac{1}{t_1 t_2}$$

$$m_1 m_2 = -1 \quad \text{from (ii')}$$

$\Rightarrow$ The tangent at A is $\perp$ to the tangent at B.

Multiplying (iii) by t_2

$$t_2 x - t_1 t_2 y + a t_1^2 t_2 = 0 \quad \text{--- (iv)}$$

Multiplying (iv) by t_1

$$t_1 x - t_1 t_2 y + a t_1^2 t_2^2 = 0 \quad \text{--- (vi)}$$

$$(v) - (vi) \Rightarrow$$

$$(t_2 - t_1)x + a(t_1^2 t_2 - t_1 t_2^2) = 0$$

$$(t_2 - t_1)x + a t_1 t_2 (t_1 - t_2) = 0$$

$$(t_2 - t_1)x - a t_1 t_2 (t_2 - t_1) = 0$$

$$(x - a t_1 t_2)(t_2 - t_1) = 0$$

$$\Rightarrow x - a t_1 t_2 = 0$$

$$\Rightarrow x - a t_1 t_2 = 0 \quad \text{from (ii')}$$

$$\Rightarrow x + a = 0$$

Hence the tangents at the ends of a focal chord of a parabola intersect at right angle on the directrix.

Q#18. (a) Show that the tangent at the vertex of a diameter of a parabola is parallel to the chords bisected by the diameter.

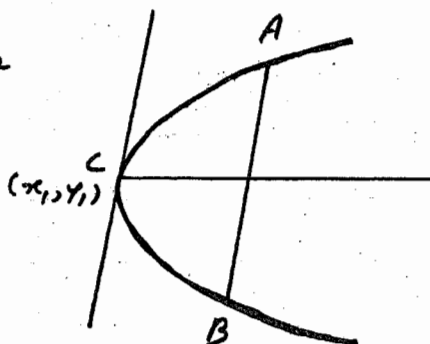
Soln. Let us consider the case of the parabola $y^2 = 4ax$ --- (i)

Let AB be the one of the // chords whose eq. is

$$y = mx + c$$

We know that the equation of the diameter is

$$y = \frac{2a}{m} \quad \text{--- (ii) where } m \text{ is the slope of the chord.}$$



If the pt. $C(x_1, y_1)$ lies on the diameter, then

$$y_1 = \frac{2a}{m} \quad \text{--- (A)}$$

$$\Rightarrow m = \frac{2a}{y_1} \quad \text{--- (B)}$$

which is the slope of $\overline{AB}$.

We know that the equation of tangent at any point of the parabola is

$$yy_1 = 2a(x + x_1)$$

Now eq. of the tangent at $C(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1)$$

$$yy_1 = 2ax + 2ax_1$$

$$2ax - yy_1 - 2ax_1 = 0 \quad \text{--- (ii)}$$

Let $m_1 = \text{slope of the tangent at } C = - \frac{\text{Coeff. of } x}{\text{Coeff. of } y}$

$$m_1 = - \frac{2a}{-y_1} = \frac{2a}{y_1} \quad \text{--- (iv)}$$

Eq. B and (iv) $\Rightarrow$

$$m = m_1$$

$\Rightarrow$ Tangent at C is parallel to the chord $\overline{AB}$.

(b) Prove that the tangents at the ^{ends} of any chord of a parabola meet on the diameter which bisects the Chord.

Soln. Let us consider the chord $\overline{AB}$ of the parabola

$$y^2 = 4ax \quad \text{--- (i)}$$

Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2) be respectively.

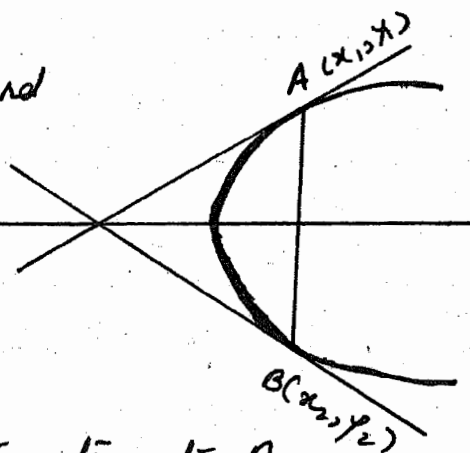
Now equations of the tangents at A and B will

$$yy_1 = 2a(x + x_1)$$

$$yy_2 = 2a(x + x_2)$$

Subtracting

$$yy_1 - yy_2 = 2a(x + x_1) - 2a(x + x_2)$$



$$y(y_1 - y_2) = 2ax + 2ax_1 - 2ax - 2ax_2$$

$$y(y_1 - y_2) = 2a(x_1 - x_2)$$

$$\Rightarrow y = \frac{2a(x_1 - x_2)}{(y_1 - y_2)}$$

$$\Rightarrow y = 2a \cdot \frac{x_2 - x_1}{y_2 - y_1} \quad \text{--- (ii)}$$

If m is the slope of $\overline{AB}$, then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{1}{m} = \frac{x_2 - x_1}{y_2 - y_1}$$

Put in (ii)

$$\Rightarrow y = \frac{2a}{m}$$

which is the eq. of the diameter.

Hence the tangents at the ends of any chord of a parabola meet on the diameter which bisect the chord.

☺ #19. Find the condition that straight line

$$lx + my + n = 0$$

may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the coordinates of the point of contact.

Soln.

$$\text{Here Line} = lx + my + n = 0 \quad \text{--- (1)}$$

$$\text{and ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- (2)}$$

We know that the eq. of the tangent at $p(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{--- (3)}$$

$\therefore$ (1) and (3) are the tangents at the same point

$\therefore$ Comparing (1) and (3)

$$\frac{\frac{x_1}{a^2}}{l} = \frac{\frac{y_1}{b^2}}{m} = \frac{-1}{n}$$

$$\frac{x_1}{a^2 l} = \frac{y_1}{b^2 m} = -\frac{1}{n} \quad \text{--- (4)}$$

4) $\Rightarrow$

$$\frac{x_1}{a^2 l} = -\frac{1}{n}$$

$$x_1 = -\frac{a^2 l}{n}$$

and

$$\frac{y_1}{b^2 m} = -\frac{1}{n}$$

$$y_1 = -\frac{b^2 m}{n}$$

Hence the point of contact is $(-\frac{a^2 l}{n}, -\frac{b^2 m}{n})$

$\therefore$ the point lie on the line

$$\therefore l(-\frac{a^2 l}{n}) + m(-\frac{b^2 m}{n}) + n = 0$$

$$-\frac{a^2 l^2}{n} + (-\frac{b^2 m^2}{n}) + n = 0$$

$$-\frac{a^2 l^2 + m^2 b^2}{n} + n = 0$$

$$\Rightarrow n = \frac{a^2 l^2 + m^2 b^2}{n}$$

$$n^2 = a^2 l^2 + m^2 b^2$$

is the required condition.

Q # 20. Show that the condition that normals at three points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may be concurrent is

$$\begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0$$

Soln. The given eq. of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- (1)}$$

We know that the eq. of the normal at (x_1, y_1) is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

x by x_1, y_1

$$a^2 y_1 + b^2 y x_1 = (a^2 + b^2) x_1 y_1 \quad \text{--- (2)}$$

Similarly the equations of the normals at (x_2, y_2) and (x_3, y_3) are

$$a^2 x y_2 + b^2 y x_2 = (a^2 - b^2)(x_2 y_2) \quad \text{--- (3)}$$

and $a^2 x y_3 + b^2 y x_3 = (a^2 - b^2)(x_3 y_3) \quad \text{--- (4)}$

$\Rightarrow$ The condition is

$$\begin{vmatrix} a^2 y_1 & b^2 x_1 & (a^2 - b^2) x_1 y_1 \\ a^2 y_2 & b^2 x_2 & (a^2 - b^2) x_2 y_2 \\ a^2 y_3 & b^2 x_3 & (a^2 - b^2) x_3 y_3 \end{vmatrix} = 0$$

$$a^2 b^2 (a^2 - b^2) \begin{vmatrix} y_1 & x_1 & x_1 y_1 \\ y_2 & x_2 & x_2 y_2 \\ y_3 & x_3 & x_3 y_3 \end{vmatrix} = 0$$

$$- a^2 b^4 (a^2 - b^2) \begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0$$

Is the required condition.

Q#21. If a Tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre C, meets the major and minor axes in T and t prove that

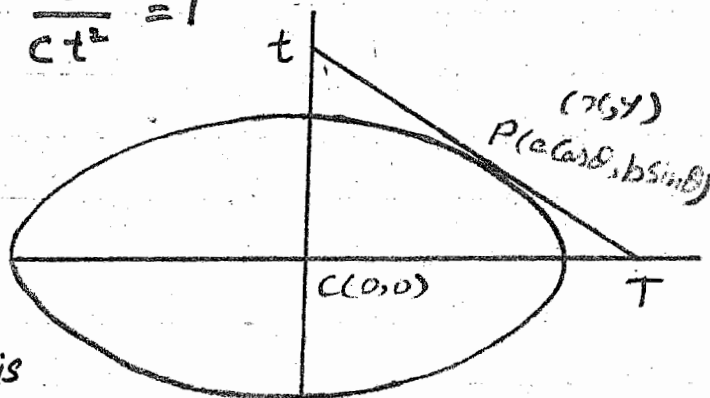
$$\frac{a^2}{CT^2} + \frac{b^2}{Ct^2} = 1$$

Soln:

Here eq. of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- (1)}$

eq. of the tangent at any point of the ellipse is

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1 \quad \text{--- (2)}$$



Intersection of tangent with major Axis:-

$$\Rightarrow y = 0$$

Put in (2)

$$\frac{x x_1}{a^2} + 0 = 1$$

$$\Rightarrow x_1 = \frac{a^2}{x} \Rightarrow x = \frac{a^2}{x_1}$$

$\Rightarrow$ The coordinates of T are $(\frac{a^2}{x_1}, 0)$.

X^n of tangent with minor axis :-

$$\Rightarrow x = 0$$

put in (2)

$$0 + \frac{y y_1}{b^2} = 1$$

$$\Rightarrow y = \frac{b^2}{y_1}$$

$\Rightarrow$ The coordinates of t are $(0, \frac{b^2}{y_1})$.

Now

$$CT = (\frac{a^2}{x_1} - 0) + (0 - 0) \Rightarrow CT = \frac{a^2}{x_1}$$

Squaring $(CT)^2 = \frac{a^4}{x_1^2}$ — (3)

and $ct = \frac{b^2}{y_1}$

$$(ct)^2 = \frac{b^4}{y_1^2}$$
 — (4)

$$3) \Rightarrow \frac{x_1^2}{a^2} = \frac{a^2}{CT^2}$$
 — (5)

$$4) \Rightarrow \frac{y_1^2}{b^2} = \frac{b^2}{ct^2}$$
 — (6)

$$3+4 \Rightarrow$$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{a^2}{CT^2} + \frac{b^2}{ct^2}$$

$$\Rightarrow \frac{a^2}{CT^2} + \frac{b^2}{ct^2} = 1$$

is as required.

Q#22. Show that the locus of the point of intersection of tangents at two points on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 \lambda$$

where 2λ is the difference of the eccentric angles of two points.

Soln. Here Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ——— (1)

Difference in eccentric angles = 2λ

Target: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 \lambda$

Let $P = (a \cos \theta, b \sin \theta)$, $Q = (a \cos \theta', b \sin \theta')$

be any pts. on (1) with θ and θ' as the eccentric angles of P and Q . Then eq. of the tangent at $P(a \cos \theta, b \sin \theta)$ to the ellipse is

$$\frac{x \cdot a \cos \theta}{a^2} + \frac{y \cdot b \sin \theta}{b^2} = 1 \quad \text{———— (2)}$$

$$\frac{\cos \theta}{a} x + \frac{\sin \theta}{b} y = 1$$

$$x \frac{\cos \theta}{a} + \frac{\sin \theta}{b} y - 1 = 0 \quad \text{———— (3)}$$

Like wise the equation of tangent at Q

$$\frac{\cos \theta'}{a} x + \frac{\sin \theta'}{b} y - 1 = 0 \quad \text{———— (4)}$$

Intersection of the tangents

$$\Rightarrow \frac{x/a}{-\sin \theta + \sin \theta'} = \frac{-y/b}{-\cos \theta + \cos \theta'} = \frac{1}{\cos \theta \sin \theta' - \cos \theta' \sin \theta}$$

$$\Rightarrow \frac{x}{a} = \frac{\sin \theta' - \sin \theta}{\sin(\theta' - \theta)} ; \frac{y}{b} = \frac{\cos \theta - \cos \theta'}{\sin(\theta' - \theta)}$$

$$\therefore \theta' - \theta = 2\lambda$$

$$\therefore \frac{x^2}{a^2} = \frac{(\sin \theta' - \sin \theta)^2}{\sin^2 2\lambda} \quad \text{———— (5)}$$

and $\frac{y^2}{b^2} = \frac{(\cos \theta - \cos \theta')^2}{\sin^2 2\lambda} \quad \text{———— (6)}$

~~$$\cos \theta \frac{x}{a} + \sin \theta \frac{y}{b} - 1 = 0$$~~

~~$$\cos \theta' \frac{x}{a} + \sin \theta' \frac{y}{b} - 1 = 0$$~~

(5) + (6) $\Rightarrow$

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{(\sin \theta' - \sin \theta)^2}{\sin^2 2\lambda} + \frac{(\cos \theta - \cos \theta')^2}{\sin^2 2\lambda} \\ &= \frac{(\sin \theta' - \sin \theta)^2 + (\cos \theta - \cos \theta')^2}{\sin^2 2\lambda} \\ &= \frac{\sin^2 \theta' + \sin^2 \theta - 2 \sin \theta \sin \theta' + \cos^2 \theta + \cos^2 \theta' - 2 \cos \theta \cos \theta'}{\sin^2 2\lambda} \\ &= \frac{2 - 2(\cos \theta' \cos \theta + \sin \theta' \sin \theta)}{\sin^2 2\lambda} \\ &= \frac{2(1 - \cos(\theta' - \theta))}{\sin^2 2\lambda} \\ &= \frac{2(1 - \cos 2\lambda)}{(2 \sin \lambda \cos \lambda)^2} \\ &= \frac{2 \cdot 2 \sin^2 \lambda}{4 \sin^2 \lambda \cos^2 \lambda} \\ &= \frac{1}{\cos^2 \lambda} \end{aligned}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 \lambda$$

Is as required.

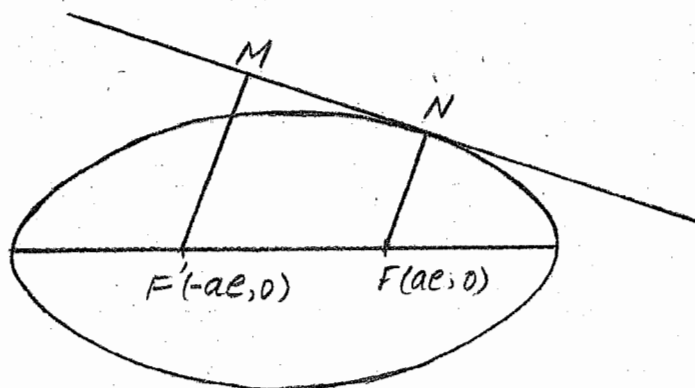
Q#23. Show that locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle and product of the ~~(lengths)~~ lengths of perpendiculars is equal to square on the semi-minor axis.

Soln.

Here ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We know that the square of the tangent at any point on the ellipse can be written in the form



$$y = mx + \sqrt{a^2 m^2 + b^2} \quad \text{--- (i)}$$

$\therefore$ Slope of the tangent = m

$\therefore$ Slope of the perpendicular to the tangent = $-\frac{1}{m}$

Eq. of the perpendicular from $F(ae, 0)$ to the tangent is

$$(y - 0) = -\frac{1}{m}(x - ae)$$

$$\Rightarrow my = -x + ae$$

$$\Rightarrow x + my = ae \quad \text{--- (ii)}$$

Elimination of m

$$1) \Rightarrow y - mx = \sqrt{a^2 m^2 + b^2} \quad \text{--- (iii)}$$

Squaring and adding (ii) and (iii)

$$(x + my)^2 + (y - mx)^2 = a^2 e^2 + (\sqrt{a^2 m^2 + b^2})^2$$

$$\Rightarrow x^2 + m^2 y^2 + 2xmy + y^2 + m^2 x^2 - 2ymx = a^2 e^2 + a^2 m^2 + b^2$$

$$(1 + m^2)x^2 + (1 + m^2)y^2 = a^2(e^2 + m^2) + b^2$$

$$(1 + m^2)(x^2 + y^2) = a^2 m^2 + a^2 e^2 + a^2 - a^2 e^2 \quad \therefore b^2 = a^2 - a^2 e^2$$

$$(1 + m^2)(x^2 + y^2) = a^2 m^2 + a^2$$

$$(1 + m^2)(x^2 + y^2) = a^2(1 + m^2)$$

$$x^2 + y^2 = a^2 \quad \text{--- (iv)}$$

Let Product of $\perp$ s FN and F'M on the tangent = b^2

$$\text{Then } 1) \Rightarrow xm - y + \sqrt{a^2 m^2 + b^2} = 0$$

Now $|FN|$ = Distance of F from the tangent

$$\begin{aligned} &= \frac{|mae - 0 + \sqrt{a^2 m^2 + b^2}|}{\sqrt{m^2 + 1}} \\ &= \frac{|mae + \sqrt{a^2 m^2 + b^2}|}{\sqrt{m^2 + 1}} \end{aligned}$$

Similarly

$$|F'M| = \frac{|-mae + \sqrt{a^2 m^2 + b^2}|}{\sqrt{1 + m^2}}$$

$$|FN| \cdot |F'M| = \frac{(mae + \sqrt{a^2 m^2 + b^2})(\sqrt{a^2 m^2 + b^2} - mae)}{m^2 + 1}$$

$$|FN| \cdot |F'M| = \frac{a^2 m^2 + b^2 - m^2 a^2 e^2}{m^2 + 1}$$

for ellipse

$$\begin{aligned}
 &= \frac{a^2 m^2 + b^2 + m^2(b^2 - a^2)}{1 + m^2} \\
 &= \frac{a^2 m^2 + b^2 + m^2 b^2 - a^2 m^2}{1 + m^2} \\
 &= \frac{b^2(1 + m^2)}{1 + m^2}
 \end{aligned}$$

$$\begin{aligned}
 b^2 &= a^2 - a^2 e^2 \\
 b^2 - a^2 &= -a^2 e^2 \\
 \text{and } a^2 e^2 &= a^2 - b^2 \\
 &\text{for ellipse}
 \end{aligned}$$

$$|FN| \cdot |F'M| = b^2$$

(iv) is a circle having the centre at $O(0,0)$ and radius a and is the auxiliary circle, and the product of the lengths of perpendiculars is equal to the square of the semi minor axis.

① #24. Prove that the area enclosed by the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant.

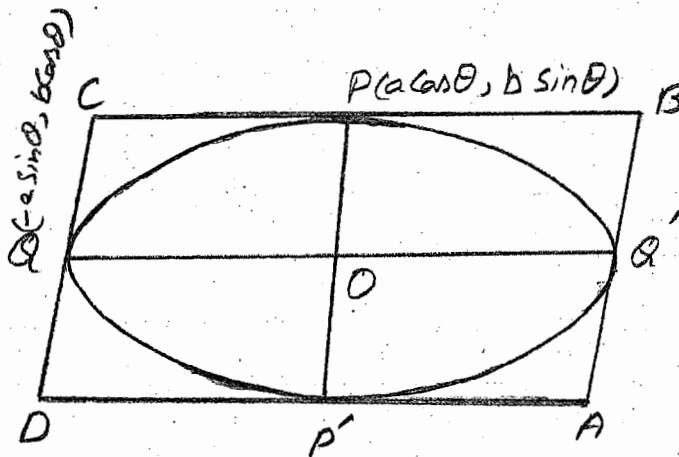
Soln.

Let POP' and QQQ' be the conjugate diameters then

$$P = (a \cos \theta, b \sin \theta)$$

$$Q = (-a \sin \theta, b \cos \theta)$$

$ABCD$ is a //gram due to the tangent drawn at the pts. P, Q, P' and Q' .



Now Area of the //gram $ABCD = 4$ [Area of the //gram $OPCQ$]

$$\text{Vector Area of the //gram } OPCQ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin \theta & b \cos \theta & 0 \\ a \cos \theta & b \sin \theta & 0 \end{vmatrix}$$

$$= \hat{k} (-ab \sin^2 \theta - ab \cos^2 \theta)$$

$$= -ab (\sin^2 \theta + \cos^2 \theta) \hat{k}$$

$$= -ab \hat{k}$$

$$\text{Required area of the //gram } OPCQ = \sqrt{(-ab)^2}$$

$$= ab$$

Put in (i)

$$\Rightarrow \text{Area of the parallelogram } ABCD = 4(ab)$$

$$= 4ab$$

$$= \text{Constant}$$

Hence the area enclosed by the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant.

Q#25. The hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1$ are said to be conjugate

to each other. If e and e' are eccentricities of a hyperbola and its conjugate, prove that

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

Soln: The given equations of the hyperbolas are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{--- (i)}$$

and

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1 \quad \text{--- (ii)}$$

If e is the eccentricity of (i) then we know that

$$a^2 + b^2 = a^2 e^2$$

$$\Rightarrow b^2 = a^2 e^2 - a^2$$

$$b^2 = a^2 (e^2 - 1) \quad \text{--- (iii)}$$

Now (ii) $\Rightarrow$

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1$$

If e' is the eccentricity of this hyperbola

$$\text{Then } a'^2 = b'^2 (e'^2 - 1) \quad \text{--- (iv)}$$

(iii) $\times$ (iv)

$$a^2 b'^2 = a^2 b^2 (e^2 - 1) (e'^2 - 1)$$

$$\frac{a^2 b'^2}{a^2 b^2} = (e^2 - 1) (e'^2 - 1)$$

$$1 = e^2 e'^2 - e^2 - e'^2 + 1$$

$$0 = e^2 e'^2 - e^2 - e'^2$$

$$e^2 + e'^2 = e^2 e'^2$$

$\div$ by $e^2 (e')^2$, we have

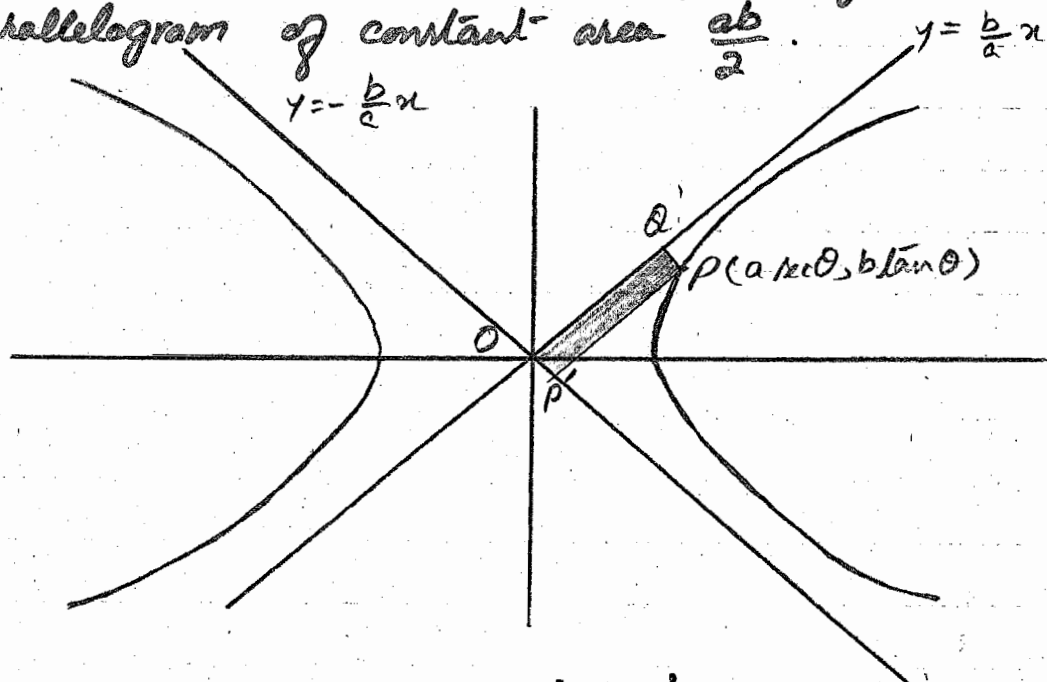
$$\frac{1}{(e')^2} + \frac{1}{e^2} = 1$$

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

Is as required.

Q#26. Show that the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and the lines drawn from any point on the hyperbola parallel to the asymptotes form a parallelogram of constant area $\frac{ab}{2}$.



Soln.

Here hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ——— (1)

and eq. of the asymptote is $y = \pm \frac{b}{a} x$

Let $P = (a \sec \theta, b \tan \theta)$ be any pt. on the hyperbola.

Let the line drawn at P // to the asymptotes meet the asymptotes at P' and Q.

Eq. of line through P or drawn from P // to the

asymptote $y = \frac{b}{a} x$

$\therefore$ The asymptote and line drawn from P are $\parallel$.

$\therefore$ The slope of the asymptote and line are equal

i.e the slope of line $= \frac{b}{a}$

Now Eq. of the line drawn from P is

$$y - b \tan \theta = \frac{b}{a} (x - a \sec \theta) \quad \text{--- (2)}$$

Also eq. of the asymptote is

$$y = -\frac{b}{a} x \quad \text{--- (3)}$$

P' is the pt. of intersection of (2) and (3)

Put (3) in (2)

$$\Rightarrow -\frac{b}{a} x - b \tan \theta = \frac{b}{a} x - b \sec \theta$$

$$b \sec \theta - b \tan \theta = \frac{b}{a} x + \frac{b}{a} x$$

$$b (\sec \theta - \tan \theta) = 2 \frac{b}{a} x$$

$$\Rightarrow x = \frac{a}{2} (\sec \theta - \tan \theta) \quad \text{--- (4)}$$

Put (4) in (3)

$$y = -\frac{b}{a} \cdot \frac{a}{2} (\sec \theta - \tan \theta)$$

$$y = -\frac{b}{2} (\sec \theta - \tan \theta)$$

Hence the coordinates of P' are

$$\frac{a}{2} (\sec \theta - \tan \theta), -\frac{b}{2} (\sec \theta - \tan \theta)$$

Now

$$\text{vector area of the } \parallel^{am} OP'PQ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \sec \theta & b \tan \theta & 0 \\ \frac{a}{2} (\sec \theta - \tan \theta) & -\frac{b}{2} (\sec \theta - \tan \theta) & 0 \end{vmatrix}$$

$$= \left[-\frac{ab}{2} (\sec^2 \theta - \sec \theta \tan \theta) - \frac{ab}{2} (\sec \theta \tan \theta - \tan^2 \theta) \right] \hat{k}$$

$$= \left[-\frac{ab}{2} (\sec^2 \theta - \sec \theta \tan \theta + \sec \theta \tan \theta - \tan^2 \theta) \right] \hat{k}$$

$$= -\frac{ab}{2} \hat{k}$$

$$\therefore \text{Hence area of the } \parallel^{am} OP'PQ = \sqrt{\left(-\frac{ab}{2}\right)^2} = \frac{ab}{2}$$

i.e the area of $\parallel^{am} OP'PQ = \frac{ab}{2}$
Is as required.

Q#27. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at the point 't' such that $t^3 t' = -1$

Note. The point $(ct, \frac{c}{t})$ on $xy = c^2$ is referred to as the point 't'.

Soln. The equation of the rectangular hyperbola is

$$xy = c^2 \quad \text{--- (i)}$$

We find the eq. of normal at the pt.

$$t(ct, \frac{c}{t}).$$

Here

$$y = \frac{c}{x}$$

$$\frac{dy}{dx} = -\frac{c}{x^2}$$

$$\frac{dy}{dx} \bigg|_t = -\frac{c}{c^2 t^2} = -\frac{1}{t^2}$$

Now the equation of the normal

$$y - y_1 = -\frac{1}{\frac{dy}{dx} \big|_t} (x - x_1)$$

$$\Rightarrow y - \frac{c}{t} = -\frac{1}{-\frac{1}{t^2}} (x - ct)$$

$$y - \frac{c}{t} = t^2 (x - ct)$$

$$t^2 y - ct = t^2 (x - ct)$$

$$ty - c = t^3 x - ct^4$$

$$t^3 x - ty + c - ct^4 = 0 \quad \text{--- (i)}$$

if this normal meets the hyperbola at $t'(ct', \frac{c}{t'})$ then

$$t^3 \cdot ct' - t \cdot \frac{c}{t'} + c - ct^4 = 0$$

$$t^3 ct'^2 - tc + \frac{c}{t'} - ct^4 t' = 0$$
~~$$t^3 ct'^2 - tc + \frac{c}{t'} - ct^4 t' = 0$$~~

~~Q# 28. Prove that if P is any point on a hyperbola~~

$$ct'^2 + ct' - ct't^4 - ct = 0$$

$$ct'(t't^3 + 1) - ct(t't^3 + 1) = 0$$

$$(ct' - ct)(t't^3 + 1) = 0$$

$$c(t' - t)(t't^3 + 1) = 0$$

$$\Rightarrow t't^3 + 1 = 0 \quad \because t' \neq t$$

$$t't^3 = -1$$

$$\text{or } t^3 t' = -1$$

is as required.

Q# 28. Prove that if P is any point on a hyperbola with foci F and F', the tangent at P bisects the angle F'PF.

Sol.

Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{--- (i)}$$

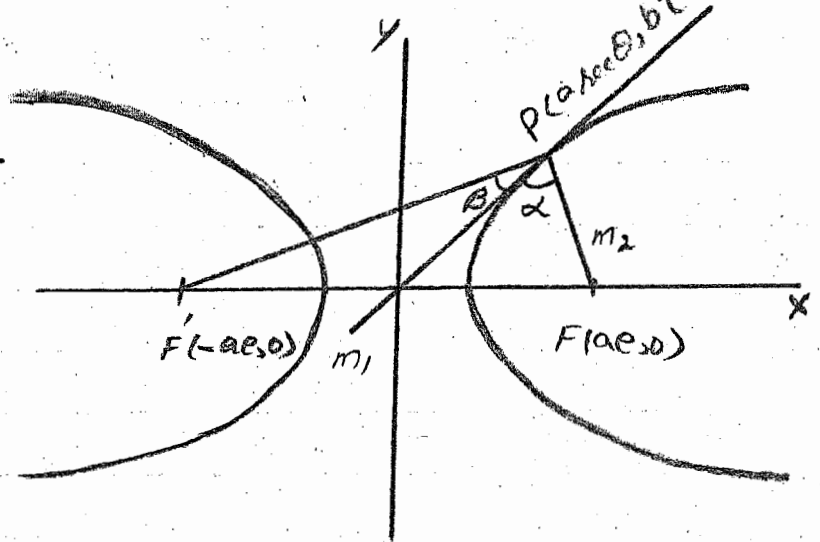
having the pt.

$P(a \sec \theta, b \tan \theta)$ on it.

$\Rightarrow$

$$\frac{\partial x}{\partial t} - \frac{\partial y}{\partial t} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = - \frac{b^2 x}{a^2 y}$$



$m_1 = \text{slope of the tangent at } P = \left(\frac{dy}{dx} \right)_P$

$$m_1 = - \frac{b^2 a \sec \theta}{a^2 b \tan \theta}$$

$$m_1 = \frac{b \sec \theta}{a \tan \theta} \quad \text{--- (ii)}$$

$$m_2 = \text{slope of } \overline{PF} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{b \tan \theta - 0}{a \sec \theta - ae}$$

$$m_2 = \frac{b \tan \theta}{a \sec \theta - ae}$$

$$\text{Now } \tan \alpha = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$$

$$\tan \alpha = \frac{\frac{b \tan \theta}{a \sec \theta - ae}}{\frac{b \sec \theta}{a \tan \theta}}$$

$$1 + \frac{b \tan \theta}{a \sec \theta - ae} \cdot \frac{b \sec \theta}{a \tan \theta}$$

$$\frac{ab \tan^2 \theta - b \sec \theta (a \sec \theta - ae)}{(a \sec \theta - ae)(a \tan \theta)}$$

$$\tan \alpha = \frac{(a \sec \theta - ae)(a \tan \theta) + b \tan \theta b \sec \theta}{(a \sec \theta - ae)(a \tan \theta)}$$

$$\tan \alpha = \frac{ab \tan^2 \theta - ab \sec^2 \theta + abe \sec \theta}{a^2 \sec \theta \tan \theta - a^2 e \tan \theta + b^2 \tan \theta \sec \theta}$$

$$\tan \alpha = \frac{ab(\tan^2 \theta - \sec^2 \theta) + abe \sec \theta}{(a^2 + b^2) \sec \theta \tan \theta - a^2 e \tan \theta}$$

$$\tan \alpha = \frac{(-1)ab + abe \sec \theta}{a^2 e^2 \tan \theta \sec \theta - a^2 e \tan \theta} \quad \because \text{for hyperbola } a^2 + b^2 = a^2 e^2$$

$$\tan \alpha = \frac{ab(e \sec \theta - 1)}{a^2 e \tan \theta (e \sec \theta - 1)}$$

$$\tan \alpha = \frac{b}{ae \tan \theta}$$

$$\Rightarrow \alpha = \tan^{-1} \left(\frac{b}{ae \tan \theta} \right)$$

Similarly for β , only replace ae by $-ae$ to get-

$$\beta = \tan^{-1} \left(\frac{b}{-ae \tan \theta} \right)$$

$$\Rightarrow \alpha = \beta$$

which is the required.

Q# 27. Find an equation of a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

in the form

$$\frac{x}{a} \cosh \theta - \frac{y}{b} \sinh \theta = 1$$

Show that the product of lengths of perpendiculars on it from the foci is constant.

Soln.

The given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore P(a \cosh \theta, b \sinh \theta)$ pt. lies on it.

$\therefore$ Eq. of tangent at P is given by

$$\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$$

$$\frac{x}{a^2} a \cosh \theta - \frac{y}{b^2} b \sinh \theta = 1$$

$$\Rightarrow \frac{x}{a} \cosh \theta - \frac{y}{b} \sinh \theta = 1 \quad \text{--- (i)}$$

is the required eq. of the tangent.

Now let d_1 and d_2

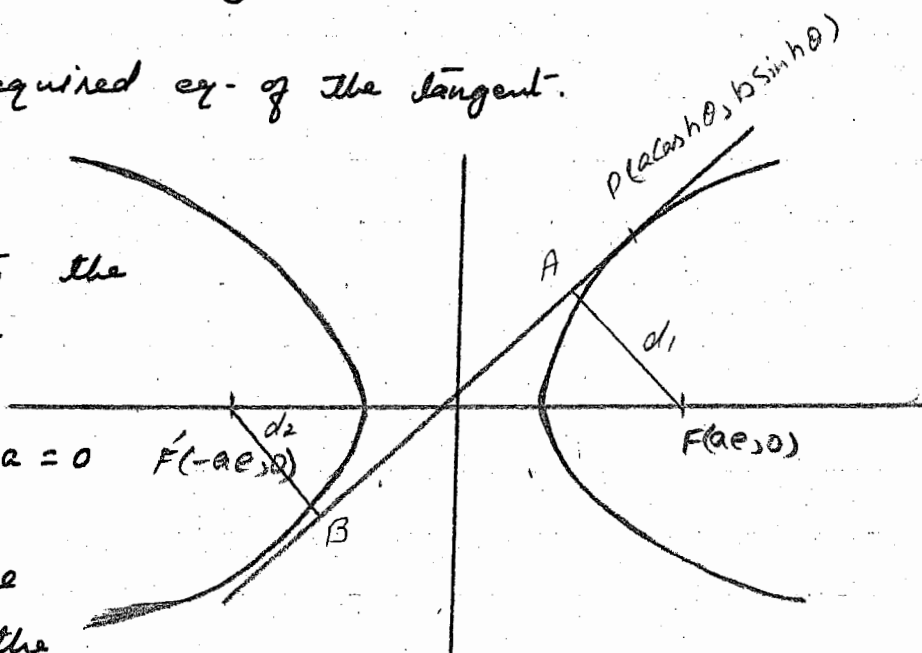
are the perpendicular distances from F, F' to the tangent line, then from (i)

$$b x \cosh \theta - a y \sinh \theta - b a = 0$$

Now

$d_1 = |FA| =$ Distance of $F(ae, 0)$ from the tangent line.

$$d_1 = \frac{|bae \cosh \theta - 0 - ab|}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}}$$



Now $d_2 = |FB| = \text{Distance of } F(-ae, 0) \text{ from the tangent line}$

$$d_2 = \frac{|-bae \cosh \theta - 0 - ab|}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}}$$

Hence $d_1 = \frac{ab(e \cosh \theta - 1)}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}}$

and

$$d_2 = \frac{ab(e \cosh \theta + 1)}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}}$$

Now

$$\begin{aligned} d_1 d_2 &= \frac{ab(e \cosh \theta - 1) \cdot ab(e \cosh \theta + 1)}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta} \cdot \sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}} \\ &= \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta} \Rightarrow \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{b^2 \cosh^2 \theta + a^2 (\cosh^2 \theta - 1)} \\ &= \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{b^2 \cosh^2 \theta + a^2 \cosh^2 \theta - a^2} \Rightarrow \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{(a^2 + b^2) \cosh^2 \theta - a^2} \\ &= \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{a^2 e^2 \cosh^2 \theta - a^2} \\ &= \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{a^2 (e^2 \cosh^2 \theta - 1)} \\ &= b^2 \\ &= \text{constant.} \end{aligned}$$

Is as required.

Q#30. Find an equation of a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the form

$$\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2.$$

Prove that the normal is external bisector of the angle between the focal distances of it's foot.

Soln:

The eq. of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (1)$$

Diff. (i) w.r.t to 'x'

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

$$\left. \frac{dy}{dx} \right|_P = \frac{b^2 a \sec \theta}{a^2 b \tan \theta} = \frac{b}{a \sin \theta}$$

Hence the slope of the normal is $-\frac{a \sin \theta}{b}$ and equation of the normal at P is

$$y - b \tan \theta = -\frac{a \sin \theta}{b} (x - a \sec \theta)$$

$$by - b^2 \tan \theta = -ax \sin \theta + a^2 \frac{\sin \theta}{\cos \theta}$$

$$by - b^2 \tan \theta = -ax \sin \theta + a^2 \tan \theta$$

$$by + ax \sin \theta = a^2 \tan \theta + b^2 \tan \theta$$

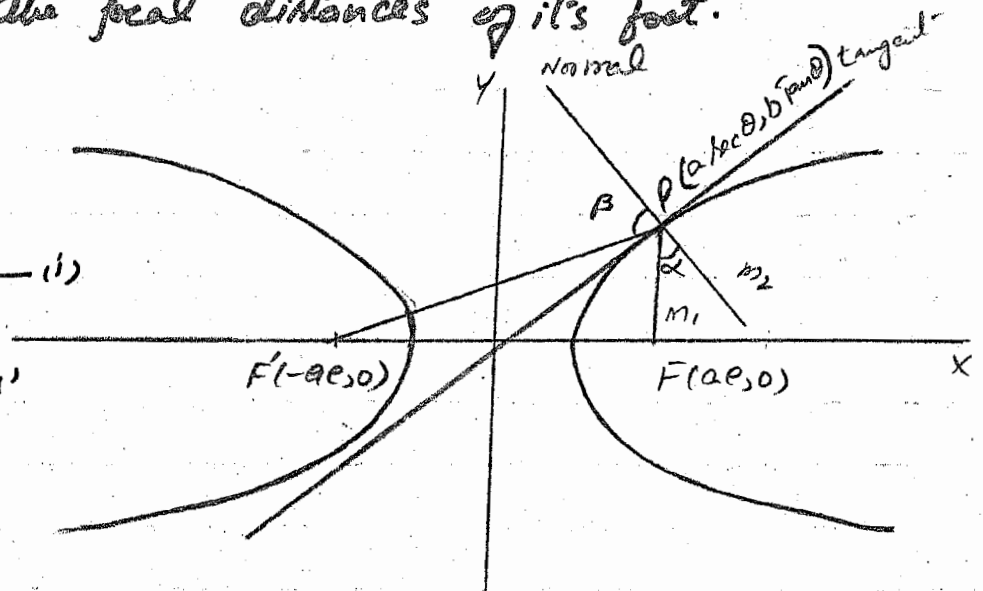
$$by + ax \sin \theta = (a^2 + b^2) \tan \theta$$

$$\frac{by}{\tan \theta} + \frac{ax \sin \theta}{\frac{\sin \theta}{\cos \theta}} = a^2 + b^2$$

$$\frac{by}{\tan \theta} + ax \sin \theta \cdot \frac{\cos \theta}{\sin \theta} = a^2 + b^2$$

$$\frac{by}{\tan \theta} + \frac{ax}{\sec \theta} = a^2 + b^2$$

Is the required eq. of normal.



[Additional Work.]

Question. Define homogeneous equation of Degree 2. Prove that this eq. always represents two straight lines. Find the angle b/w these two straight lines also deduce a condition so that the straight lines are

i) parallel

ii) perpendicular.

Solution.

Def. The eq. of the form

$$ax^2 + 2hxy + by^2 = 0$$

where a, h and b are not simultaneously zero, is called homogeneous eq. of Degree 2.

Now we show that this eq. always represents two straight lines.

$\therefore$ Homogeneous eq. of Degree 2 is

$$ax^2 + 2hxy + by^2 = 0 \quad \text{--- (1)}$$

$$\Rightarrow by^2 + 2hxy + ax^2 = 0$$

$$\div \text{ by } x^2 \quad b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

which is quadratic in $\frac{y}{x}$. So by using Quadratic formula, we have

$$\frac{y}{x} = \frac{-2h \pm \sqrt{4h^2 - 4ba}}{2b}$$

$$\frac{y}{x} = \frac{-2h \pm 2\sqrt{h^2 - ab}}{2b}$$

$$\frac{y}{x} = \frac{-h \pm \sqrt{h^2 - ab}}{b}$$

$$y = \frac{-h \pm \sqrt{h^2 - ab}}{b} \cdot x$$

So the given homogeneous eq. of Degree 2 will represent two straight lines which are given by

$$y = \frac{-h + \sqrt{h^2 - ab}}{b} \cdot x \quad \text{--- (2)}$$

and

$$y = \frac{-h - \sqrt{h^2 - ab}}{b} \cdot x \quad \text{--- (3)}$$

Now from (2) and (3) it is clear that the slopes of the lines are

$$m_1 = \frac{-h + \sqrt{h^2 - ab}}{b}$$

$$m_2 = \frac{-h - \sqrt{h^2 - ab}}{b}$$

if θ is the required angle then we know that-

$$\begin{aligned} \tan \theta &= \frac{|m_1 - m_2|}{1 + m_1 m_2} \\ &= \frac{\left| \frac{-h + \sqrt{h^2 - ab}}{b} - \frac{-h - \sqrt{h^2 - ab}}{b} \right|}{1 + \left(\frac{-h + \sqrt{h^2 - ab}}{b} \right) \left(\frac{-h - \sqrt{h^2 - ab}}{b} \right)} \\ &= \frac{\frac{2\sqrt{h^2 - ab}}{b}}{\frac{-h + \sqrt{h^2 - ab} + h + \sqrt{h^2 - ab}}{b}} \\ &= \frac{2\sqrt{h^2 - ab}}{b^2 + (-h + \sqrt{h^2 - ab})(-h - \sqrt{h^2 - ab})} \\ &= \frac{2\sqrt{h^2 - ab}}{b} \times \frac{b^2}{b^2 + (-h + \sqrt{h^2 - ab})(-h - \sqrt{h^2 - ab})} \\ &= \frac{2b\sqrt{h^2 - ab}}{b^2 + h^2 - (h^2 - ab)} \\ &= \frac{2b\sqrt{h^2 - ab}}{b^2 + h^2 - h^2 + ab} \\ &= \frac{2b\sqrt{h^2 - ab}}{b(a + b)} \\ \tan \theta &= \frac{2\sqrt{h^2 - ab}}{a + b} \quad \text{--- (4)} \end{aligned}$$

$$\text{Hence } \theta = \tan^{-1} \left(\frac{2\sqrt{h^2 - ab}}{a + b} \right)$$

Now if $\theta = 0$

$$\Rightarrow \tan \theta = 0$$

Put in (4)

$$\Rightarrow \frac{2\sqrt{h^2 - ab}}{a+b} = 0$$

$$\sqrt{h^2 - ab} = 0$$

$$h^2 = ab$$

Is the required condition for lines to be parallel.

Now if $\theta = 90^\circ$

$$\Rightarrow \tan 90^\circ = \infty$$

Put in (v)

$$\frac{2\sqrt{h^2 - ab}}{a+b} = \infty$$

$$\Rightarrow a+b=0$$

is the required condition for perpendicularity.

Question. (Theorem 6.15)

Show that the general Eq. of 2nd in x and y always represents a Conic section.

Ans. We know that the general Eq. of 2nd degree in x, y is given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{--- (1)}$$

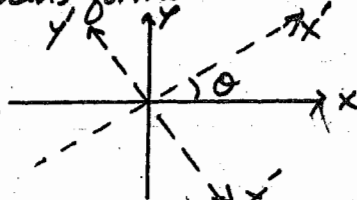
where a, h and b are not simultaneously zero.

if the axis of coordinates rotate through an angle

θ . Then we know that the eqs. of transformations are

$$x = x' \cos \theta - y' \sin \theta$$

$$y = x' \sin \theta + y' \cos \theta$$



So from (1) we have

$$a(x' \cos \theta - y' \sin \theta)^2 + 2h(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) + 2g(x' \cos \theta - y' \sin \theta) + 2f(x' \sin \theta + y' \cos \theta) + c = 0$$

$$a(x'^2 \cos^2 \theta + y'^2 \sin^2 \theta - 2x'y' \cos \theta \sin \theta) + 2h(x'^2 \cos \theta \sin \theta + x'y' \cos^2 \theta - x'y' \sin^2 \theta - y'^2 \sin \theta \cos \theta) + b(x'^2 \sin^2 \theta + y'^2 \cos^2 \theta + 2x'y' \sin \theta \cos \theta) + 2gx' \cos \theta - 2gy' \sin \theta + 2fx' \sin \theta + 2fy' \cos \theta + c = 0$$

$$\begin{aligned}
 & a x'^2 \cos^2 \theta + a y'^2 \sin^2 \theta - 2 x' y' a \cos \theta \sin \theta + 2 h x'^2 \cos \theta \sin \theta + 2 h x' y' \cos^2 \theta - 2 h x' y' \sin^2 \theta \\
 & - 2 h y'^2 \sin \theta \cos \theta + b x'^2 \sin^2 \theta + b y'^2 \cos^2 \theta + 2 b x' y' \sin^2 \theta + b y'^2 \cos^2 \theta + 2 b x' y' \sin \theta \cos \theta \\
 & + 2 g x' \cos \theta - 2 g y' \sin \theta + 2 f x' \sin \theta + 2 f y' \cos \theta + C = 0 \\
 & x'^2 (a \cos^2 \theta + 2 h \cos \theta \sin \theta + b \sin^2 \theta) + 2 x' y' (b \sin \theta \cos \theta - h \sin^2 \theta + h \cos^2 \theta \\
 & - a \sin \theta \cos \theta) + y'^2 (a \sin^2 \theta - 2 h \sin \theta \cos \theta + b \cos^2 \theta) + 2 x' (g \cos \theta + f \sin \theta) \\
 & + 2 y' (f \cos \theta - g \sin \theta) + C = 0
 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
 & x'^2 (a \cos^2 \theta + 2 h \cos \theta \sin \theta + b \sin^2 \theta) + 2 x' y' \{ h (\cos^2 \theta - \sin^2 \theta) - (a-b) \sin \theta \cos \theta \} \\
 & + y'^2 (a \sin^2 \theta - 2 h \sin \theta \cos \theta + b \cos^2 \theta) + 2 x' (g \cos \theta + f \sin \theta) + 2 y' (f \cos \theta \\
 & - g \sin \theta) + C = 0 \quad \text{--- (2)}
 \end{aligned}$$

Now

$$\begin{aligned}
 \text{Put } & A = a \cos^2 \theta + 2 h \cos \theta \sin \theta + b \sin^2 \theta \\
 & H = h (\cos^2 \theta - \sin^2 \theta) - (a-b) \sin \theta \cos \theta \\
 & B = a \sin^2 \theta - 2 h \sin \theta \cos \theta + b \cos^2 \theta \\
 & G = g \cos \theta + f \sin \theta \\
 & F = f \cos \theta - g \sin \theta
 \end{aligned} \quad \left. \vphantom{\begin{aligned} A \\ H \\ B \\ G \\ F \end{aligned}} \right\} \text{--- (3)}$$

So (2) can be written as

$$A x'^2 + 2 H x' y' + B y'^2 + 2 G x' + 2 F y' + C = 0 \quad \text{--- (4)}$$

If we vanish term involving $x' y'$ in (2) then we have

$$\text{to set } \{ h (\cos^2 \theta - \sin^2 \theta) - (a-b) \sin \theta \cos \theta \} = 0$$

x by (2) we have

$$2 h (\cos^2 \theta - \sin^2 \theta) - (a-b) 2 \sin \theta \cos \theta = 0$$

$$2 h (\cos 2 \theta) - (a-b) \sin 2 \theta = 0$$

$$2 h \cos 2 \theta = (a-b) \sin 2 \theta$$

$$\Rightarrow \frac{\sin 2 \theta}{\cos 2 \theta} = \frac{2 h}{a-b}$$

$$\Rightarrow \tan 2 \theta = \frac{2 h}{a-b}$$

And then (4) can be written as

$$A x'^2 + B y'^2 + 2 G x' + 2 F y' + C = 0 \quad \text{--- (5)}$$

$$A x'^2 + 2Gx' + B y'^2 + 2Fy' = -C$$

$$A \left(x'^2 + \frac{2Gx'}{A} \right) + B \left(y'^2 + \frac{2Fy'}{B} \right) = -C$$

$$A \left(x'^2 + \frac{2G}{A} x' + \frac{G^2}{A^2} - \frac{G^2}{A^2} \right) + B \left(y'^2 + \frac{2F}{B} y' + \frac{F^2}{B^2} - \frac{F^2}{B^2} \right) = -C$$

$$A \left[\left(x' + \frac{G}{A} \right)^2 - \frac{G^2}{A^2} \right] + B \left[\left(y' + \frac{F}{B} \right)^2 - \frac{F^2}{B^2} \right] = -C$$

$$A \left(x' + \frac{G}{A} \right)^2 - \frac{G^2}{A} + B \left(y' + \frac{F}{B} \right)^2 - \frac{F^2}{B} = -C$$

$$A \left(x' + \frac{G}{A} \right)^2 + B \left(y' + \frac{F}{B} \right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C \quad \text{--- (6)}$$

Case (1). if $AB \neq 0$

$$\Rightarrow A \neq 0, B \neq 0$$

So from (6), we have

$$A x^2 + B y^2 = C \quad \text{--- (7)}$$

where $X = x' + \frac{G}{A}$, $Y = y' + \frac{F}{B}$

$$C = \frac{G^2}{A} + \frac{F^2}{B} - C$$

(7) $\div$ by C , we have

$$\frac{x^2}{C/A} + \frac{y^2}{C/B} = 1$$

$$\frac{x^2}{(\sqrt{C/A})^2} + \frac{y^2}{(\sqrt{C/B})^2} = 1 \quad \text{--- (8)}$$

Now may discuss the following cases.

Case (a). If C/A and C/B are +ve and also

$C/A = C/B$ then (8) will represent a circle of the form

$$x^2 + y^2 = a^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

Case (b). If $\frac{C}{A}$ and $\frac{C}{B}$ both are +ve and $\frac{C}{A} \neq \frac{C}{B}$ then (8) represent an ellipse.

Case (c). If both $\frac{C}{A}$ and $\frac{C}{B}$ are -ve then (8) will represent an imaginary ellipse.

Case (d). If $\frac{C}{A}$ and $\frac{C}{B}$ are of opposite sign then (8) will represent a hyperbola.

Case (E). If $C = 0$ then (8) can be written as

$$Ax^2 + By^2$$

which is homogeneous equation of Degree 2 will represent two straight line

Case 2. If $AB = 0$ then $A = 0$, $B = 0$ or $A = 0$, $B \neq 0$ or $A \neq 0$, $B = 0$

now we discuss three cases.

Case (A). both A and can not be zero because if so then from (5)

$$0 + 0 + 2Gx' + 2Fy' + C = 0$$

$$2Gx' + 2Fy' + C = 0$$

which is linear and so it is a contradiction.

Case (B). If $A \neq 0$, $B \neq 0$ then (5) can be written as

$$A x'^2 + 0 + 2Gx' + 2Fy' + C = 0$$

$$A \left(x'^2 + \frac{2G}{A} x' \right) = -2Fy' - C$$

$$A \left(x'^2 + \frac{2G}{A} x' + \frac{G^2}{A^2} - \frac{G^2}{A^2} \right) = -2Fy' - C$$

$$A \left[\left(x' + \frac{G}{A} \right)^2 - \frac{G^2}{A^2} \right] = -2Fy' + (-C)$$

$$A \left(x' + \frac{G}{A} \right)^2 - \frac{G^2}{A} = -2Fy' - C$$

$$A \left(x' + \frac{G}{A} \right)^2 = -2Fy' + \frac{G^2}{A} - C$$

$$A \left(x' + \frac{G}{A} \right)^2 = -2Fy' + \frac{G^2 - AC}{A}$$

$$A(x' + \frac{G}{A})^2 = -2F(y' - \frac{G^2 - AC}{2FA})$$

$$(x' + \frac{G}{A})^2 = -\frac{2F}{A}(y' - \frac{G^2 - AC}{2FA})$$

$$(x' + \frac{G}{A})^2 = -\frac{4F}{2A}(y' - \frac{G^2 - AC}{2AF})$$

$$\Rightarrow X^2 = -\frac{4F}{2A} Y$$

where

$$X = x' + \frac{G}{A}, \quad Y = y' - \frac{G^2 - AC}{2AF}$$

which represent a parabola.

Case C).

If $A \neq 0$, $B = 0$ in addition to this F is also equal to zero.

$$\text{So } 5) \Rightarrow Ax'^2 + 2Gx' + C = 0$$

then this will represent two straight lines.

Case 3).

If $A = 0$, $B \neq 0$ then

$$5) \Rightarrow 0 + By'^2 + 2Gx' + 2Fy' + C = 0$$

$$By'^2 + 2Fy' = -2Gx' - C$$

$$B(y'^2 + \frac{2F}{B}y') = -2Gx' - C$$

$$B(y'^2 + \frac{2F}{B}y' + \frac{F^2}{B^2} - \frac{F^2}{B^2}) = -2Gx' - C$$

$$B[(y' + \frac{F}{B})^2 - \frac{F^2}{B^2}] = -2Gx' - C$$

$$B(y' + \frac{F}{B})^2 - \frac{F^2}{B} = -2Gx' - C$$

$$B(y' + \frac{F}{B})^2 = -2Gx' + \frac{F^2}{B} - C$$

$$B(y' + \frac{F}{B})^2 = -2Gx' + \frac{F^2 - BC}{B}$$

$$B(y' + \frac{F}{B})^2 = -2G[x' + \frac{F^2 - BC}{2BG}]$$

$$\Rightarrow (y' + \frac{F}{B})^2 = -\frac{2G}{B}[x' + \frac{F^2 - BC}{2BG}]$$

$$(y' + \frac{F}{B})^2 = -\frac{4G}{2B}[x' + \frac{F^2 - BC}{2BG}]$$

$$\Rightarrow y^2 = -4\frac{G}{2B}x$$

where $Y = y' + \frac{F}{B}$, $X = x' - \frac{F^2 - BC}{2B^2}$
 which represents a parabola.

Case 4. If $A=0$, $B \neq 0$ and $G=0$ then

$$S) \Rightarrow B y'^2 + 2F y' + C = 0$$

Then this represents two straight lines and it is quadratic in y' .

From above discussion whatever the case the general equation of 2nd degree always represent a conic section.

In above article we may discuss the notes.

Note 1.

$$H^2 - AB = 0$$

$$\begin{aligned} h(\cos^2 \theta - \sin^2 \theta) - (a-b) \sin \theta \cos \theta - (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta)(a \sin^2 \theta \\ - 2h \sin \theta \cos \theta + b \cos^2 \theta) \\ = h \cos^2 \theta - h \sin^2 \theta - a \sin \theta \cos \theta + b \sin \theta \cos \theta - a^2 \cos^2 \theta \sin^2 \theta - 2ha \cos^3 \theta \sin \theta \\ - ab \cos^4 \theta - 2ha \cos \theta \sin^3 \theta + 4h^2 \sin^2 \theta \cos^2 \theta - 2hb \cos^3 \theta \sin \theta \\ - ba \sin^4 \theta + 2bh \sin^3 \theta \cos \theta - b^2 \sin^2 \theta \cos^2 \theta \\ = h^2 - ab \end{aligned}$$

Similarly we may show that

$$A+B = a+b$$

these are called invariants of rotation.

Note 2. The invariants of rotation provide a rule to identify the conic which is as follows.

i) If $h^2 - ab > 0$

Then conic will be hyperbola

ii) If $h^2 - ab < 0$

Then conic will be an ellipse

iii) If $h^2 - ab = 0$

Then conic will be a parabola.

Theorem. Consider that $F_1(c, 0)$; $F_2(-c, 0)$ are the foci of an ellipse s.t. the sum of the distances of all points on the ellipse to the foci is $2a$ then prove that equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Proof:- Consider that $P(x, y)$ is a pt. on the ellipse then given that

$$|PF_1| + |PF_2| = 2a \quad \text{--- I}$$

$$|PF_1| = \sqrt{(x-c)^2 + (y-0)^2}$$

$$|PF_2| = \sqrt{(x+c)^2 + (y-0)^2}$$

Put in (i)

$$\sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

$$\sqrt{x^2 + c^2 - 2cx + y^2} + \sqrt{x^2 + c^2 + 2cx + y^2} = 2a$$

$$\sqrt{x^2 + c^2 - 2cx + y^2} = 2a - \sqrt{x^2 + c^2 + 2cx + y^2}$$

$$x^2 + c^2 - 2cx + y^2 = [2a - \sqrt{x^2 + c^2 + 2cx + y^2}]^2$$

$$x^2 + c^2 - 2cx + y^2 = 4a^2 + x^2 + c^2 + 2cx + y^2 - 2a \cdot 2\sqrt{x^2 + c^2 + 2cx + y^2}$$

$$-2cx = 4a^2 + 2cx - 2(2a)\sqrt{x^2 + c^2 + 2cx + y^2}$$

$$-2cx - 4a^2 - 2cx = -4a\sqrt{x^2 + c^2 + 2cx + y^2}$$

$$-4(cx + a^2) = -4a\sqrt{x^2 + c^2 + 2cx + y^2}$$

$$cx + a^2 = a\sqrt{x^2 + c^2 + 2cx + y^2}$$

$$c^2x^2 + a^4 + 2a^2cx = a^2x^2 + a^2c^2 + 2ca^2x + a^2y^2$$

$$a^2x^2 - c^2x^2 + a^2y^2 + a^2c^2 - a^4 = 0$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

$$\div \text{ by } (a^2 - c^2)a^2$$

$\Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore a^2 - c^2 = b^2$$

Theorem. Consider that $F_1(c, 0)$ and $F_2(-c, 0)$ are the foci of a hyperbola s.t. difference between the distances from any arbitrary pt. on this hyperbola to the foci is $2a$. Then prove that eq. of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Proof:

Prove your self.

Questions

Find the equation of the tangent and equation of the normal to the following curves at the given pts.

- i) $x^2 + y^2 = a^2$ at (x_1, y_1)
- ii) $y^2 = 4ax$ at (x_1, y_1)
- iii) $y^2 = 4ax$ at $(at^2, 2at)$
- iv) $x^2 = 4ay$ at (x_1, y_1)
- v) $x^2 = 4ay$ at $(2at, at^2)$
- vi) $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1)
- vii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1)
- viii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \sin \theta, b \cos \theta)$
- ix) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \cos \theta, b \sin \theta)$
- x) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1)
- xi) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$
- xii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \cosh \theta, b \sinh \theta)$

Questions

Find the equation of the tangent and normal at the given pts.

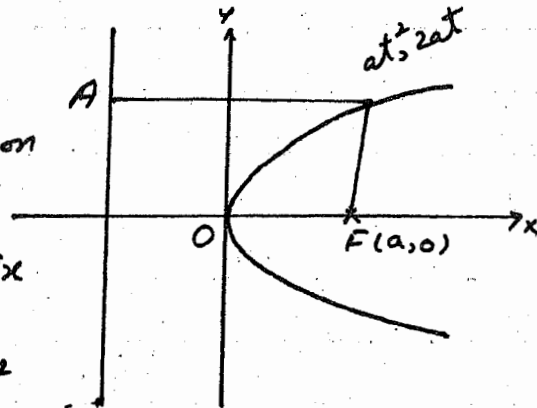
- i) $x = at^2, y = 2at$
- ii) $x = a \cos \theta, y = b \sin \theta$
- iii) $x = a \sec \theta, y = b \tan \theta$
- iv) $x = a \cosh \theta, y = b \sinh \theta$

Theorem.

Prove distance of a pt. P on the parabola from its focus is the same as distance of P from the directrix i.e. $|PF| = |PA|$

Proof:

Let $P(at^2, 2at)$ be a pt. on the parabola $y^2 = 4ax$ having the focus $F(a, 0)$ and directrix $x + a = 0$.



$$\begin{aligned} \text{Then } |PF|^2 &= (at^2 - a)^2 + (2at - 0)^2 \\ &= a^2t^4 + a^2 - 2a^2t^2 + 4a^2t^2 \quad x+a=0 \\ &= a^2t^4 + a^2 + 2a^2t^2 \\ &= (a + at^2)^2 \end{aligned}$$

$$\Rightarrow |PF| = a + at^2 \quad \text{--- (1)}$$

$|PA|$ = Distance of the pt. $P(at^2, 2at)$ from the line $x + a = 0$

$$= \frac{|at^2 + a|}{\sqrt{1^2 + 0^2}}$$

$$\left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

$$|PA| = a + at^2 \quad \text{--- (2)}$$

(1) and (2) $\Rightarrow$

$$|PF| = |PA|.$$

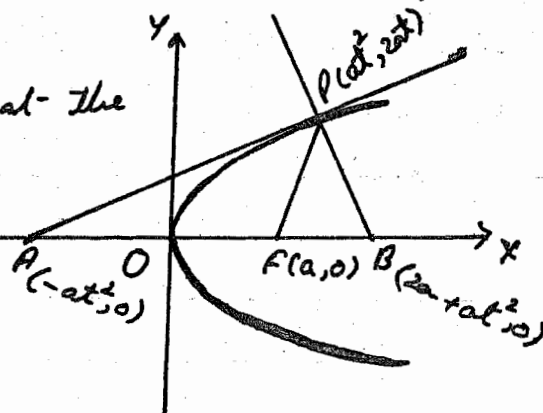
As required.

Theorem. If the tangent and the normal at a pt. P of a parabola meet x -axis at A and B respectively then prove that

$$|FP| = |FA| = |FB| \text{ where } F \text{ is the focus.}$$

Proof:

Let the tangent and normal at the pt. $P(at^2, 2at)$ of the parabola $y^2 = 4ax$ meet x -axis at A and B resp. $F(a, 0)$ is the focus of the parabola.



We know that equation of the tangent at P is

$$x - ty + at^2 = 0$$

(see exp. 8)

For the coordinates of A put $y = 0$ then

$$x - 0 + at^2 = 0$$

$$x = -at^2$$

$\Rightarrow$ Coordinates of A $(-at^2, 0)$

Similarly equation of the normal at P is

$$tx + y - 2at - at^3 = 0$$

For the coordinates of B put $y = 0$, then

$$tx + 0 - 2at - at^3 = 0$$

$$tx = 2at + at^3$$

$$x = 2a + at^2$$

$\Rightarrow$ Coordinates of B are $(2a + at^2, 0)$

Now

$$|FP| = \sqrt{(at^2 - a)^2 + (2at - 0)^2}$$

$$= a + at^2 \quad \text{--- (1)}$$

$$|FA| = \sqrt{(a + at^2)^2 + (0 - 0)^2}$$

$$= a + at^2 \quad \text{--- (2)}$$

$$|FB| = \sqrt{(2a + at^2 - a)^2 + (0 - 0)^2}$$

$$= \sqrt{(a + at^2)^2}$$

$$= a + at^2 \quad \text{--- (3)}$$

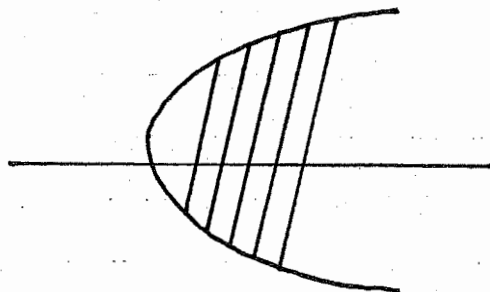
(1), (2) and (3) $\Rightarrow$

$$|FP| = |FA| = |FB|$$

as required.

Diameter of a Parabola.

The locus of the middle pts. of parallel chords of a parabola is called the Diameter of the parabola.



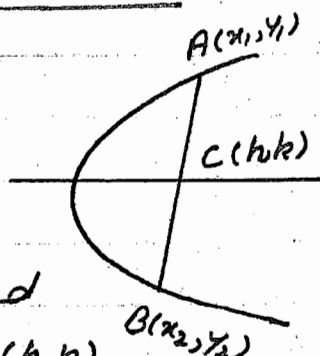
Equation of the Diameter of the Parabola.

Consider the parabola

$$y^2 = 4ax \quad (1)$$

Let AB be one of the parallel chords.

We suppose that the coordinates of A and B are (x_1, y_1) and (x_2, y_2) resp. Let $C(h, k)$ be the mid point of this chord.



Consider that the equation of this chord is

$$y = mx + c$$

$$\Rightarrow mx = y - c$$

$$x = \frac{y - c}{m} \quad (2)$$

Put in (1)

$$y^2 = 4a \cdot \frac{y - c}{m}$$

$$my^2 = 4ay - 4ac$$

$$my^2 - 4ay - 4ac = 0$$

which is quadratic in y . Thus if y_1 and y_2 are the roots of this eq. then

$$y_1 + y_2 = -\frac{-4a}{m}$$

$$y_1 + y_2 = \frac{4a}{m}$$

$$\frac{y_1 + y_2}{2} = \frac{2a}{m} \quad (3)$$

$$\left| \begin{array}{l} ax^2 + bx + c = 0 \\ x_1 + x_2 = -\frac{b}{a} \end{array} \right.$$

$C(h, k)$ is the mid pt. of AB

$$\therefore h = \frac{x_1 + x_2}{2} \text{ and } k = \frac{y_1 + y_2}{2}$$

Put in (3)

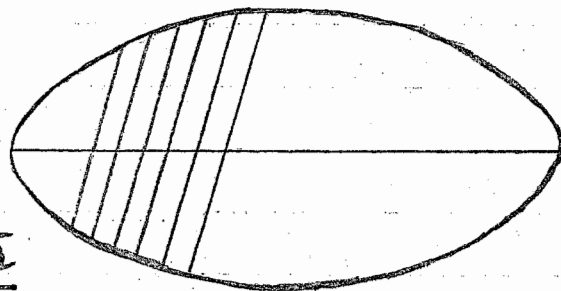
$$k = \frac{2a}{m}$$

Hence equation of the diameter

$$y = \frac{2a}{m}$$

Diameter of an ellipse :-

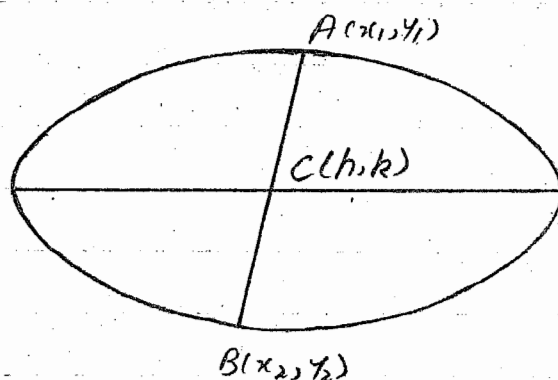
The locus of the middle pts. of parallel chords of an ellipse is called diameter of the ellipse.



Equation of the Diameter of an Ellipse :-

Consider that $\overline{AB}$ is one of the parallel chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ——— (1)

Let A and B has coordinates (x_1, y_1) and (x_2, y_2) resp.



$C(h, k)$ is the mid point of $\overline{AB}$. Then $h = \frac{x_1 + x_2}{2}$ and $k = \frac{y_1 + y_2}{2}$
we consider that eq. of the chord AB is ——— (2)

$$y = mx + C \quad \text{——— (3)}$$

Put in (1)

$$\begin{aligned} \frac{x^2}{a^2} + \frac{(mx + C)^2}{b^2} &= 1 \\ b^2 x^2 + a^2 (mx + C)^2 &= a^2 b^2 \\ b^2 x^2 + a^2 m^2 x^2 + a^2 C^2 + 2a^2 cmx - a^2 b^2 &= 0 \\ (b^2 + a^2 m^2) x^2 + 2mca^2 x + a^2 C^2 - a^2 b^2 &= 0 \end{aligned}$$

which is quadratic in x .

$\therefore$ If x_1 and x_2 are the roots of the equation then

$$x_1 + x_2 = - \frac{2mca^2}{b^2 + a^2 m^2}$$

$$\frac{x_1 + x_2}{2} = - \frac{mca^2}{b^2 + a^2 m^2}$$

$$\Rightarrow h = - \frac{mca^2}{b^2 + a^2 m^2} \quad \text{——— (4)}$$

$\therefore C(h, k)$ lies on AB whose equation is

$$y = mx + C$$

Then

$$K = mh + C$$

$$K - mh = C$$

Put this value of C in (4)

$$\Rightarrow h = - \frac{ma^2(K - mh)}{b^2 + a^2m^2}$$

$$h(b^2 + a^2m^2) = -ma^2(K - mh)$$

$$hb^2 + h^2a^2m^2 = -ma^2K + m^2a^2h$$

$$hb^2 = -ma^2K$$

$$K = - \frac{hb^2}{ma^2}$$

Thus the equation of the diameter is

$$y = \frac{-b^2}{ma^2} x$$

Theorem.

1) Prove that distance of a pt. P on an ellipse from the focus = e times its distance from the corresponding directrix.

2) Also prove that

$$|PF| + |PF'| = \text{Constant}$$

Proof:-

1) Let the pt. $P(a \cos \theta, b \sin \theta)$ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

having the focus $(ae, 0)$

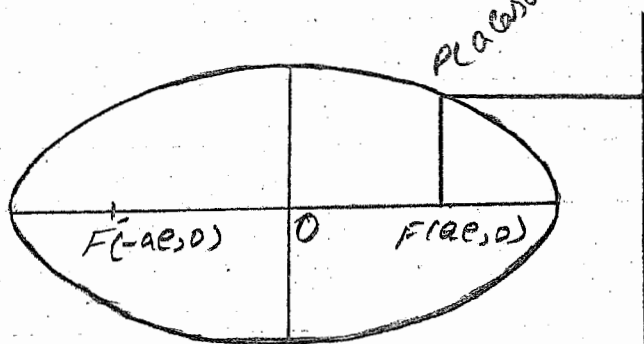
and $F'(-ae, 0)$. Centre at

$O(0, 0)$ and $x = \pm \frac{a}{e}$

its directrix.

$$x = -\frac{a}{e}$$

$$x = \frac{a}{e}$$



$$\text{Now } |PF|^2 = (ae - a \cos \theta)^2 + (0 - b \sin \theta)^2$$

$$= a^2e^2 + a^2 \cos^2 \theta - 2ae \cos \theta + b^2 \sin^2 \theta$$

$$= a^2 \cos^2 \theta + a^2e^2 - 2ae \cos \theta + b^2(1 - \cos^2 \theta)$$

$$= a^2 \cos^2 \theta + a^2e^2 - 2ae \cos \theta + b^2 - b^2 \cos^2 \theta$$

$$= (a^2 - b^2) \cos^2 \theta + b^2 + a^2e^2 - 2ae \cos \theta \quad (2)$$

$$= (a^2 - b^2) \cos^2 \theta + a^2 - 2a^2e \cos \theta$$

$$= a^2e^2 \cos^2 \theta + a^2 - 2a^2e \cos \theta$$

$$= a^2(e^2 \cos^2 \theta + 1 - 2e \cos \theta)$$

$$= a^2 - b^2 = a^2e^2$$

$$a^2 = b^2 + e^2a^2$$

$$|PF|^2 = a^2 (e \cos \theta - 1)^2$$

$$\Rightarrow |PF| = a (e \cos \theta - 1) \quad \text{--- (3)}$$

Now equation of the directrix is

$$x = \frac{a}{e}$$

$$ex = a$$

$$ex - a = 0 \quad \text{--- (4)}$$

Now $|PA|$ = Distance of the pt. P from the directrix.

$$|PA| = \frac{|e a \cos \theta - a|}{\sqrt{e^2 + 0^2}}$$

$$|PA| = \frac{a (e \cos \theta - 1)}{e}$$

$$e|PA| = a (e \cos \theta - 1)$$

$$e|PA| = |PF|$$

by (3)

$$\Rightarrow |PF| = e|PA|$$

Hence distance of pt. P on an ellipse from the focus = e times its distance from the corresponding directrix.

2).

$$\therefore |PF|^2 = a^2 (e \cos \theta - 1)^2 \quad \text{--- (5)}$$

$$\Rightarrow |PF|^2 = a^2 (1 - e \cos \theta)^2$$

$$|PF| = a (1 - e \cos \theta) \quad \text{--- (6)}$$

$$\text{Similarly } |PF'| = a (1 + e \cos \theta) \quad \text{--- (7)}$$

$$(7) + (6) \Rightarrow$$

$$|PF| + |PF'| = a (1 - e \cos \theta) + a (1 + e \cos \theta)$$

$$= a (1 - e \cos \theta + 1 + e \cos \theta)$$

$$= 2a$$

$$= \text{Centr.}$$

$$= \text{Length of the major axis of the ellipse.}$$

Question.

Find the locus of the intersection of normals to the parabola $y^2 = 4ax$ inclined at right angle to each other.

Solution.

The equation of the parabola is

$$y^2 = 4ax \quad \text{--- (1)}$$

Then the equation of the normal is

$$y = mx - 2am - am^3$$

if the pt. $P(x_1, y_1)$ lies on it then

$$y_1 = mx_1 - 2am - am^3$$

$$\Rightarrow am^3 + am^2 + (2a - x_1)m + y_1 = 0 \quad \text{--- (2)}$$

Now if m_1, m_2 and m_3 are the roots then

$$m_1 m_2 m_3 = (-1)^3 \frac{y_1}{a} \quad \text{--- (3)}$$

Since given that two of the normals are $\perp$

$$\therefore m_1 m_2 = -1 \quad \text{put in (3)}$$

$$-m_3 = -\frac{y_1}{a}$$

$$\Rightarrow m_3 = \frac{y_1}{a}$$

But m_3 is the root of (2)

$$\therefore a \cdot \frac{y_1^3}{a^3} + (2a - x_1) \frac{y_1}{a} + y_1 = 0$$

$$\frac{y_1^3}{a^2} + (2a - x_1) \frac{y_1}{a} + y_1 = 0$$

$$\frac{y_1^3}{a^2} + (2a - x_1) \frac{y_1}{a} + y_1 = 0$$

$$y_1^3 + 2a^2 y_1 - ax_1 y_1 + a^2 y_1 = 0$$

$$y_1^3 + 3a^2 y_1 - ax_1 y_1 = 0$$

Thus the locus of the pt. (x_1, y_1) is

$$y^3 + 3a^2 y - axy = 0$$

$$y^3 + 3a^2 y - axy = 0$$

$$y(y^2 + 3a^2 - ax) = 0$$

$$\Rightarrow y = 0, \quad y^2 + 3a^2 - ax = 0$$

$$\Rightarrow y = 0, \quad y^2 = ax - 3a^2$$

$$\Rightarrow y = 0, \quad y^2 = a(x - 3a)$$

as required.

$$\text{if } ax^3 + bx^2 + (x+d) = 0$$

and x_1, x_2, x_3 be the roots, then

$$x_1 + x_2 + x_3 = (-1) \frac{b}{a}$$

$$x_1 x_2 + x_1 x_3 + x_2 x_3 = (-1)^2 \frac{c}{a}$$

$$x_1 x_2 x_3 = (-1)^3 \frac{d}{a}$$

Question. Show that the tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.

Solution.

Consider that the focal chord AB of the parabola $y^2 = 4ax$ — (1)

We know that the coordinates of the extremities are as shown.

Diff. (1), we have

$$2y \frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y} \quad \text{--- (2)}$$

Now

$$m_1 = \text{slope of the tangent at A}$$

$$m_1 = \left. \frac{dy}{dx} \right|_{(at^2, 2at)} = \frac{2a}{2at}$$

$$m_1 = \frac{1}{t} \quad \text{--- (3)}$$

Now $m_2 = \text{slope of the tangent at B}$

$$m_2 = \left. \frac{dy}{dx} \right|_{\left(\frac{a}{t^2}, -\frac{2a}{t}\right)} = \frac{2a}{-\frac{2a}{t}}$$

$$m_2 = -t \quad \text{--- (4)}$$

$$\text{let } m_3 = \text{slope of the normal at B}$$

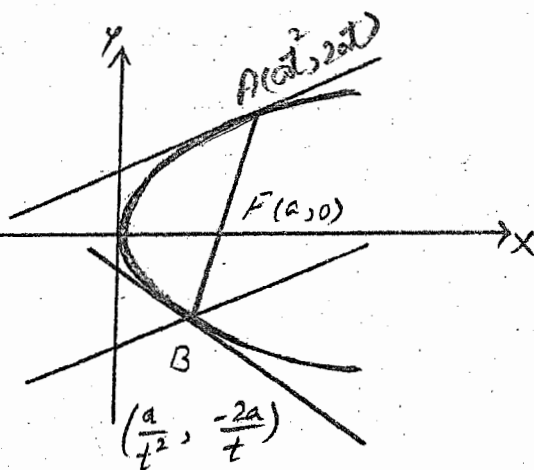
$$= -\frac{1}{m_2}$$

$$= -\frac{1}{-t} = \frac{1}{t} \quad \text{--- (5)}$$

From (3) and (5)

$$m_1 = m_3$$

$\Rightarrow$ The tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.



Question.

Prove that for tangent to a parabola intersect on the directrix and the chord joining the pts. of contact passes through the focus.

Solution.

We know that the equation of the tangent to the parabola

$$y^2 = 4ax \text{ is}$$

$$ty = x + at^2$$

$$x - ty = at^2$$

$$x - ty - at^2 = 0 \quad (1)$$

$$m_1 = \text{slope of the tangent} = - \frac{\text{C-eff. of } x}{\text{C-eff. of } y}$$

$$= - \frac{1}{-t}$$

$$= \frac{1}{t}$$

$$\text{Now } m_2 = \text{slope of the tangent to (1)} \\ = -t$$

and so the equation of this tangent is

$$x + \frac{1}{t}y + \frac{a}{t^2} = 0 \quad (\text{replace by } -\frac{1}{t})$$

$$t^2x + ty + a = 0 \quad (2)$$

(1)+(2)

$$x - ty + at^2 = 0$$

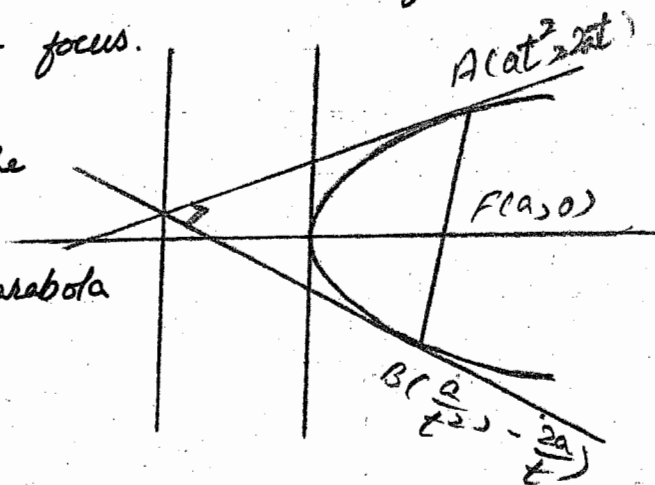
$$t^2x + ty + a = 0$$

$$\hline x(1+t^2) + a(1+t^2) = 0$$

$$\Rightarrow (1+t^2)(x+a) = 0$$

$\Rightarrow x+a=0$ which is the equation of the directrix.

Hence the for tangent to a parabola intersect on directrix.



Now points of contact are

$$A(at^2, 2at) \text{ and } B\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$$

Eq. of the line passing through A and B is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$y-y_1 = \frac{y_2-y_1}{x_2-x_1} (x-x_1)$$

Putting the values

$$y-2at = \frac{-\frac{2a}{t} - 2at}{\frac{a}{t^2} - at^2} (x-at^2)$$

$$y-2at = \frac{\frac{-2a - 2at^2}{t}}{\frac{1 - at^4}{t^2}} (x-at^2)$$

$$\Rightarrow y-2at = \frac{-2a(1+t^2)}{t} \cdot \frac{t^2}{a(1-t^4)} (x-at^2)$$

$$y-2at = \frac{-2(1+t^2)t}{(1-t^2)(1+t^2)} (x-at^2)$$

$$y-2at = \frac{-2t}{1-t^2} (x-at^2) \quad \text{--- (4)}$$

Now the focus is $(a, 0)$ put in (4)

$$0-2at = -\frac{2t}{1-t^2} (a-at^2)$$

$$-2at = -\frac{2t}{1-t^2} a(1-t^2)$$

$$-2at = -2at$$

$\Rightarrow$ The chord joining A and B passes through the focus.

Question. If the tangent at P of a parabola meets the directrix at K, then prove that PFK is right angle, where F is the focus.

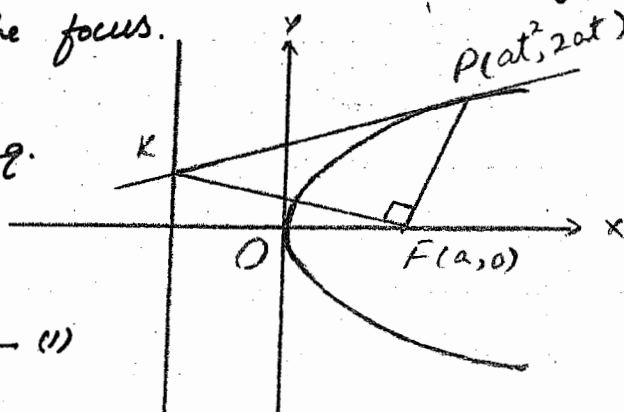
Solution

We know that the eq. of the tangent to the parabola at P is

$$ty = x + at^2 \quad \text{--- (1)}$$

but at K, $x = -a$

put in (1)



$$x+a=0$$

$$-a - ty + at^2 = 0$$

$$yt = at^2 - a$$

$$y = \frac{a(t^2 - 1)}{t} \Rightarrow \text{the coordinate of the}$$

Point K.

$$\left(-a, \frac{at^2 - a}{t}\right)$$

Now $m_1 = \text{slope of } \overline{PF} = \frac{0 - \frac{at^2 - a}{t}}{a - at^2} = \frac{-\frac{at^2 - a}{t}}{a - at^2}$

also

$$m_2 = \text{slope of } \overline{FK} = \frac{\frac{at^2 - a}{t} - 0}{-a - a}$$

$$= \frac{at^2 - a}{-2at}$$

$$= \frac{-a + at^2}{-2at} = \frac{a - at^2}{2at}$$

Consider

$$m_1 \cdot m_2 = \cancel{\frac{at^2 - a}{t}} \cdot \frac{-2at}{a - at^2} \times \frac{a - at^2}{2at}$$

$$= -1$$

$$\Rightarrow \overline{PF} \perp \overline{FK}$$

$$\angle PFK = 90^\circ$$

as required.

Question. If I is the tangent line at $P(x_1, y_1)$ of a parabola $y^2 = 4ax$ and if K is a line through P $\parallel$ to x -axis. Show that the measure of the angle b/w K and I is equal to the measure of the angle between I and $\overline{PF}$.

Solution.

Consider that the tangent at the pt. (x_1, y_1) of the parabola

$$y^2 = 4ax \text{ meet } x\text{-axis}$$

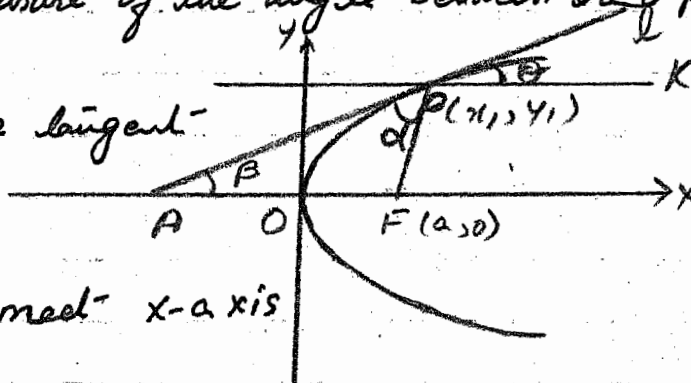
at A . Now

$$|PF| = |FA|$$

($\because$ Reflective prop.)

$$\Rightarrow \triangle AFP \text{ is isosceles}$$

$$\Rightarrow \alpha = \beta \quad (1)$$



Also by elementary geometry

$$\beta = 0$$

Put in (1) $\Rightarrow$

$$\alpha = 0$$

as required

Question.

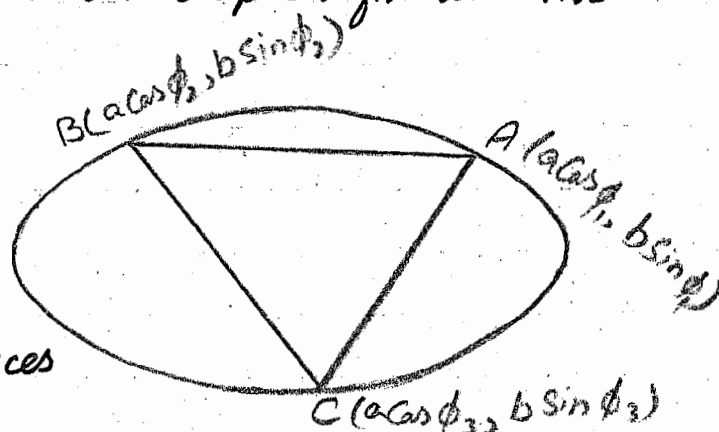
If ϕ_1, ϕ_2 and ϕ_3 are the eccentric angles of vertices of a Δ inscribed in an ellipse, find the Area.

Solution.

Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If A, B, C are the vertices of the said triangle, the eccentric angles of the vertices are given.



$\Rightarrow$ The coordinates of these values will be as shown.

$\therefore$ Area of ΔABC is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} a \cos \phi_1 & b \sin \phi_1 & 1 \\ a \cos \phi_2 & b \sin \phi_2 & 1 \\ a \cos \phi_3 & b \sin \phi_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left[a \cos \phi_1 \{ b \sin \phi_2 - b \sin \phi_3 \} - b \sin \phi_1 \{ a \cos \phi_2 - a \cos \phi_3 \} + 1 \{ ab \cos \phi_2 \sin \phi_3 - ab \sin \phi_2 \cos \phi_3 \} \right]$$

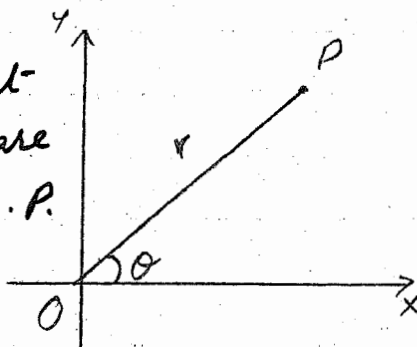
Simplify to get result.

[Additional work is over]

Exercise 6.3

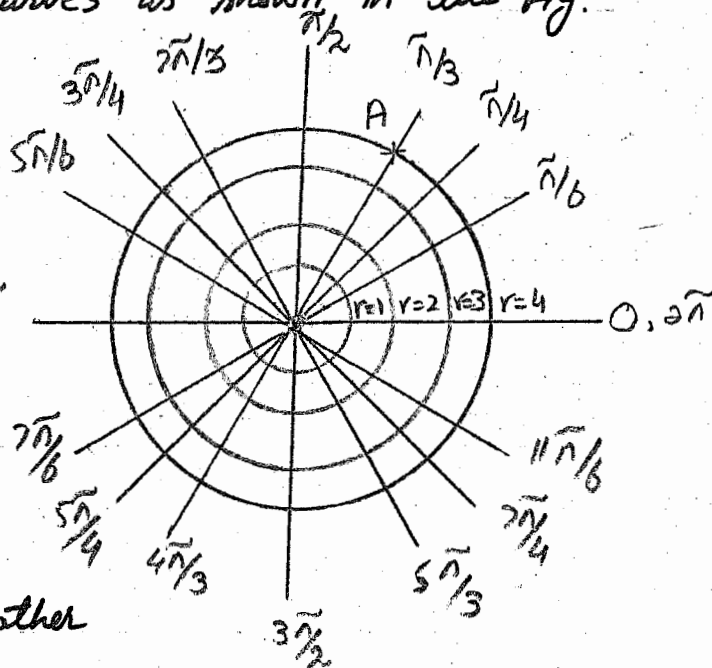
Polar Coordinates.

If P is a point, such that $|OP| = r$ and $\angle POX = \theta$ then (r, θ) are called Polar coordinates of the pt. P . The curves in (r, θ) are called the Polar curves.



To draw a graph of polar curves, we select a pt. O of the plane, which is called the pole then we draw the family of curves as shown in the Fig. and also a family of straight lines through the pole as in the Fig.

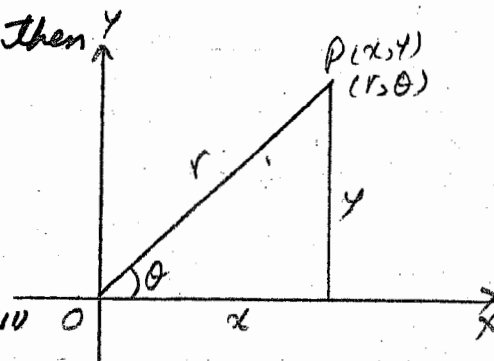
Now if a pt. A has $\tilde{r}$ the Polar coordinates, $(4, \tilde{r}_3)$, then the position of this pt. A is shown in the Fig. Similarly all other pts. can be drawn.



Relation between Polar and Cartesian Coordinates:-

Consider the pt. P whose cartesian coordinates are (x, y) and polar coordinates are (r, θ) , then
 $|OA| = x$; $|AP| = y$; $|OP| = r$ and
 $\angle AOP = \theta$

Then $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$
 $\Rightarrow x = r \cos \theta$ and $y = r \sin \theta$
(i) $x^2 + y^2 = r^2$ — (i)



$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$\Rightarrow r = \sqrt{x^2 + y^2} \quad \text{--- (ii)}$$

$$(ii) \div (i) \Rightarrow$$

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta$$

$$\Rightarrow \theta = \tan^{-1} \frac{y}{x} \quad \text{--- (iv)}$$

(i), (ii) gives cartesian in form of polar and (iii), (iv) polar in form of cartesian.

Note:-

$$\sin \theta = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}$$

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}$$

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Written by

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Exercise 6.3.

In Problems 1-8, express the given equations in rectangular coordinates.

Question 1.

Soln.

$$r^2 = a^2 \sin 2\theta \quad \text{--- (1)}$$

$$r^2 = 2a^2 \sin \theta \cos \theta \quad \text{--- (2)}$$

$$\therefore r^2 = x^2 + y^2, \quad \sin \theta = \frac{y}{r} \text{ and } \cos \theta = \frac{x}{r}$$

$\therefore$ (2) becomes

$$r^2 = 2a^2 \frac{y}{r} \cdot \frac{x}{r}$$

$$r^4 = 2a^2 xy$$

$$(x^2 + y^2)^2 = 2a^2 xy$$

Question 2.

Soln.

$$r^4 \sin 4\theta = a^4$$

$$r^4 = \frac{a^4}{\sin 4\theta}$$

$$r^4 = \frac{a^4}{2 \sin 2\theta \cos 2\theta}$$

$$r^4 = \frac{a^4}{4 \sin \theta \cos \theta (\cos^2 \theta - \sin^2 \theta)} \quad \text{--- (1)}$$

$$\therefore \sin \theta = \frac{y}{r} \text{ and } \cos \theta = \frac{x}{r}$$

$$\therefore (1) \Rightarrow = \frac{a^4}{4 \left(\frac{y}{r}\right) \left(\frac{x}{r}\right) \left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)}$$

$$= \frac{a^4}{\frac{4xy}{r^2} \left(\frac{x^2 - y^2}{r^2}\right)}$$

$$r^4 = \frac{a^4 r^4}{4xy(x^2 - y^2)}$$

$$\Rightarrow 4xy(x^2 - y^2) = a^4$$

Question 3.

$$r^2 = a^2 \cos 2\theta$$

Soln.

$$r^2 = a^2 (\cos^2 \theta - \sin^2 \theta)$$

$$r^2 = a^2 \left(\frac{x^2}{r^2} - \frac{y^2}{r^2} \right)$$

$$r^2 = a^2 \left(\frac{r^2 - y^2}{r^2} \right)$$

$$r^4 = a^2 (x^2 - y^2)$$

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2)$$

$$\therefore \frac{x}{r} = \cos \theta, \frac{y}{r} = \sin \theta$$

$$\therefore r^2 = x^2 + y^2$$

Question 4.

$$r = 2a \sin \theta \sec \theta$$

Soln.

$$r = 2a \frac{y}{r} \cdot \frac{r}{x}$$

$$\therefore \sin \theta = \frac{y}{r}, \sec \theta = \frac{r}{x}$$

$$r^2 = \frac{2a y^2}{x}$$

$$x(x^2 + y^2) = 2a y^2$$

Question 5.

$$r = 1 - \cos \theta$$

Soln.

$$r = 1 - \frac{x}{r}$$

$$\therefore \cos \theta = \frac{x}{r}$$

$$r = \frac{r - x}{r}$$

$$r^2 = r - x$$

$\Rightarrow$

$$x^2 + y^2 = \sqrt{x^2 + y^2} - x$$

$$\therefore r^2 = x^2 + y^2$$

$$x^2 + y^2 + x = \sqrt{x^2 + y^2}$$

$$\Rightarrow r = \sqrt{x^2 + y^2}$$

$$(x^2 + y^2 + x)^2 = x^2 + y^2$$

Question 6.

$$r^2 (4 \sin^2 \theta - 9 \cos^2 \theta) = 36$$

Soln.

$$r^2 \left(4 \frac{y^2}{r^2} - 9 \frac{x^2}{r^2} \right) = 36$$

$$r^2 (4y^2 - 9x^2) = 36$$

$$4y^2 - 9x^2 = 36$$

$$\frac{4y^2}{36} - \frac{9x^2}{36} = 1$$

$$\Rightarrow \frac{y^2}{9} - \frac{x^2}{4} = 1$$

Question 7.

Soln.

$$r = \frac{8}{2 - \cos \theta}$$

$$r = \frac{8}{2 - \frac{x}{r}} = \frac{8}{\frac{2r - x}{r}}$$

$$r = \frac{8r}{2r - x} \Rightarrow 1 = \frac{8}{2r - x}$$

$$\Rightarrow 2r - x = 8$$

$$\Rightarrow 2r = x + 8$$

$$2\sqrt{x^2+y^2} = x + 8$$

$$(2)^2(x^2+y^2) = x^2 + 64 + 16x$$

$$4x^2 + 4y^2 = x^2 + 64 + 16x$$

$$\Rightarrow 3x^2 + 4y^2 - 16x - 64 = 0$$

$$\therefore r = \sqrt{x^2+y^2}$$

$$r = 2\sin\theta + 3\cos\theta$$

$$r = 2\frac{y}{r} + 3\frac{x}{r}$$

$$\therefore \sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\Rightarrow r^2 = 2y + 3x$$

$$\Rightarrow x^2 + y^2 = 2y + 3x$$

$$\therefore r^2 = x^2 + y^2$$

Transform the given equations in polar coordinates. (9-15)

Question 9.

$$xy = a \quad \text{--- (1)}$$

Soln.

$$\therefore x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\therefore (1) \Rightarrow r\cos\theta \cdot r\sin\theta = a$$

$$r^2 \sin\theta \cos\theta = a$$

$$\Rightarrow r^2 = \frac{a}{\sin\theta \cos\theta}$$

$$\Rightarrow r^2 = a \operatorname{cosec}\theta \sec\theta$$

Question 10

$$y^2 = 4x$$

Soln.

$$r^2 \sin^2\theta = 4r\cos\theta$$

$$r(r\sin^2\theta - 4\cos\theta) = 0$$

$$r\sin^2\theta - 4\cos\theta = 0$$

$$\therefore y = r\sin\theta$$

$$x = r\cos\theta$$

Question 11

$$y = \frac{x}{x+1}$$

$$r\sin\theta = \frac{r\cos\theta}{r\cos\theta + 1}$$

$$r\sin\theta(r\cos\theta + 1) = r\cos\theta$$

$$\sin\theta(r\cos\theta + 1) = \cos\theta$$

$$r\sin\theta\cos\theta + \sin\theta = \cos\theta$$

$$r\sin\theta\cos\theta = \cos\theta - \sin\theta$$

$$r = \frac{\cos\theta - \sin\theta}{\sin\theta\cos\theta}$$

$$\Rightarrow r = \frac{1}{\sin \theta} - \frac{1}{\cos \theta}$$

$$r = \operatorname{cosec} \theta - \sec \theta.$$

Question 12.

$$x^2 + y^2 - 8x + 6y + 7 = 0 \quad \text{--- (1)}$$

Soln.

$$\therefore r^2 = x^2 + y^2, \quad x = r \cos \theta, \quad y = r \sin \theta$$

$$\therefore (1) \Rightarrow r^2 - 8r \cos \theta + 6r \sin \theta + 7 = 0$$

$$r^2 - 2r(4 \cos \theta + 3 \sin \theta) + 7 = 0$$

Question 13.

$$(x^2 + y^2) y^2 = a^2 x^2$$

Soln.

$$r^2 \cdot r^2 \sin^2 \theta = a^2 r^2 \cos^2 \theta$$

$$\therefore x^2 + y^2 = r^2$$

$$r^2 \sin^2 \theta = a^2 \cos^2 \theta$$

$$y = r \sin \theta$$

$$r^2 = \frac{a^2 \cos^2 \theta}{\sin^2 \theta}$$

$$x = r \cos \theta.$$

$$\Rightarrow r^2 = a^2 \cot^2 \theta$$

$$\Rightarrow r = a \cot \theta$$

Question 14.

$$x^3 + 4x^2 + xy^2 - 4y^3 = 0$$

Soln.

$$r^3 \cos^3 \theta + 4r^2 \cos^2 \theta + r \cos \theta \cdot r^2 \sin^2 \theta - 4r^3 \sin^3 \theta = 0$$

$$r^2(r \cos^3 \theta + r \cos \theta \sin^2 \theta + 4 \cos^2 \theta - 4 \sin^3 \theta) = 0$$

$$r \cos^3 \theta + 4(\cos^2 \theta - \sin^2 \theta) + r \sin^2 \theta \cos \theta = 0$$

$$r \cos^3 \theta + 4 \cos^2 \theta + r \sin^2 \theta \cos \theta = 0$$

Question 15.

$$x^4 + 2x^2y^2 + y^4 - 6x^2y + 2y^3 = 0$$

Soln.

$$(x^2 + y^2)^2 - 2y(3x^2 + y^2) = 0$$

$$r^4 - 2r \sin \theta (3r^2 \cos^2 \theta + r^2 \sin^2 \theta) = 0$$

$$r^4 - 2r^3 \sin \theta (3 \cos^2 \theta + \sin^2 \theta) = 0$$

$$r - 2 \sin \theta (3 \cos^2 \theta + \sin^2 \theta) = 0$$

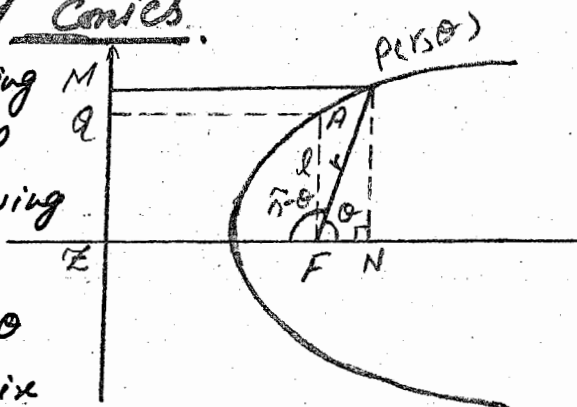
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Polar Equations of Conics.

Consider the conic having M
the focus at the pole. let P Q
be a pt. on the conic having
the coordinates (r, θ) $\mathbb{R}$


$$\Rightarrow |FP| = r \text{ and } P\hat{F}N = 0$$

And ZM be the Directrix

of the conic and $|FA| = \ell$ length of semi latus rectum.
and $|PM|$ is the length of the perpendicular from P on
the directrix.

By definition of a conic

$$\frac{|PF|}{|PM|} = e \quad |PF| = e |PM|$$

$$\begin{aligned} r &= e |Z_N| \\ &= e (|Z_F| + |F_N|) \\ &= e |Z_F| + e |F_N| \end{aligned} \quad \text{--- (1)}$$

Also we have

$$\Rightarrow \frac{|AF|}{|AQ|} |AF| = e |AQ|$$

$$|AF|^2 = e |ZF| \quad \text{--- (2)}$$

$$\therefore |AF| = 2$$

$$Q = e |ZF| \quad \text{--- (3)}$$

(3) in (1) $\Rightarrow$

$$r = \frac{I}{\epsilon_0 E} = \frac{I}{\epsilon_0 \frac{Q}{A}} = \frac{I A}{\epsilon_0 Q} \quad \text{--- (4)}$$

Now from r.t. ΔPNF

$$\frac{|F_N|}{r} = \cos \theta$$

$$|F_N| = r \cos \theta \quad \text{--- (5)}$$

(5) in (4) $\Rightarrow$

$$r = l + r \cos \theta$$

$$r - r \cos \theta = l$$

$$r(1 - e^{G \sin \theta}) = l$$

$$r = \frac{l}{1 - e \cos \theta}$$

or $1 - e \cos \theta = \frac{l}{r}$

which is called polar equations of Conics.

This represents parabola, ellipse or hyperbola according as $e = 1$, $e < 1$ or $e > 1$

Deduction.

In the equation $\frac{l}{r} = 1 - e \cos \theta$ we have taken $\vec{FN}$ to be the positive direction of the initial line. If we regard $\vec{FZ}$ to be the positive direction of the initial line, then eq. of the Conic is given by

$$\begin{aligned} \frac{l}{r} &= 1 - e \cos(\bar{n} - \theta) \\ \therefore \vec{FP} \text{ makes angle } \bar{n} - \theta \text{ with } \vec{ZF} &\quad \therefore \cos(\bar{n} - \theta) = -\cos \theta \\ \Rightarrow \frac{l}{r} &= 1 + e \cos \theta \end{aligned}$$

Some useful results to recognize the Conic:-

The Conic is a parabola if $e = 1$

The Conic is an ellipse if $e < 1$

The Conic is a hyperbola if $e > 1$

Exercise 6.4

In Problems (1-6), identify and graph the given polar equations:

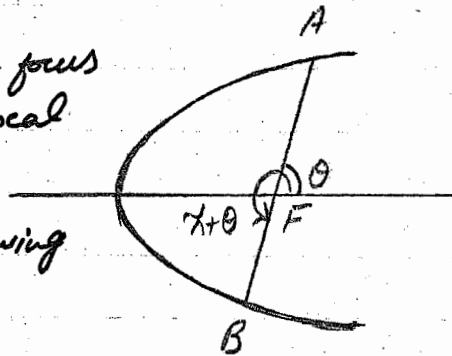
For Questions 1-6 Please see Graph Book.

Question?

Show that in any Conic the sum of the reciprocals of the segments of any focal chord is constant.

Soln.

Consider a Conic having the focus at $O(0,0)$. Let AFB be the focal chord. Then AF and FB be the segments of focal chord having the angles θ and $\pi + \theta$.



We know that-

$$r = \frac{l}{1 - e \cos \theta}$$

$$\Rightarrow |AF| = \frac{l}{1 - e \cos \theta} \quad \text{--- (1)}$$

and

$$|FB| = \frac{l}{1 - e \cos (\pi + \theta)}$$

$$|FB| = \frac{l}{1 + e \cos \theta} \quad \text{--- (2)}$$

Now sum of the reciprocals of the segments

$$\begin{aligned} \frac{1}{|AF|} + \frac{1}{|FB|} &= \frac{1 - e \cos \theta}{l} + \frac{1 + e \cos \theta}{l} \\ &= \frac{1 - e \cos \theta + 1 + e \cos \theta}{l} \\ &= \frac{2}{l} = \text{Constant.} \end{aligned}$$

Note

$$\frac{|AF| + |BF|}{|AF||BF|} = \frac{2}{l}$$

$$l = \frac{2 |AF| |BF|}{|AF| + |BF|}$$

$$= \frac{2ab}{a+b}$$

We also observe that l is the semi latus rectum is the harmonic mean between the two segments of the focal chord.

Question 8. If PP' and QQ' are two perpendicular focal chords of a conic, prove that

$$\frac{1}{|PF| \cdot |FP'|} + \frac{1}{|QF| \cdot |FQ'|} \text{ is constant}$$

Soln. Let the coordinates of P be (r, θ)

Then the coordinate of P' be $(r', \pi + \theta)$

The coordinates of Q , $(r'', \frac{\pi}{2} + \theta)$

The coordinates of Q' , $(r''', \frac{3\pi}{2} + \theta)$

Now we know that

$$\frac{1}{r} = \frac{1 - e \cos \theta}{l}$$

$$\frac{1}{|PF|} = \frac{1 - e \cos \theta}{l}$$

$$\text{and } \frac{1}{|FP'|} = \frac{1 - e \cos(\pi + \theta)}{l} = \frac{1 + e \cos \theta}{l}$$

$$\text{now } \frac{1}{|PF| \cdot |FP'|} = \frac{1 - e \cos \theta}{l} \cdot \frac{1 + e \cos \theta}{l}$$

$$= \frac{1 - e^2 \cos^2 \theta}{l^2} \quad \text{--- (1)}$$

Also

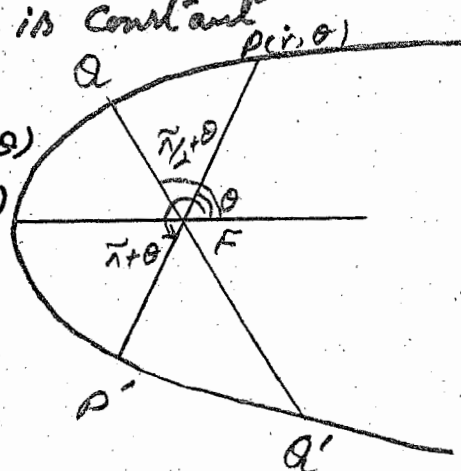
$$\frac{1}{|QF|} = \frac{1 - e \cos(\frac{\pi}{2} + \theta)}{l}$$

$$= \frac{1 + e \sin \theta}{l}$$

and

$$\frac{1}{|FQ'|} = \frac{1 - e \cos(\frac{3\pi}{2} + \theta)}{l}$$

$$= \frac{1 - e \sin \theta}{l}$$



$$\begin{aligned} \text{Now } \frac{1}{|QF| \cdot |FQ'|} &= \frac{1+e \sin \theta}{l} \cdot \frac{1-e \sin \theta}{l} \\ &= \frac{1-e^2 \sin^2 \theta}{l^2} \quad \text{--- (2)} \end{aligned}$$

(1) + (2) $\Rightarrow$

$$\begin{aligned} \frac{1}{|PF| \cdot |FP'|} + \frac{1}{|QF| \cdot |FQ'|} &= \frac{1-e^2 \cos^2 \theta}{l^2} + \frac{1-e^2 \sin^2 \theta}{l^2} \\ &= \frac{1-e^2 \cos^2 \theta + 1-e^2 \sin^2 \theta}{l^2} \\ &= \frac{2-e^2 (\cos^2 \theta + \sin^2 \theta)}{l^2} \\ &= \frac{2-e^2}{l^2} = \text{Constant} \end{aligned}$$

Question 9.

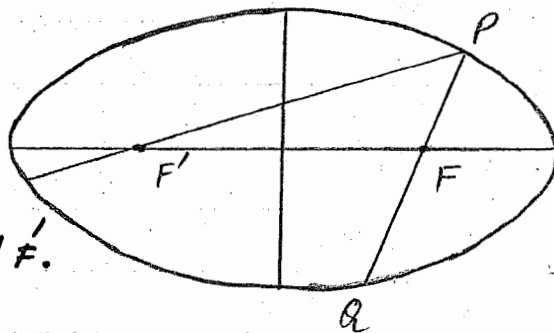
If PFQ , $PF'R$ be two chords of an ellipse through the foci F, F' , show that

$$\frac{|PF|}{|FQ|} + \frac{|PF'|}{|F'R|} \text{ is independent of}$$

the position of P .

Soln.

Consider that PFQ and $PF'R$ are the two focal chords of the ellipse having the foci at F and F' .



$\therefore PFQ$ is a focal chord

$\therefore PF$ and FQ are its two segments

$\therefore l$ is the H.M. between PF and FQ where l is the semi latus rectum

$$\therefore \frac{1}{|PF|} + \frac{1}{|FQ|} = \frac{2}{l}$$

$\times$ by $|PF|$

$$\Rightarrow 1 + \frac{|PF|}{|FQ|} = \frac{2|PF|}{l} \quad \text{--- (1)}$$

$$\begin{aligned} \text{H.M.} &= \frac{2ab}{a+b} \\ \frac{1}{a} + \frac{1}{b} &= \frac{2}{H} \end{aligned}$$

Also $PF'R$ is a focal chord

$\therefore$ therefore $|PF'|$ and $|F'R|$ are its two segments

Therefore proceeding as above

$$1 + \frac{|PF|}{|FQ|} = \frac{2|PF'|}{l} \quad (2)$$

(1) + (2) $\Rightarrow$

$$2 + \frac{|PF|}{|FQ|} + \frac{|PF'|}{|F'Q|} = \frac{2}{l} (|PF| + |PF'|)$$

$$\frac{|PF|}{|FQ|} + \frac{|PF'|}{|F'Q|} = \frac{2}{l} (2a) - 2$$

$$\frac{|PF|}{|FQ|} + \frac{|PF'|}{|F'Q|} = \frac{4a}{l} - 2 = \text{Constant. if } l \text{ is equal to } 2a.$$

In Ellipse if we take a point on it. If we sum up the distance of this point from F and F' i.e. $|PF| + |PF'|$ then

Express each of the given equations in polar form and find the eccentricity and equation of the directrix.

Question 10

$$y^2 = 4 - 4x$$

$$r^2 \sin^2 \theta = 4 - 4r \cos \theta$$

$$\text{put } x = r \cos \theta$$

$$\sin^2 \theta r^2 + 4 \cos \theta r - 4 = 0$$

$$y = r \sin \theta$$

which is quadratic in r

$$\text{So } r = \frac{-4 \cos \theta \pm \sqrt{16 \cos^2 \theta - 4 \sin^2 \theta (-4)}}{2 \sin^2 \theta}$$

$$r = \frac{4(-\cos \theta \pm \sqrt{\cos^2 \theta + \sin^2 \theta})}{2 \sin^2 \theta}$$

$$r = \frac{4(-\cos \theta \pm 1)}{2 \sin^2 \theta}$$

$$r = \frac{2(-\cos \theta + 1)}{1 - \cos^2 \theta}$$

Neglecting -ve sign

$$r = \frac{2(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$$

$$r = \frac{2}{1 + \cos \theta}$$

$$\Rightarrow e = 1$$

This is a parabola with eccentricity, $e = 1$

eq. of Directrix

we know that eq. of the directrix of parabola is $x + a = 0$ latus rectum = $4a$
semi latus rectum = $2a$

$$\therefore l = 2a$$

$$\Rightarrow 2 = 2a$$

$$a = 1$$

$\therefore$ The distance of the directrix line from the focus is equal to $2a$. And the form of the Conic focus is at the origin.

$\therefore$ The directrix line is at a distance of $2a$ from the focus

$$\text{i.e. } x = 2a$$

$$\Rightarrow x = 2$$

$$r \cos \theta = 2$$

$$r = 2 \sec \theta$$

$$\left| x' = \frac{l}{e} \right.$$

Question 11.

$$3y^2 - 16y - x^2 + 16 = 0$$

$$3r^2 \sin^2 \theta - 16r \sin \theta - r^2 \cos^2 \theta + 16 = 0$$

$$\text{Put } x = r \cos \theta$$

$$(3 \sin^2 \theta - \cos^2 \theta) r^2 - 16 \sin \theta r + 16 = 0$$

$$y = r \sin \theta$$

$$r = \frac{16 \sin \theta \pm \sqrt{256 \sin^2 \theta - 4(3 \sin^2 \theta - \cos^2 \theta)(16)}}{2(3 \sin^2 \theta - \cos^2 \theta)}$$

$$r = \frac{16 \sin \theta \pm \sqrt{256 \sin^2 \theta - 192 \sin^2 \theta + 64 \cos^2 \theta}}{2(3 \sin^2 \theta - \cos^2 \theta)}$$

$$r = \frac{16 \sin \theta \pm \sqrt{64 \sin^2 \theta + 64 \cos^2 \theta}}{2(3 \sin^2 \theta - \cos^2 \theta)}$$

$$r = \frac{16 \sin \theta \pm 8 \sqrt{\sin^2 \theta + \cos^2 \theta}}{2(3 \sin^2 \theta - (1 - \sin^2 \theta))}$$

$$r = \frac{16 \sin \theta \pm 8}{2(3 \sin^2 \theta - 1 + \sin^2 \theta)}$$

$$r = \frac{8(2 \sin \theta \pm 1)}{2(4 \sin^2 \theta - 1)}$$

$$r = \frac{4(2 \sin \theta + 1)}{4 \sin^2 \theta - 1}$$

Neglecting the sign.

$$r = \frac{4(2 \sin \theta + 1)}{(2 \sin \theta - 1)(2 \sin \theta + 1)}$$

$$r = \frac{4}{2 \sin \theta - 1}$$

$$r = \frac{4}{-(1-2\sin\theta)}$$

$$r = \frac{-4}{1-2\cos(\pi/2-\theta)}$$

$$\Rightarrow e = 2, > 1$$

Thus this is a hyperbola with eccentricity, $e = 2$

Equation of a directrix is $x = 2$

$$r \sin \theta = 2$$

$$r = 2 \csc \theta$$

Question 12. $8x^2 + 9y^2 + 4x - 4 = 0$

The equation in polar form is

$$8r^2 \cos^2 \theta + 9r^2 \sin^2 \theta + 4r \cos \theta - 4 = 0$$

$$(8\cos^2 \theta + 9\sin^2 \theta)r^2 + 4\cos \theta - 4 = 0$$

$$r = \frac{-4\cos \theta \pm \sqrt{16\cos^2 \theta - 4(8\cos^2 \theta + 9\sin^2 \theta)(-4)}}{2(8\cos^2 \theta + 9\sin^2 \theta)}$$

$$r = \frac{-4\cos \theta \pm \sqrt{16\cos^2 \theta + 128\cos^2 \theta + 144\sin^2 \theta}}{2(8\cos^2 \theta + 9\sin^2 \theta)}$$

$$r = \frac{-4\cos \theta \pm \sqrt{144\cos^2 \theta + 144\sin^2 \theta}}{2(8\cos^2 \theta + 9\sin^2 \theta)}$$

$$r = \frac{-4\cos \theta \pm \sqrt{144}}{2(8\cos^2 \theta + 9\sin^2 \theta)}$$

$$r = \frac{-4\cos \theta \pm 12}{2(8\cos^2 \theta + 9\sin^2 \theta)}$$

$$r = \frac{-2\cos \theta \pm 6}{8\cos^2 \theta + 9\sin^2 \theta}$$

$$r = \frac{6 - 2\cos \theta}{8\cos^2 \theta + 9(1 - \cos^2 \theta)}$$

$$r = \frac{6 - 2\cos \theta}{8\cos^2 \theta + 9 - 9\cos^2 \theta}$$

$$r = \frac{6 - 2\cos \theta}{9 - \cos^2 \theta}$$

$$r = \frac{2(3 - \cos \theta)}{(3 + \cos \theta)(3 - \cos \theta)}$$

Eg. 7 Directrix

$$x + a = 0$$

$$\therefore l = -4 \Rightarrow \frac{l}{a} = 2 \Rightarrow a = -2$$

$$\Rightarrow x - 2 = 0$$

$$\Rightarrow x = 2$$

$$\Rightarrow r = \frac{2}{3 + \cos \theta}$$

$$r = \frac{2}{3(1 + \frac{1}{3} \cos \theta)}$$

$$r = \frac{2/3}{1 + \frac{1}{3} \cos \theta}$$

$$\Rightarrow e = \frac{1}{3} < 1$$

thus the conic is ellipse

Eq. of the directrix is $x = 2$

$$\Rightarrow r \cos \theta = 2$$

$$r = 2 \sec \theta$$

$$\frac{\frac{2}{3}}{\frac{1}{3}}$$

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Exercise 6.5

Sketching of Curves in Polar Coordinates:

To sketch the graph of a curve whose equation is given in polar coordinates, the following properties of the curve should be examined.

I. Symmetry about the initial line: (x-axis)

The curve is symmetric about the initial line if the curve remains the same on changing (r, θ) to $(r, -\theta)$ or $(-r, \pi - \theta)$ e.g.

$$r = a \cos \theta$$

$$r = a \cos(-\theta)$$

$$r = a \cos \theta$$

$\therefore$ There is no change in the curve

$\therefore$ It is symm. about the initial line.

II. Symmetry about the line $\theta = \frac{\pi}{2}$: (y-axis)

The curve is symmetric about the line $\theta = \frac{\pi}{2}$ (y-axis) if it remains the same on replacing (r, θ) by $(-r, -\theta)$ or $(r, \pi - \theta)$

III. Symmetry about the pole:

The curve is symmetric w.r.t. Pole (origin) if there is no change in the curve on replacing (r, θ) by $(-r, \theta)$.

e.g.

$$r^2 = a \sin \frac{\theta}{2}$$

$$(-r)^2 = a \sin \frac{\theta}{2}$$

$$r^2 = a \sin \frac{\theta}{2}$$

$\therefore$ There is no change in the curve

$\therefore$ It is symmetric about pole.

IV Position of the Pole relative to the Curve.

Check whether the pole lies on the curve by putting $r=0$ in the given eq. and find the corresponding value of θ .

I Table of Values.

Construct a sufficiently complete table of values. This can be a great help in sketching the graph of a curve.

Exercise 6.5.

Sketch the graph of each of the curves.
(1-15)

Please see these questions on
Graph Book.

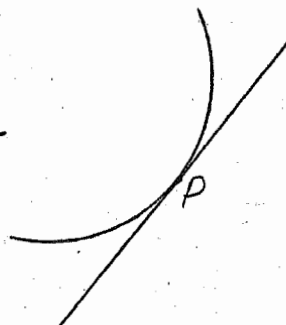
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Exercise 6.6

Slope of a Curve.

The slope of a curve at a point on the curve is the slope of the tangent line to the curve at that point.



Theorem.

If $P(r, \theta)$, $r \neq 0$ is a point on the curve defined by $r = f(\theta)$. Then the angle between the radius vector and tangent at P in the direction of increasing θ has measured ψ given by $\tan \psi = \frac{r}{dr/d\theta}$.

OR Give geometrical significance of derivative in polar coordinates.

OR Prove that $\tan \psi = \frac{r}{dr/d\theta}$

Proof: Let $P(r, \theta)$ and

$Q(r+\delta r, \theta+\delta \theta)$ be any adjacent points on the curve $r = f(\theta)$.

Let ψ be the angle between the radius vector and the tangent at P .

Further let $\angle PQR = \alpha$

Then $\angle OPQ = \tilde{n} - \alpha$

and $\angle QPR = \tilde{n} - (\tilde{n} - \alpha + \delta \theta)$

$\angle QPR = \alpha - \delta \theta$

Apply the law of sine to $\triangle OPQ$

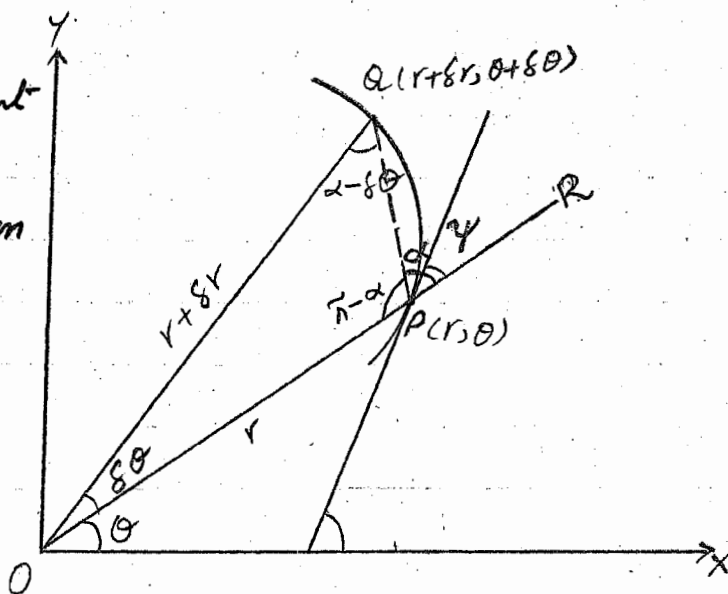
$$\frac{r + \delta r}{\sin(\tilde{n} - \alpha)} = \frac{r}{\sin(\alpha - \delta \theta)}$$

$$r \sin(\tilde{n} - \alpha) = (r + \delta r) \sin(\alpha - \delta \theta)$$

$$r \sin \alpha = (r + \delta r) (\sin \alpha \cos \delta \theta - \cos \alpha \sin \delta \theta)$$

$$r \sin \alpha = r \sin \alpha \cos \delta \theta - r \cos \alpha \sin \delta \theta + \delta r \sin \alpha \cos \delta \theta - \delta r \cos \alpha \sin \delta \theta$$

$$r \sin \alpha - r \sin \alpha \cos \delta \theta = \delta r \sin \alpha \cos \delta \theta - (r + \delta r) \cos \alpha \sin \delta \theta$$



$$r \sin \alpha (1 - \cos \delta \theta) = \delta r \sin \alpha \cos \delta \theta - (r + \delta r) \cos \alpha \sin \delta \theta.$$

Dividing by $\delta \theta$

$$r \sin \alpha \frac{(1 - \cos \delta \theta)}{\delta \theta} = \frac{\delta r}{\delta \theta} [\sin \alpha \cos \delta \theta] - \frac{(r + \delta r)}{\delta \theta} \cos \alpha \sin \delta \theta.$$

$$\lim_{\delta \theta \rightarrow 0} \frac{r \sin \alpha (1 - \cos \delta \theta)}{\delta \theta} = \lim_{\delta \theta \rightarrow 0} \frac{\delta r}{\delta \theta} [\sin \alpha \cos \delta \theta] - \lim_{\delta \theta \rightarrow 0} \frac{(r + \delta r)}{\delta \theta} \cos \alpha \sin \delta \theta$$

[As point Q tends to P the chord PQ coincides with the tangent at P and $\alpha \rightarrow \psi$] ($\therefore \delta \theta \rightarrow 0$)

$$\lim_{\delta \theta \rightarrow 0} \frac{1 - \cos \delta \theta}{\delta \theta} = 0 \quad [\text{By using L'Hospital Rule}]$$

$$\text{So } \Rightarrow \quad 0 = \frac{dr}{d\theta} \sin \psi \cos 0 - (r + 0) \cos \psi \cdot 1 \quad \left[\lim_{\delta \theta \rightarrow 0} \frac{\sin \delta \theta}{\delta \theta} = 1 \right]$$

$$0 = \frac{dr}{d\theta} \sin \psi - r \cos \psi$$

$$\Rightarrow \quad \frac{dr}{d\theta} \sin \psi = r \cos \psi$$

$$\frac{\sin \psi}{\cos \psi} = \frac{r}{\frac{dr}{d\theta}}$$

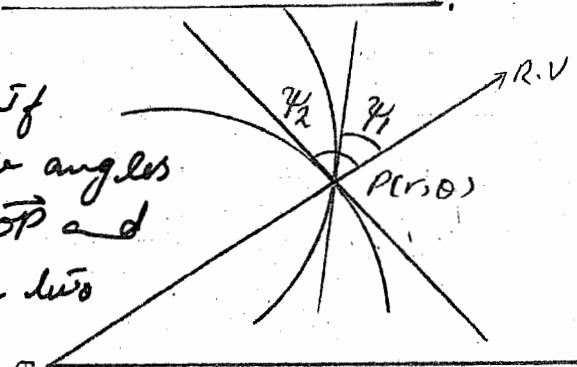
$$\tan \psi = r \frac{d\theta}{dr}$$

Is as required.

Angle of Intersection of Two Polar Curves.

Suppose that two curves intersect at a point $P = (r, \theta)$. If ψ_1 and ψ_2 are measures of the angles between the radius vector $\vec{OP}$ and the tangents to each of the two curves at P, then measure of their angle of intersection is

$$|\psi_1 - \psi_2|.$$



Length of Perpendicular From the pole to a Tangent.

Let ON be the perpendicular from O to the tangent at $P(r, \theta)$ of the curve defined by $r = f(\theta)$

Let $ON = p$ then from r.t ΔONP

$$\frac{ON}{OP} = \sin \psi$$

$$\frac{p}{r} = \sin \psi$$

$$\Rightarrow p = r \sin \psi$$

Is the required length of perpendicular from pole to tangent.

Now if α is the inclination of tangent.

Then by elementary geometry

$$\alpha = \theta + \psi$$

Theorem

If (r, θ) be the coordinates of a pt. P on the curve and T be the tangent at that point. If α be the inclination of the tangent T and ψ be the angle between the radius vector and tangent then prove that-

I) $\alpha = \theta + \psi$

II) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$

where p is the length of perpendicular from pole to the tangent.

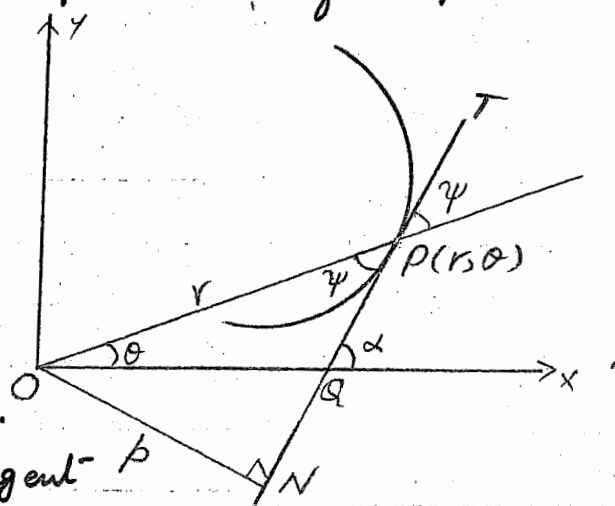
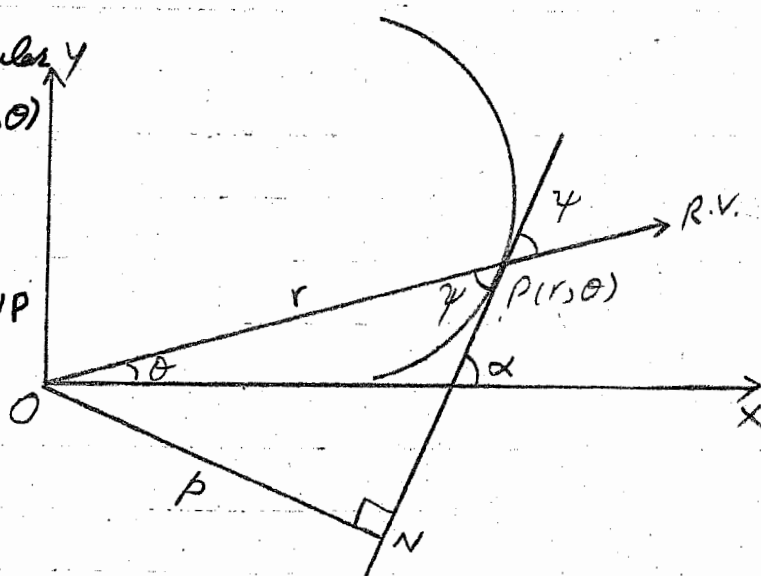
Proof Consider the pt. $P(r, \theta)$ on the curve $r = f(\theta)$ — " —

Then $|OP| = r$

$\angle xOP = \theta$

Let T be the tangent at $P(r, \theta)$.

α is the inclination of this tangent p



and ψ is the angle b/w the tangent and radius vector.
 p is the length of the Lar from the pole to the tangent.

I) Consider the ΔOQP , we have

$$\alpha = \theta + \psi \quad (\text{By elementary geometry})$$

II) Consider the ΔONP

$$\sin \psi = \frac{p}{r}$$

$$p = r \sin \psi$$

$$\frac{1}{p^2} = \frac{1}{r^2 \sin^2 \psi}$$

$$= \frac{1}{r^2} \operatorname{cosec}^2 \psi$$

$$= \frac{1}{r^2} (1 + \cot^2 \psi)$$

$$= \frac{1}{r^2} \left(1 + \left(\frac{1}{r} \frac{dr}{d\theta} \right)^2 \right)$$

$$\tan \psi = r \frac{d\theta}{dr}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \quad \text{--- (1)}$$

is as required.

Amendment.

$$\text{Let } u = \frac{1}{r} \quad \text{--- (2)}$$

$$\frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}$$

$$\left(\frac{du}{d\theta} \right)^2 = \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \quad \text{--- (3)}$$

Put (2) and (3) in (1)

$$\Rightarrow \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta} \right)^2$$

we can prove
 this relation directly
 by saying $\because p = r \sin \psi$
 and proceed as above.

Pedal Eq. of Polar Curves.

To obtain the pedal equation of a curve defined by $r = f(\theta)$, we eliminate θ between

$$r = f(\theta) \text{ and } \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2.$$

Some times it is convenient to obtain pedal Eq. by eliminating θ and ψ from $r = f(\theta)$,

$$\tan \psi = r \frac{d\theta}{dr} \text{ and } p = r \sin \psi.$$

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If you have any question, ask it at

<http://forum.mathcity.org>

Exercise 6.6

Exercise 6.6

Find ψ for each of the given curves (1-4)

Q#1.

$$r = a(1 - \cos \theta)$$

$$\frac{dr}{d\theta} = a \sin \theta$$

Now $\tan \psi = \frac{r}{dr/d\theta}$

$$\tan \psi = \frac{a(1 - \cos \theta)}{a \sin \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

$$\tan \psi = \frac{2 \sin^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}$$

$$\tan \psi = \frac{\sin \theta/2}{\cos \theta/2}$$

$$\tan \psi = \tan \theta/2$$

$$\Rightarrow \psi = \theta/2$$

Q#2.

$$r = -5 \csc \theta \cot \theta$$

$$\frac{dr}{d\theta} = 5 \csc \theta \cot \theta$$

Now

$$\tan \psi = \frac{r}{dr/d\theta}$$

$$\tan \psi = \frac{-5 \csc \theta \cot \theta}{5 \csc \theta \cot \theta}$$

$$\tan \psi = -\tan \theta = \tan (\pi - \theta)$$

$$\psi = \pi - \theta$$

$\Rightarrow$

Q#3.

$$\frac{2a}{r} = 1 + \sin \theta$$

$$r = \frac{2a}{1 + \sin \theta}$$

Diff. w.r.t. θ

$$\frac{dr}{d\theta} = \frac{-2a \cos \theta}{(1 + \sin \theta)^2}$$

Now $\tan \psi = r/dr/d\theta$

$$= \frac{2a}{1 + \sin \theta} \cdot \frac{(1 + \sin \theta)^2}{-2a \cos \theta}$$

$$\tan \psi = - \frac{1 + \sin \theta}{\cos \theta}$$

$$\tan \psi = + \frac{1 + 2 \sin \theta/2 \cos \theta/2}{\sin^2 \theta/2 - \cos^2 \theta/2}$$

$$\tan \psi = + \frac{\sin^2 \theta/2 + \cos^2 \theta/2 + 2 \sin \theta/2 \cos \theta/2}{\sin^2 \theta/2 - \cos^2 \theta/2}$$

$$\tan \psi = \frac{(\sin \theta/2 + \cos \theta/2)^2}{(\sin \theta/2 - \cos \theta/2)(\sin \theta/2 + \cos \theta/2)}$$

$$\tan \psi = \frac{\sin \theta/2 + \cos \theta/2}{\sin \theta/2 - \cos \theta/2}$$

$$\tan \psi = \frac{1 + \tan \theta/2}{1 - \tan \theta/2}$$

$$\tan \psi = - \frac{1 - \tan \theta/2}{1 + \tan \theta/2}$$

$$\tan \psi = - \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right)$$

$$\tan \psi = \tan \left(\pi - \frac{\pi}{4} - \frac{\theta}{2} \right)$$

$$\tan \psi = \tan \left(\frac{3\pi}{4} - \frac{\theta}{2} \right)$$

$$\Rightarrow \psi = \frac{3\pi}{4} - \frac{\theta}{2}$$

Q#4.

Diff. w.r.t. 'θ'

$$r = \frac{3}{2(1 - \cos \theta)} \Rightarrow r = \frac{3}{2 - 2 \cos \theta}$$

$$\frac{dr}{d\theta} = \frac{-3(2 \sin \theta)}{(2 - 2 \cos \theta)^2}$$

$$= \frac{-6 \sin \theta}{(2 - 2 \cos \theta)^2}$$

$$\text{Now } \tan \psi = r \frac{d\theta}{dr}$$

$$= \frac{3}{2(1 - \cos \theta)} \cdot \frac{2^2(1 - \cos \theta)^2}{-6 \sin \theta}$$

$$= \frac{1 - \cos \theta}{-\sin \theta}$$

$$\tan \left(\frac{\pi}{4} - \theta \right) = \frac{1 - \tan \theta}{1 + \tan \theta}$$

$1 + \sin \theta = 1 + 2 \sin \theta/2 \cos \theta/2$
 $\cos \theta = \cos^2 \theta/2 - \sin^2 \theta/2$

$$= - \frac{2 \sin^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}$$

$$= - \frac{\sin \theta/2}{\cos \theta/2} = - \tan \theta/2$$

$$\tan \psi = \tan (\pi - \theta/2)$$

$$\Rightarrow \psi = \pi - \frac{\theta}{2}$$

Find the measure of the angle of intersection of the given curves (5-10)

Q#5. $r=1$ and $r=2 \sin \theta$

Here $r=1$ — (1), $r=2 \sin \theta$ — (2)

Put (1) in (2) $\Rightarrow 1 = 2 \sin \theta$

$$\frac{1}{2} = \sin \theta$$

$$\Rightarrow \theta = 30^\circ$$

Diff. (1) w.r.t. ' θ ', we have

$$\frac{dr}{d\theta} = 0$$

Now $\tan \psi_1 = r \cdot \frac{d\theta}{dr}$

$$\tan \psi_1 = \frac{1}{0}$$

$$\tan \psi_1 = \infty$$

$$\psi_1 = 90^\circ$$

Diff. (2) w.r.t. ' θ '

$$\frac{dr}{d\theta} = 2 \cos \theta$$

Now

$$\tan \psi_2 = \frac{r}{dr/d\theta}$$

$$\tan \psi_2 = \frac{2 \sin \theta}{2 \cos \theta}$$

$$\tan \psi_2 = \tan \theta$$

$$\psi_2 = \theta$$

$$\psi_2 = 30^\circ$$

$$\therefore \theta = 30^\circ$$

Now

$$|\psi_1 - \psi_2|$$

$$|90^\circ - 30^\circ| = 60^\circ$$

Angle b/w the curves at $\theta = 30^\circ$ is 60°

Q# 6. $r = a\theta$ — (i) and $r\theta = a$ — (ii)

Put (i) in (ii) $\Rightarrow$

$$a\theta \cdot \theta = a$$

$$\theta^2 = 1$$

$$\Rightarrow \theta = \pm 1 \quad \text{--- iii}$$

Diff. (i) w.r.t. θ

$$\frac{dr}{d\theta} = a$$

Now $\tan \psi_1 = \frac{r}{dr/d\theta}$

$$\tan \psi_1 = \frac{a\theta}{a}$$

$$\tan \psi_1 = \theta \quad \text{--- iv}$$

at $\theta = 1$

$$\tan \psi_1 = 1$$

$$\psi_1 = 45^\circ$$

Diff. (ii) w.r.t. θ

$$\frac{dr}{d\theta} \cdot \theta + r = 0$$

$$\frac{dr}{d\theta} \cdot \theta = -r$$

$$\frac{dr}{d\theta} \cdot \theta = -\frac{a}{\theta}$$

$$\frac{dr}{d\theta} = -\frac{a}{\theta^2}$$

Now $\tan \psi_2 = r \cdot \frac{d\theta}{dr}$

$$\tan \psi_2 = \frac{a/\theta}{-a/\theta^2}$$

$$\tan \psi_2 = -\frac{a}{\theta} \cdot \frac{\theta^2}{a}$$

$$\tan \psi_2 = -\theta \quad \text{--- v}$$

at $\theta = 1$

$$\tan \psi_2 = -1$$

$$\psi_2 = 135^\circ$$

If the required angle is α then

$$\alpha = |\psi_1 - \psi_2|$$

$$= |45^\circ - 135^\circ|$$

$$= 90^\circ$$

Now if $\theta = -1$

then $\Rightarrow \tan \psi_1 = -1$

$$\psi_1 = 135^\circ$$

$$\Rightarrow \tan \psi_2 = -(-1)$$

$$\psi_2 = 45^\circ$$

$$\text{angle of intersection} = |\psi_1 - \psi_2|$$

$$= |135^\circ - 45^\circ|$$

$$= 90^\circ$$

Q#7.

$$r = \frac{a\theta}{1+\theta}$$

$$\text{and } r = \frac{a}{1+\theta^2}$$

$$r = \frac{a\theta}{1+\theta} \quad \text{--- (i)}$$

$$r = \frac{a}{1+\theta^2} \quad \text{--- (ii)}$$

Diff. w.r.t. θ .

$$\frac{dr}{d\theta} = \frac{(1+\theta)a - a\theta(1)}{(1+\theta)^2}$$

$$= \frac{a(1+\theta-\theta)}{(1+\theta)^2}$$

$$= \frac{a}{(1+\theta)^2}$$

Diff. w.r.t. ' θ '

$$\frac{dr}{d\theta} = - \frac{a \cdot 2\theta}{(1+\theta^2)^2}$$

$$= - \frac{2a\theta}{(1+\theta^2)^2}$$

$$\text{Now } \tan \psi_2 = \frac{a}{1+\theta^2} \cdot \frac{(1+\theta^2)^2}{2a\theta}$$

$$= - \frac{1+\theta^2}{2\theta} \quad \text{--- (iv)}$$

$$\tan \psi_1 = \frac{r}{dr/d\theta}$$

$$= \frac{a\theta}{1+\theta} \cdot \frac{(1+\theta)^2}{a}$$

$$= \theta(1+\theta) \quad \text{--- (iii)}$$

Now (i) and (2)

$$\Rightarrow \frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$$

$$\theta(1+\theta^2) = 1+\theta$$

$$\theta + \theta^3 = 1+\theta$$

$$\theta^3 = 1$$

$$\theta^3 = 1$$

Put in (3) and (iv)

$$\tan \psi_1 = 1(1+1)$$

$$\tan \psi_1 = 2$$

$$\psi_1 = 63.43^\circ$$

$$\tan \psi_2 = - \frac{1+1}{2(1)}$$

$$= -1$$

$$\psi_2 = 135^\circ$$

$$\text{Required angle} = |\psi_1 - \psi_2|$$

$$= |63.43 - 135|$$

$$= 71.57$$

Q#8

$$r = ae^{\theta} \quad \text{and} \quad re^{\theta} = b$$

Diff. w.r.t. θ

$$\frac{dr}{d\theta} = ae^{\theta}$$

So

$$\begin{aligned} \tan \psi_1 &= r \cdot \frac{d\theta}{dr} \\ &= ae^{\theta} \cdot \frac{1}{ae^{\theta}} \end{aligned}$$

$$\begin{aligned} \tan \psi_1 &= 1 \\ \psi_1 &= 45^\circ \end{aligned}$$

$$r = \frac{b}{e^{\theta}}$$

Diff. w.r.t. θ

$$\frac{dr}{d\theta} = -\frac{b}{e^{\theta}}$$

So

$$\tan \psi_2 = \frac{b}{e^{\theta}} \cdot -\frac{e^{\theta}}{b}$$

$$\tan \psi_2 = -1$$

$$\psi_2 = 135^\circ$$

$$\begin{aligned} \text{Required angle} &= |\psi_1 - \psi_2| \\ &= |45^\circ - 135^\circ| \\ &= 90^\circ \end{aligned}$$

Q#9

$$r = a(1 - \cos \theta) \quad \text{and} \quad r = \frac{a}{1 - \cos \theta}$$

$$r = a(1 - \cos \theta) \quad \text{--- (I)}$$

Diff. w.r.t. θ

$$\frac{dr}{d\theta} = a \sin \theta$$

$$\text{So } \tan \psi_1 = \frac{r}{dr/d\theta}$$

$$\tan \psi_1 = \frac{a(1 - \cos \theta)}{a \sin \theta}$$

$$\tan \psi_1 = \frac{1 - \cos \theta}{\sin \theta}$$

$$\tan \psi_1 = \frac{2 \sin^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}$$

$$= \tan \theta/2$$

$$\Rightarrow \psi_1 = \theta/2 \quad \text{--- (II)}$$

$$(I) \quad \text{and} \quad (II) \quad \Rightarrow$$

$$\begin{aligned} a(1 - \cos \theta) &= \frac{a}{1 - \cos \theta} \\ (1 - \cos \theta)^2 &= 1 \end{aligned}$$

$$1 - \cos \theta = \pm 1$$

$$r = a(1 - \cos \theta)^{-1} \quad \text{--- (I')}$$

Diff. w.r.t. θ

$$\frac{dr}{d\theta} = -\frac{a \sin \theta}{(1 - \cos \theta)^2}$$

$$\text{So } \tan \psi_2 = \frac{r}{dr/d\theta}$$

$$\tan \psi_2 = \frac{a}{1 - \cos \theta} \cdot \frac{(1 - \cos \theta)^2}{-a \sin \theta}$$

$$\tan \psi_2 = -\frac{1 - \cos \theta}{\sin \theta}$$

$$\tan \psi_2 = -\frac{2 \sin^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}$$

$$\tan \psi_2 = -\tan \theta/2$$

$$\tan \psi_2 = \tan (\pi - \theta/2)$$

$$\psi_2 = \pi - \theta/2 \quad \text{--- (IV)}$$

$$\Rightarrow 1 - \cos \theta = 1 \quad \text{and} \quad 1 - \cos \theta = -1$$

$$\Rightarrow \cos \theta = 0 \quad \text{and} \quad \cos \theta = 2$$

$$\theta = 90^\circ \quad \text{and} \quad \theta = \infty \quad \text{which is not accepted.}$$

Now III and IV $\Rightarrow$

$$\text{at } \theta = 90^\circ \quad \psi_1 = 90^\circ/2$$

$$\Rightarrow \psi_1 = 45^\circ$$

$$\text{at } \theta = 90^\circ \quad \psi_2 = \pi - 90^\circ/2$$

$$\psi_2 = \pi - 45^\circ$$

$$\begin{aligned} \text{So Required angle} &= |\psi_1 - \psi_2| \\ &= |45^\circ - (\pi - 45^\circ)| \\ &= |45^\circ - \pi + 45^\circ| \\ &= |90^\circ - \pi| \\ &= |\pi/2| \\ &= \pi/2 \end{aligned}$$

Is the required angle of intersection.

Q# 10

$$r = \cos 2\theta \quad \text{and} \quad r = \sin \theta \quad \text{at } \left(\frac{1}{2}, \frac{\pi}{6}\right)$$

$$\text{Here } r = \cos 2\theta \quad \text{--- (I)}$$

$$r = \sin \theta \quad \text{--- II}$$

Diff. (I) w.r.t. 'θ'

$$\frac{dr}{d\theta} = -2 \sin 2\theta$$

$$\text{So } \tan \psi_1 = \frac{r}{dr/d\theta}$$

$$\tan \psi_1 = \frac{\cos 2\theta}{-2 \sin 2\theta}$$

$$\tan \psi_1 = -\frac{1}{2} \cot 2\theta$$

$$\begin{aligned} \text{Now } (\tan \psi_1)_{\left(\frac{1}{2}, \frac{\pi}{6}\right)} &= -\frac{1}{2} \cot 2\left(\frac{\pi}{6}\right) \\ &= -\frac{1}{2} \cot \frac{\pi}{3} \\ &= -\frac{1}{2} (.577) \\ &= -.288 \end{aligned}$$

$$\psi_1 = -\tan^{-1}(.288)$$

$$\psi_1 = -\tan^{-1}(.288)$$

$$\psi_1 = -16.1^\circ$$

Diff. w.r.t. 'θ'

$$\frac{dr}{d\theta} = \cos \theta$$

$$\text{So } \tan \psi_2 = \frac{r}{dr/d\theta}$$

$$\tan \psi_2 = \frac{\sin \theta}{\cos \theta}$$

$$\tan \psi_2 = \tan \theta$$

$$\psi_2 = \theta$$

$$\psi_2 = \pi/6$$

$$\psi_2 = 30^\circ$$

Now

$$\begin{aligned}\text{Required angle} &= |\psi_1 - \psi_2| \\ &= |-16.1 - 30| \\ &= |-46.1| \\ &= 46.1^\circ\end{aligned}$$

Find the pedal Equation of the given curves.

Q.11.

$$\frac{l}{r} = 1 + e \cos \theta \quad \text{--- I}$$

We know that pedal eq. of any polar curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \quad \text{--- II}$$

Now

Diff. (I) w.r.t. θ

$$-\frac{l}{r^2} \frac{dr}{d\theta} = -e \sin \theta$$

$$\Rightarrow \frac{1}{r^2} \frac{dr}{d\theta} = \frac{e \sin \theta}{l}$$

$$\Rightarrow \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 = \frac{e^2 \sin^2 \theta}{l^2} \quad \text{--- III}$$

Put III in II $\Rightarrow$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2 \sin^2 \theta}{l^2} \quad \text{--- IV}$$

Now I) $\Rightarrow$

$$\begin{aligned}e \cos \theta &= \frac{l}{r} - 1 = \frac{l-r}{r} \\ \Rightarrow \cos \theta &= \frac{l-r}{er} \quad \text{--- V}\end{aligned}$$

Rewriting (4) in the form

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2 (1 - \cos^2 \theta)}{l^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} \left(1 - \left(\frac{l-r}{er} \right)^2 \right) \quad \text{from V}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} \left[1 - \frac{l^2 + r^2 - 2lr}{e^2 r^2} \right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} \left[\frac{e^2 r^2 - l^2 - r^2 + 2lr}{e^2 r^2} \right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^4 r^2}{l^2 e^2 r^2} - \frac{e^2 l^2}{l^2 e^2 r^2} - \frac{e^2 r^2}{l^2 e^2 r^2} + \frac{2e^2 lr}{l^2 e^2 r^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{1}{r^2} - \frac{1}{l^2} + \frac{2}{rl}$$

$$\frac{1}{p^2} = \frac{e^2}{l^2} - \frac{1}{l^2} + \frac{2}{rl}$$

$$\frac{1}{p^2} = \frac{e^2 - 1}{l^2} + \frac{2}{rl}$$

Is the required pedal Equation.

Q.12.

$$r = a \theta \quad \text{--- I}$$

And $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$ --- II

Diff. (I) w.r.t. 'θ'

$$\frac{dr}{d\theta} = a$$

Put in II =>

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} a^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}$$

which is the required pedal Equation.

Q.13.

$$r = a \sin m\theta. \quad \text{--- I}$$

And $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$ --- II

Diff. (I) w.r.t. 'θ'

$$\frac{dr}{d\theta} = am \cos m\theta$$

Put in II, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} (am \cos m\theta)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} a^2 m^2 \cos^2 m\theta. \quad \text{--- III}$$

But I) => $r = a \sin m\theta$

$$\sin m\theta = \frac{r}{a} \quad \text{--- IV}$$

Rewriting III in the form

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} a^2 m^2 (1 - \sin^2 m\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} a^2 m^2 (1 - (\frac{r}{a})^2) \quad \text{from (iv)}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left[a^2 m^2 - \frac{a^2 m^2 r^2}{a^2} \right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left[a^2 m^2 - m^2 r^2 \right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2 m^2}{r^4} - \frac{m^2 r^2}{r^4}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2 m^2}{r^4} - \frac{m^2}{r^2}$$

$$\frac{1}{p^2} = \frac{r^2 + a^2 m^2 - m^2 r^2}{r^4}$$

$$r^4 = p^2 [r^2 + a^2 m^2 - m^2 r^2]$$

$$r^4 = p^2 [r^2 (1 - m^2) + a^2 m^2]$$

It is the required pedal Eq.

$$r = a + b \cos \theta \quad \text{--- (1)}$$

③ # 14.

Diff. w.r.t. 'θ'

$$\frac{dr}{d\theta} = -b \sin \theta \quad \text{--- (2)}$$

we know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} (-b \sin \theta)^2 \quad \text{from (2)}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} b^2 \sin^2 \theta$$

$$\frac{1}{p^2} = \frac{r^2 + b^2 \sin^2 \theta}{r^4}$$

$$\frac{1}{p^2} = \frac{(a + b \cos \theta)^2 + b^2 \sin^2 \theta}{r^4} \quad \text{from (1)}$$

$$\frac{1}{p^2} = \frac{1}{r^4} \left[a^2 + b^2 \cos^2 \theta + 2ab \cos \theta + b^2 \sin^2 \theta \right]$$

$$\frac{1}{p^2} = \frac{1}{r^4} \left[a^2 + b^2 (\cos^2 \theta + \sin^2 \theta) + 2a (b \cos \theta) \right]$$

$$\frac{1}{p^2} = \frac{1}{r^4} (a^2 + b^2 + 2a(r-a))$$

$$\frac{1}{p^2} = \frac{1}{r^4} (a^2 + b^2 + 2a^2 + 2ar)$$

$$\frac{1}{p^2} = \frac{1}{r^4} (b^2 - a^2 + 2ar)$$

$$r^4 = p^2 (b^2 - a^2 + 2ar)$$

which is the required pedal equation.

Q # 15.

$$r = a(1 - \sin \theta) \quad \text{--- (1)}$$

Diff. w.r.t. 'θ'

$$\frac{dr}{d\theta} = -a \cos \theta \quad \text{--- (2)}$$

We know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} (-a \cos \theta)^2 \quad \text{from (2)}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} (a^2 \cos^2 \theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} \cos^2 \theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} (1 - \sin^2 \theta) \quad \text{--- (3)}$$

$$\text{Now } 0 \Rightarrow \frac{r}{a} = 1 - \sin \theta$$

$$\sin \theta = 1 - \frac{r}{a}$$

$$\sin \theta = \frac{a-r}{a}$$

Put in (3) $\Rightarrow$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} \left(1 - \left(\frac{a-r}{a} \right)^2 \right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} \left[1 - \frac{a^2 + r^2 - 2ar}{a^2} \right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} \left[\frac{a^2 - a^2 - r^2 + 2ar}{a^2} \right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4} \left(-\frac{r^2 + 2ar}{a^2} \right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(-r^2 + 2ar \right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{2ar}{r^4} - \frac{r^2}{r^4}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{2a}{r^3} - \frac{1}{r^2}$$

$$\frac{1}{p^2} = \frac{2a}{r^3}$$

$$r^3 = 2ap^2$$

Is the required Pedal Equation.

Q# 16. Show that the curves $r^m = a^m \cos m\theta$ and $r^m = a^m \sin m\theta$ cut each other orthogonally.

Soln.

Here $r^m = a^m \cos m\theta$ ——— ①

$r^m = a^m \sin m\theta$ ——— ②

Taking ln of both sides of ①

$$\Rightarrow m \ln r = m \ln a + \ln \cos m\theta$$

Diff. w.r.t. 'θ'

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\cos m\theta} \cdot -\sin m\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\tan m\theta$$

Reciprocating

$$r \frac{d\theta}{dr} = -\cot m\theta$$

$$\tan \psi_1 = -\cot m\theta$$

$$\tan \psi_1 = \tan \left(\frac{\pi}{2} + m\theta \right)$$

$$\psi_1 = m\theta \quad \text{————— ③}$$

Now Taking ln of both sides of ②

$$\Rightarrow m \ln r = m \ln a + \ln \sin m\theta$$

Diff. w.r.t. 'θ'

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{1}{\sin m\theta} \cdot \cos m\theta$$

$$\begin{aligned}\Rightarrow \frac{1}{r} \frac{dr}{d\theta} &= \cot m\theta \\ \Rightarrow r \frac{d\theta}{dr} &= \tan m\theta \\ \Rightarrow \tan \psi_2 &= \tan m\theta \\ \Rightarrow \psi_2 &= m\theta\end{aligned}$$

$$\text{Now } |\psi_1 - \psi_2| = \left| \frac{\pi}{2} + m\theta - m\theta \right| = \frac{\pi}{2}$$

Thus the curves cut each other orthogonally.

⑤ #17. Show that the tangents to the Cardioid

$$r = a(1 + \cos \theta)$$

at the point $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ are respectively parallel and perpendicular to the initial line.

Soln: Here $r = a(1 + \cos \theta)$ — (1)

Diff. w.r.t. ' θ ', we have

$$\begin{aligned}\frac{dr}{d\theta} &= -a \sin \theta \\ \text{So } \tan \psi &= r \frac{d\theta}{dr}\end{aligned}$$

$$\tan \psi = \frac{a(1 + \cos \theta)}{-a \sin \theta} = - \frac{1 + \cos \theta}{\sin \theta}$$

$$\tan \psi = - \frac{2 \cos^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}$$

$$\tan \psi = - \cot \theta/2$$

$$\tan \psi = \tan (\pi/2 + \theta/2)$$

$$\Rightarrow \psi = \pi/2 + \theta/2 \quad \text{--- (2)}$$

Now

$$\alpha = \theta + \psi$$

$$\psi = \alpha - \theta$$

$$\frac{\pi}{2} + \frac{\theta}{2} = \alpha - \theta$$

$$\Rightarrow \alpha = \frac{\pi}{2} + \frac{3\theta}{2} \quad \text{--- (3)}$$

Case (1).

$$\text{If } \theta = \frac{\pi}{3} \text{ then } (3) \Rightarrow$$

$$\alpha = \frac{\pi}{2} + \frac{3}{2} \cdot \frac{\pi}{3}$$

$$\alpha = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Note: i) the tangent will be to x-axis if $\alpha = 0$ or 180°

(ii) the tangent line will be to x-axis if $\alpha = 90^\circ$ or 270° .

$\Rightarrow$ the tangent at $\theta = \pi/3$ is || to the initial line.

Case 2.

If $\theta = \frac{2\pi}{3}$ then $3) \Rightarrow$

$$\begin{aligned}\alpha &= \frac{3 \cdot \frac{2\pi}{3}}{2} + \frac{\pi}{2} \\ &= \pi + \frac{\pi}{2} \\ &= \frac{3\pi}{2}\end{aligned}$$

$\Rightarrow$ The tangent is $\perp$ to the initial line.

Q#18. Show that $\tan \psi = \frac{x \frac{dy}{dx} - y}{y \frac{dy}{dx} + x}$

Soln.

We know that

$$\alpha = \theta + \psi$$

$$\psi = \alpha - \theta$$

$$\Rightarrow \tan \psi = \tan(\alpha - \theta)$$

$$\tan \psi = \frac{\tan \alpha - \tan \theta}{1 + \tan \alpha \tan \theta}$$

But we know that

$$\tan \alpha = \frac{dy}{dx} \quad \text{and} \quad \tan \theta = \frac{y}{x}$$

So

$$\tan \psi = \frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{dy}{dx} \cdot \frac{y}{x}}$$

$$\tan \psi = \frac{\frac{1}{x} [x \frac{dy}{dx} - y]}{\frac{1}{x} [x + y \frac{dy}{dx}]}$$

$$\tan \psi = \frac{x \frac{dy}{dx} - y}{x + y \frac{dy}{dx}}$$

$$\Rightarrow \tan \psi = \frac{x \frac{dy}{dx} - y}{y \frac{dy}{dx} + x}$$

is as required.

Q19. Show that at any point of the Lemniscate
 $r^2 = a^2 \cos 2\theta$, $0 \leq \theta \leq \frac{\pi}{4}$

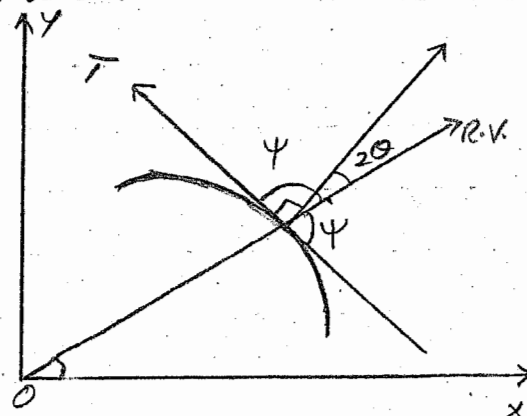
the measure of the angle between the radius vector and outward-pointed normal is 2θ .

Soln. Here

$$r^2 = a^2 \cos 2\theta \quad (1)$$

$$2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$



$$\begin{aligned} \text{Now } \tan \psi &= r \frac{d\theta}{dr} \\ &= r \cdot \frac{1}{-\frac{a^2 \sin 2\theta}{r}} \end{aligned}$$

$$= -\frac{r^2}{a^2 \sin 2\theta} = -\frac{a^2 \cos 2\theta}{a^2 \sin 2\theta}$$

$$= -\cot 2\theta$$

$$\tan \psi = \tan \left(\frac{\pi}{2} + 2\theta \right)$$

$$\Rightarrow \psi = \frac{\pi}{2} + 2\theta$$

Now if α is the required angle then

$$\alpha = \left| \left(\frac{\pi}{2} + 2\theta \right) - \frac{\pi}{2} \right| \quad \left| \psi - \frac{\pi}{2} \right|$$

$$\Rightarrow \alpha = 2\theta$$

Hence the measure of the angle b/w the radius vector and outward normal is 2θ .

Q#20. Find an equation (in rectangular coordinates) of the tangent to:

i) $r = a \sin 2\theta$ at $\left(1, \frac{\pi}{4} \right)$.

We know that in rectangular coordinates

$$x = r \cos \theta \quad (2)$$

$$y = r \sin \theta \quad (3)$$

then the point become $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

Put (1) into (2) and (3), we have

$$x = \sin 2\theta \cos \theta$$

$$\Rightarrow x = 2 \sin \theta \cos \theta \cdot \cos \theta$$

$$\Rightarrow x = 2 \sin \theta \cos^2 \theta \quad \text{--- (4)}$$

and $y = \sin 2\theta \sin \theta$

$$y = 2 \sin \theta \cos \theta \sin \theta$$

$$y = 2 \sin^2 \theta \cos \theta \quad \text{--- (5)}$$

Diff. (4) w.r.t. ' θ '

$$\begin{aligned} \frac{dx}{d\theta} &= 2 [\cos \theta \cos^2 \theta + \sin \theta \cdot 2 \cos \theta (-\sin \theta)] \\ &= 2 \cos^3 \theta - 4 \sin^2 \theta \cos \theta \quad \text{--- (6)} \end{aligned}$$

Diff. (5) w.r.t. ' θ '

$$\begin{aligned} \frac{dy}{d\theta} &= 2 [2 \sin \theta \cos \theta \cos \theta + \sin^2 \theta (-\cos \theta)] \\ &= 4 \sin \theta \cos^2 \theta - 2 \sin^3 \theta \quad \text{--- (7)} \end{aligned}$$

From (6) and (7)

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{4 \sin \theta \cos^2 \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - 4 \sin^2 \theta \cos \theta}$$

$$\frac{dy}{dx} \Big|_{(1, \pi/4)} = \frac{4 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2} - 2 \cdot \left(\frac{1}{\sqrt{2}}\right)^3}{2 \cdot \left(\frac{1}{\sqrt{2}}\right)^3 - 4 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2} - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} - \sqrt{2}}$$

$$\frac{dy}{dx} \Big|_{(1, \pi/4)} = \frac{2-1}{1-2} = \frac{1}{-1} = -1$$

So equation of the tangent is given by

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{\sqrt{2}} = -1 \left(x - \frac{1}{\sqrt{2}} \right)$$

$$x + y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$x + y = \frac{1+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$x + y = \sqrt{2}$$

which is the required eq. of line tangent to
 $r = \sin 2\theta$ at $(1, \pi/4)$

Q#20

$$r = 1 + \cos \theta \quad \text{at } (1, \pi/2)$$

(iii) Soln:

Changing the given eq. in rectangular coordinates

$$x = r \cos \theta \quad \text{--- (2)}$$

$$y = r \sin \theta \quad \text{--- (3)}$$

the the point become $(0, 1)$

$$2) \Rightarrow x = (1 + \cos \theta) \cos \theta$$

$$x = \cos \theta + \cos^2 \theta \quad \text{--- (4)}$$

$$3) \Rightarrow y = (1 + \cos \theta) \sin \theta$$

$$y = \sin \theta + \sin \theta \cos \theta \quad \text{--- (5)}$$

Diff. (4) w.r.t. ' θ '

$$\frac{dx}{d\theta} = -\sin \theta - 2 \cos \theta \sin \theta \quad \text{--- (6)}$$

Diff. (5) w.r.t. ' θ '

$$\frac{dy}{d\theta} = \cos \theta - \sin^2 \theta + \cos^2 \theta \quad \text{--- (7)}$$

So by chain Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{\cos \theta - \sin^2 \theta + \cos^2 \theta}{-\sin \theta - 2 \cos \theta \sin \theta}$$

$$m = \left. \frac{dy}{dx} \right|_{(1, \pi/2)} = \frac{0 - 1 + 0}{-1 - 0} = 1$$

We know that the eq. of the tangent is given by

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 1(x - 0)$$

$$y - 1 = x$$

$$x - y + 1 = 0$$

Is the required eq. of the line tangent to
 $r = 1 + \cos \theta$ at $(1, \pi/2)$

Exercise 6.7

Parametric representation of curves

Let $y = f(x)$ ——— I

be any curve in rectangular coordinates.

Then $x = x(t)$ and $y = y(t)$

represent parametric form of (I)

We know that Eq. of the circle is

$$x^2 + y^2 = a^2$$

Parametric form of this circle is

$$x = a \cos t, \quad y = a \sin t.$$

Similarly parametric form the parabola

$$y^2 = 4ax$$

is

$$x = at^2, \quad y = 2at.$$

Similarly parametric form of the Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$x = a \cos \theta, \quad y = b \sin \theta.$$

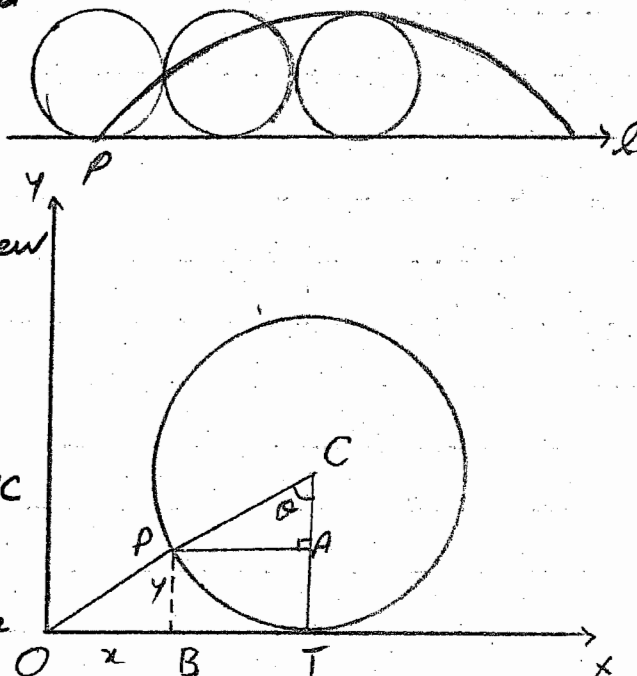
The Cycloid.

Def: The cycloid is a curve described by a point marked on the circumference of a circle as it rolls without slipping or sliding along a fixed right line.

Let the line l be taken as x -axis and let the marked point P be at O . Let the new position be as shown.

Draw PA and $PB \perp$ to CT and x -axis.

Let $\angle PCA = \theta$ and $CT = a = PC$ the radius of the circle. Let the coordinates of P are (x, y) distance we note that the distance travelled along x -axis = OT



and Distance along the arc of the circle = PT

$$OT = \widehat{PT} = a\theta$$

$$OT = a\theta$$

$$l = r\theta$$

$$x = OB = OT - BT$$

$$x = OT - PA$$

$$x = a\theta - a \sin\theta$$

$$\therefore PT = BT$$

$$\frac{PA}{PC} = \sin\theta$$

$$PA = \sin\theta \cdot PC$$

Now

$$y = PB = AT$$

$$y = CT - CA$$

$$y = a - a \cos\theta$$

$$\text{So } x = a(\theta - \sin\theta)$$

$$y = a(1 - \cos\theta)$$

are the parametric equations of the cycloid.

Written by

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Available at <http://www.MathCity.org>

If you have any question, ask it at

<http://forum.mathcity.org>

Exercise 6.7

Written by
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Exercise 6.7.

Find parametric eqs. of the given curves (1-3)

Q#1

$$r = a \sin 2\theta, \quad 0 \leq \theta \leq 2\pi$$

Soln.

For parametric eqs.

Put $x = r \cos \theta$

$$\Rightarrow x = a \sin 2\theta \cos \theta$$

and $y = r \sin \theta$

$$\Rightarrow y = a \sin 2\theta \sin \theta$$

Hence $x = a \sin 2\theta \cos \theta$

$y = a \sin 2\theta \sin \theta$ are the required

parametric eqs.

Q#2

Soln.

$$r = \theta$$

$$x = r \cos \theta$$

$$x = \theta \cos \theta$$

and $y = r \sin \theta$

$$y = \theta \sin \theta$$

So parametric eqs. of $r = \theta$ are

$$x = \theta \cos \theta$$

$$y = \theta \sin \theta$$

Q#3

Soln.

$\therefore$ For parametric eqs.

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\Rightarrow x = (2 + 3 \sin \theta) \cos \theta, \quad y = (2 + 3 \sin \theta) \sin \theta$$

are the required parametric eqs.

Q#4

Show that the equations

$$x = a + r \cos \theta, \quad y = b + r \sin \theta$$

are parametric eqs. for a circle with centre (a, b) and radius $|r|$.

Soln.

Here $x = a + r \cos \theta$ _____ (1)

$$y = b + r \sin \theta$$
 _____ (2)

$$1) \Rightarrow x - a = r \cos \theta$$
 _____ (3)

$$2) \Rightarrow y - b = r \sin \theta$$
 _____ (4)

Now $(3)^2 + (4)^2 \Rightarrow$

$$(x-a)^2 + (y-b)^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$(x-a)^2 + (y-b)^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$(x-a)^2 + (y-b)^2 = r^2$$

which represent a circle with centre at (a, b) and radius $|r|$.

⑤ #5 Show that the curve whose parametric eqs. are

$$\left. \begin{aligned} x &= a \cos \theta + h \\ y &= b \sin \theta + k \end{aligned} \right\}, \quad 0 \leq \theta \leq 2\pi$$

is an ellipse with centre (h, k)

Soln:

Here $x = a \cos \theta + h$ _____ (1)

$y = b \sin \theta + k$ _____ (2)

1) $\Rightarrow x - h = a \cos \theta$

$\frac{x-h}{a} = \cos \theta$ _____ (3)

2) $\Rightarrow y - k = b \sin \theta$

$\frac{y-k}{b} = \sin \theta$ _____ (4)

Now $(3)^2 + (4)^2 \Rightarrow$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \cos^2 \theta + \sin^2 \theta$$

$$\Rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

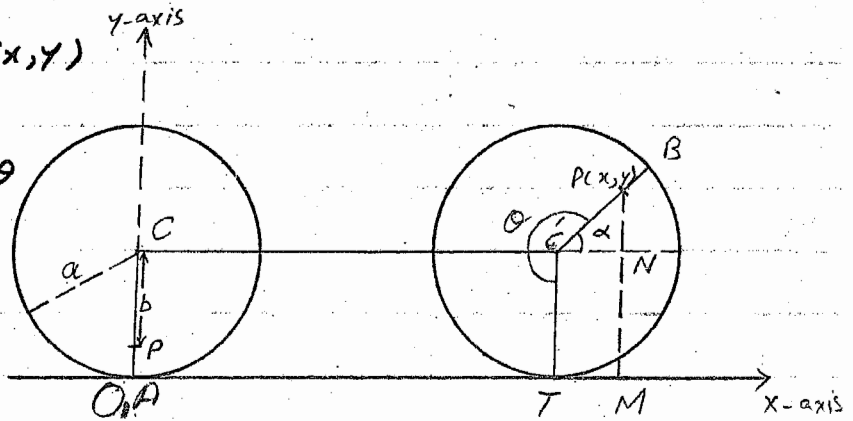
which represent an ellipse with centre (h, k) .

⑥ #6. A wheel of radius a rolls on a straight line without slipping or sliding. Let P be a fixed point on the wheel, at a distance b from the centre of the wheel. Find parametric eqs. of the curve described by the point P . The curve is called a trochoid. Hence deduce parametric equations for a cycloid.

Soln: Let the wheel be initially in contact with the line at O . Where the line is x -axis, $CP = b$. Let the wheel rotate through an angle θ as reflected in the end fig.

Let $\widehat{PCN} = \alpha$ and $P = (x, y)$
 then $\widehat{BCT} = \theta$

and $OT = TB = a\theta \quad \therefore l = r\theta$
 (because the radial
 distance of P is equal
 to the distance along
 x-axis)



Now $x = OT + TM$

$x = OT + C'N$

$x = a\theta + b \cos \alpha \quad \text{--- (1)}$

and $y = PN + NM$

$\therefore NM = C'T = a$

$y = b \sin \alpha + a \quad \text{--- (2)}$

$\therefore \alpha + \theta = \frac{3\pi}{2}$

$\Rightarrow \alpha = \frac{3\pi}{2} - \theta$

Putting $\alpha = \frac{3\pi}{2} - \theta$ into (1) and (2), we have

$x = a\theta + b \cos(\frac{3\pi}{2} - \theta) \Rightarrow x = a\theta - b \sin \theta$

$y = b \sin(\frac{3\pi}{2} - \theta) + a \Rightarrow y = -b \cos \theta + a$

Hence the parametric eqs. of trochoid are

$x = a(\theta) - b \sin \theta \quad \text{--- (3)}$

$y = a - b \cos \theta \quad \text{--- (4)}$

Put $b = a$ in (3) and (4), we get

$x = a\theta - a \sin \theta \Rightarrow x = a(\theta - \sin \theta)$

$y = a - a \cos \theta \Rightarrow y = a(1 - \cos \theta)$

So $x = a(\theta - \sin \theta)$

$y = a(1 - \cos \theta)$

are eqs. of a cycloid.

Q#7. Find the points at which $r = 1 + \cos \theta$ has horizontal and vertical tangents.

Here $r = 1 + \cos \theta$ ——— (1)

Diff. w.r.t. ' θ ' $\frac{dr}{d\theta} = -\sin \theta$

$\Rightarrow \frac{d\theta}{dr} = -\frac{1}{\sin \theta}$

So $\tan \psi = r \frac{d\theta}{dr}$
 $= 1 + \cos \theta \cdot -\frac{1}{\sin \theta}$

$= -\frac{2 \cos^2 \theta/2}{2 \sin \theta/2 \cos \theta/2}$

$= -\cot \theta/2$

$= \tan(\frac{\pi}{2} + \theta/2)$

$\Rightarrow \psi = \frac{\pi}{2} + \theta/2$

But $\alpha = \theta + \psi$

$\Rightarrow \alpha = \theta + \frac{\pi}{2} + \theta/2$

$\alpha = \frac{3\theta}{2} + \frac{\pi}{2}$ ——— (2)

For horizontal tangents.

For these tangents we may have the following two cases.

Case 1. If $\alpha = 0$ Put in (2)

$2) \Rightarrow 0 = \frac{3\theta}{2} + \frac{\pi}{2}$

$0 = 3\theta + \pi$

$\theta = -\frac{\pi}{3} = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\Rightarrow$ at $\theta = -\frac{\pi}{3}$ the tangent is $\parallel$ to x-axis.

Put $\frac{5\pi}{3}$ in (1)

$\Rightarrow r = \frac{3}{2}$

$\Rightarrow$ the tangent is horizontal at $(\frac{3}{2}, \frac{5\pi}{3})$

Case 2. If $\alpha = \pi$

Put in (2) $\Rightarrow \pi = \frac{3\theta}{2} + \frac{\pi}{2}$

$\frac{3\theta}{2} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow$ at $\theta = \frac{\pi}{3}$ the tangent is horizontal.

Put $\frac{\pi}{3}$ in (1)

$\Rightarrow r = \frac{3}{2}$

$\Rightarrow$ the tangent is horizontal at $(\frac{3}{2}, \frac{\pi}{3})$

For Vertical Tangents

Case 1

Two cases arise here

if $\alpha = \frac{\pi}{2}$, put in (2)

$$\Rightarrow \frac{\pi}{2} = \frac{3\theta}{2} + \frac{\pi}{2}$$

$$\Rightarrow \frac{3\theta}{2} = 0$$

$$\Rightarrow \theta = 0$$

$\Rightarrow$ at $\theta = 0$ the tangent is vertical

Case 2

if $\alpha = \frac{3\pi}{2}$, put in (2)

$$\Rightarrow \frac{3\pi}{2} = \frac{3\theta}{2} + \frac{\pi}{2}$$

$$\Rightarrow \frac{3\theta}{2} = \frac{3\pi}{2} - \frac{\pi}{2}$$

$$\Rightarrow 3\theta = 3\pi - \pi$$

$$\Rightarrow 3\theta = 2\pi$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

$\Rightarrow$ at $\theta = \frac{2\pi}{3}$ the tangent is vertical.

Q#8 Find the points on the curve

$$x(t) = t^2 + 4, \quad y(t) = 3t^2 - 6t + 2$$

where tangents are horizontal and vertical.

Soln:

Here $x = t^2 + 4$ ——— (i)

and $y = 3t^2 - 6t + 2$ ——— (ii)

Now Diff. (i) w.r.t. t

$$\frac{dx}{dt} = 2t \quad \text{——— (iii)}$$

Also Diff. (ii) w.r.t. t

$$\frac{dy}{dt} = 6t - 6 \quad \text{——— (iv)}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t-6}{2t} = \frac{6(t-1)}{2t}$$

$$\frac{dy}{dx} = \frac{3(t-1)}{t} \quad \text{——— (v)}$$

Now for horizontal tangent

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dt} = 0 \quad \text{and} \quad \frac{dx}{dt} \neq 0$$

Put $\theta = 0$ in (1)

$\Rightarrow r = 2$

$\Rightarrow$ the tangent is vertical at $(2, 0)$

Put $\theta = \frac{2\pi}{3}$ in (1)

$\Rightarrow r = \frac{1}{2}$

$\Rightarrow$ the tangent is vertical at $(\frac{1}{2}, \frac{2\pi}{3})$

So for horizontal tangent-

$$\frac{dy}{dt} = 0$$

$$6t - 6 = 0$$

$$6t = 6$$

$$\Rightarrow t = 1$$

Put $t = 1$ in (1) and (2)

$$\Rightarrow x = 1 + 4 = 5$$

$$\text{and } y = 3 - 6 + 2 = -1$$

$\Rightarrow$ at $(5, -1)$ the tangent is $\parallel$ to x-axis.

For Vertical Tangent-

$$\frac{dy}{dx} = \infty$$

$$\Rightarrow \frac{dx}{dt} = 0$$

$$2t = 0$$

$$t = 0$$

Put in (1) and (2)

$$\Rightarrow x = 0 + 4 = 4$$

$$y = 0 - 0 + 2 = 2$$

$\Rightarrow$ at $(4, 2)$ the tangent is vertical.

Find equations of the tangent and Normal to each of the given curves at indicated point. (9-11)

Q#9

Soln.

$$x = 2a \cos \theta - a \cos 2\theta, \quad y = 2a \sin \theta - a \sin 2\theta \text{ at } \theta = \frac{\pi}{2}$$

$$\text{Here } x = 2a \cos \theta - a \cos 2\theta \quad \text{--- (i)}$$

$$y = 2a \sin \theta - a \sin 2\theta \quad \text{--- (ii)}$$

$$\text{I) } \Rightarrow \frac{dx}{d\theta} = -2a \sin \theta + 2a \sin 2\theta$$

$$= 2a (\sin 2\theta - \sin \theta)$$

$$= 2a \cdot 2 \cos \frac{2\theta + \theta}{2} \cdot \sin \frac{2\theta - \theta}{2}$$

$$= 4a \cos \frac{3\theta}{2} \sin \frac{\theta}{2} \quad \text{--- (iii)}$$

$$\text{II) } \Rightarrow \frac{dy}{d\theta} = 2a \cos \theta - 2a \cos 2\theta$$

$$= 2a (\cos \theta - \cos 2\theta)$$

$$= 2a \cdot 2 \sin \frac{\theta + 2\theta}{2} \sin \frac{\theta - 2\theta}{2}$$

$$\begin{aligned}\frac{dy}{d\theta} &= -4a \sin \frac{3\theta}{2} \sin \left(-\frac{\theta}{2}\right) \\ &= 4a \sin \frac{3\theta}{2} \sin \frac{\theta}{2}\end{aligned}$$

Now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{4a \sin \frac{3\theta}{2} \sin \frac{\theta}{2}}{4a \cos \frac{3\theta}{2} \sin \frac{\theta}{2}}$$

$$\frac{dy}{dx} = \frac{\sin \frac{3\theta}{2}}{\cos \frac{3\theta}{2}}$$

$$\frac{dy}{dx} = \tan \frac{3\theta}{2}$$

$$\left. \frac{dy}{dx} \right|_{\pi/2} = \tan \frac{3}{2} \cdot \frac{\pi}{2}$$

$$\left. \frac{dy}{dx} \right|_{\pi/2} = \tan \frac{3\pi}{4} = -1$$

Also $x_1 = 2a \cos \frac{\pi}{2} - a \cos 2 \cdot \frac{\pi}{2} = 0 - a(-1) = a$

$$y_1 = 2a \sin \frac{\pi}{2} - a \sin 2 \cdot \frac{\pi}{2} = 2a - 0 = 2a$$

Thus the eq. of the tangent is

$$y - y_1 = \left. \frac{dy}{dx} \right|_p (x - x_1)$$

$$y - 2a = -1(x - a)$$

$$y - 2a = -x + a$$

$$x + y - 2a - a = 0$$

$$x + y - 3a = 0$$

IN the eq. of the tangent at $\pi/2$

Now eq. of the normal

$$y - y_1 = -\frac{1}{dy/dx} (x - x_1)$$

$$y - 2a = -\frac{1}{-1} (x - a)$$

$$y - 2a = x - a$$

$$x - y - a + 2a = 0$$

$$x - y + a = 0$$

Is the eq. of the tangent at $\pi/2$

③ #10.
soln:

$$x = \frac{2at^2}{1+t^2}, \quad y = \frac{2at^3}{1+t^2} \quad \text{at } t = \frac{1}{2}$$

$$x = \frac{2at^2}{1+t^2} \quad \text{--- (1)}$$

$$y = \frac{2at^3}{1+t^2} \quad \text{--- (2)}$$

$$1) \Rightarrow \frac{dx}{dt} = \frac{(1+t^2) \cdot 4at - 2at^2 \cdot 2t}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{4at - 4at^3 - 4at^3}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{4a(1-t^4)}{(1+t^2)^2} \quad \text{--- (3)}$$

$$2) \Rightarrow \frac{dy}{dt} = \frac{(1+t^2) \cdot 6at^2 - 2at^3 \cdot 2t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at^2 + 6at^4 - 4at^4}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at^2 + 2at^4}{(1+t^2)^2}$$

$$= \frac{2a(3t^2 + t^4)}{(1+t^2)^2}$$

Now

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2a(3t^2 + t^4)}{(1+t^2)^2} \cdot \frac{(1+t^2)^2}{4a(1-t^4)}$$

$$\frac{dy}{dx} = \frac{2a(3t + t^3)}{4a(1-t^4)}$$

$$\frac{dy}{dx} = \frac{3t + t^3}{2}$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{1}{2}} = \frac{3(\frac{1}{2}) + (\frac{1}{2})^3}{2}$$

$$= \frac{\frac{3}{2} + \frac{1}{8}}{2} = \frac{\frac{12+1}{8}}{2}$$

$$= \frac{13}{16}$$

Now

$$\begin{aligned}
 x_1 &= \frac{2a(1/2)^2}{1+(1/2)^2}, & y_1 &= \frac{2a(1/2)^3}{1+(1/2)^2} \\
 x_1 &= \frac{2a \cdot 1/4}{1+1/4}, & y_1 &= \frac{2a(1/8)}{1+1/4} \\
 x_1 &= \frac{a/2}{5/4}, & y_1 &= \frac{a/4}{5/4} \\
 x_1 &= \frac{2a}{5}, & y_1 &= \frac{a}{5} \\
 \Rightarrow (x_1, y_1) &= \left(\frac{2a}{5}, \frac{a}{5}\right)
 \end{aligned}$$

So Eq. of the tangent at $t = \frac{1}{2}$ is

$$\begin{aligned}
 y - \frac{a}{5} &= \frac{13}{16} \left(x - \frac{2a}{5}\right) \\
 16y - \frac{16a}{5} &= 13 \left(x - \frac{2a}{5}\right)
 \end{aligned}$$

$$80y - 16a = 65x - 26a$$

$$65x - 80y - 10a = 0$$

$$13x - 16y - 2a = 0$$

which is the eq. of tangent.

Now Eq. of the Normal is

$$y - \frac{a}{5} = -\frac{16}{13} \left(x - \frac{2a}{5}\right)$$

$$65y - 13a = -80x + 32a$$

$$80x + 65y - 45a = 0$$

$$16x + 13y - 9a = 0$$

which is the eq. of the Normal at $t = \frac{1}{2}$

Q# 21.

$$\begin{aligned}
 x &= (t-1)^{3/2}, & y &= 3t & \text{at } t=5 \\
 \text{Here}
 \end{aligned}$$

$$x = (t-1)^{3/2} \quad \text{--- (I)}$$

$$y = 3t \quad \text{--- (II)}$$

Now 1) $\Rightarrow$

$$\frac{dx}{dt} = \frac{3}{2} (t-1)^{1/2}$$

2) $\Rightarrow$

$$\frac{dy}{dt} = 3$$

Now $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{3}{\frac{3}{2}\sqrt{t-1}} = \frac{2}{\sqrt{t-1}}$

$$\left. \frac{dy}{dx} \right|_{t=5} = \frac{2}{\sqrt{5-1}} = \frac{2}{\sqrt{4}} = \frac{2}{2} = 1$$

The point $P(x_1, y_1)$ at $t = 5$

$$x_1 = (5-1)^{3/2} \\ = (4)^{3/2} = 8$$

and $y_1 = 3(5)$

$$y_1 = 15$$

Hence the point $P(x_1, y_1)$ at $t = 5$ is $(8, 15)$

So the Eq. of the tangent is given by

$$y - 15 = 1(x - 8)$$

$$x - y + 7 = 0$$

Eq. of the Normal is

$$y - 15 = -1(x - 8)$$

$$x + y - 23 = 0$$

Q.12 Show that the Normal at any point of the curve $x = a \cos \theta + a \theta \sin \theta$

$$y = a \sin \theta - a \theta \cos \theta$$

is at a constant distance from the origin.

Soln: Here $x = a \cos \theta + a \theta \sin \theta$ _____ (i)

$$y = a \sin \theta - a \theta \cos \theta$$
 _____ (ii)

$$1) \Rightarrow \frac{dx}{d\theta} = -a \sin \theta + a \theta \cos \theta + a \sin \theta \\ = a \theta \cos \theta$$

$$2) \Rightarrow \frac{dy}{d\theta} = a \cos \theta + a \theta \sin \theta - a \cos \theta \\ = a \theta \sin \theta$$

But $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a \theta \sin \theta}{a \theta \cos \theta} = \tan \theta$

Now the Eq. of the Normal at $(a \cos \theta + a \theta \sin \theta, a \sin \theta - a \theta \cos \theta)$

$$y - y_1 = -\frac{1}{dy/dx} (x - x_1)$$

$$y - (a \sin \theta - a \theta \cos \theta) = -\frac{1}{\tan \theta} (x - (a \cos \theta + a \theta \sin \theta))$$

$$y - a \sin \theta + a \theta \cos \theta = - \frac{\cos \theta}{\sin \theta} [x - a \cos \theta - a \theta \sin \theta]$$

x by $\sin \theta$

$$y \sin \theta - a \sin^2 \theta + a \theta \sin \theta \cos \theta = -x \cos \theta + a \cos^2 \theta + a \theta \sin \theta \cos \theta$$

$$\Rightarrow x \cos \theta + y \sin \theta - a \sin^2 \theta - a \cos^2 \theta = 0$$

$$x \cos \theta + y \sin \theta - a (\sin^2 \theta + \cos^2 \theta) = 0$$

$$x \cos \theta + y \sin \theta - a = 0 \quad \text{--- iii}$$

which is the eq. of Normal

Now If d is the distance of this normal from $O(0,0)$ then

$$d = \frac{|0 + 0 - a|}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$$

$$d = \frac{a}{1} = a$$

= Constant.

Q #13. Prove that an equation of the Normal to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ can be written in the form

$$x \sin t - y \cos t + a \cos 2t = 0, \quad t \text{ being parameter.}$$

Soln: The parametric eq. of the astroid is

$$x = a \cos^3 t \quad \text{--- (1)} \quad \text{and} \quad y = a \sin^3 t \quad \text{--- (2)}$$

$$1) \Rightarrow \frac{dx}{dt} = -3a \sin^2 t \cos t$$

$$2) \Rightarrow \frac{dy}{dt} = 3a \cos^2 t \sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = - \frac{3a \cos^2 t \sin t}{3a \sin^2 t \cos t}$$

$$= - \frac{\cos t}{\sin t}$$

So eq. of the Normal at $(a \cos^3 t, a \sin^3 t)$

$$\text{is} \quad y - a \sin^3 t = \frac{\sin t}{\cos t} (x - a \cos^3 t)$$

$$\Rightarrow y \cos t - a \cos^4 t = x \sin t - a \sin^4 t$$

$$\Rightarrow x \sin^4 t - y \cos^4 t + a \cos^4 t - a \sin^4 t = 0$$

$$x \sin^4 t - y \cos^4 t + a (\cos^4 t - \sin^4 t) = 0$$

$$x \sin^4 t - y \cos^4 t + a [(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)] = 0$$

$$x \sin^4 t - y \cos^4 t + a (\cos 2t) = 0$$

$$x \sin^4 t - y \cos^4 t + a \cos 2t = 0$$

which is the required eq. of the Normal.

Q#14. Show that the pedal eq. of the astroid

$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta$$

is $r^2 = a^2 - 3p^2$

Soln.

Here $x = a \cos^3 \theta$ _____ (1)

$y = a \sin^3 \theta$ _____ (2)

1) $\Rightarrow \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$

2) $\Rightarrow \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$

But $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = - \frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta \sin \theta}$
 $= - \frac{\sin \theta}{\cos \theta}$

Now Eq. of the tangent at $(a \cos^3 \theta, a \sin^3 \theta)$ is given by

$$y - a \sin^3 \theta = - \frac{\sin \theta}{\cos \theta} (x - a \cos^3 \theta)$$

$$y \cos \theta - a \sin^3 \theta \cos \theta = -x \sin \theta + a \cos^3 \theta \sin \theta$$

$$\sin \theta x + \cos \theta y - a [\sin^3 \theta \cos \theta + \cos^3 \theta \sin \theta] = 0$$

$$\sin \theta x + \cos \theta y - a \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) = 0$$

$$\sin \theta x + \cos \theta y - a \sin \theta \cos \theta = 0$$

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 + 0 - a \sin \theta \cos \theta|}{\sqrt{\sin^2 \theta + \cos^2 \theta}}$$

$$p = a \sin \theta \cos \theta$$

$$\frac{p}{a} = \sin \theta \cos \theta \quad \text{--- (3)}$$

$$r^2 = x^2 + y^2$$

$$r^2 = a^2 \cos^2 \theta + a^2 \sin^2 \theta$$

$$r^2 = a^2 (\cos^2 \theta + \sin^2 \theta)$$

$$r^2 = a^2 ((\cos^2 \theta)^3 + (\sin^2 \theta)^3)$$

$$r^2 = a^2 [(\cos^2 \theta + \sin^2 \theta)^3 - 3(\cos^2 \theta \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta)]$$

$$r^2 = a^2 [1 - 3 \cos^2 \theta \sin^2 \theta] \quad \text{--- (4)}$$

Put (3) in (4)

$$r^2 = a^2 \left[1 - 3 \frac{p^2}{a^2} \right]$$

$$\Rightarrow r^2 = a^2 - 3p^2$$

Is the required pedal Equation.

Q#15. Prove that the pedal Eq. of the curve $x = 2a \cos \theta - a \cos 2\theta$, $y = 2a \sin \theta - a \sin 2\theta$ is $9(r^2 - a^2) = 8p^2$

Soln:

Here

$$x = 2a \cos \theta - a \cos 2\theta \quad \text{--- I}$$

$$y = 2a \sin \theta - a \sin 2\theta \quad \text{--- II}$$

$$I) \Rightarrow \frac{dx}{d\theta} = -2a \sin \theta + 2a \sin 2\theta$$

$$= 2a [\sin 2\theta - \sin \theta]$$

$$II) \Rightarrow \frac{dy}{d\theta} = 2a \cos \theta - 2a \cos 2\theta$$

$$= 2a [\cos \theta - \cos 2\theta]$$

$$\text{But } \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{2a [\cos \theta - \cos 2\theta]}{2a [\sin 2\theta - \sin \theta]}$$

$$= \frac{-2 \sin \frac{\theta+2\theta}{2} \sin \frac{\theta-2\theta}{2}}{2 \cos \frac{2\theta+\theta}{2} \sin \frac{2\theta-\theta}{2}}$$

$$= \frac{\sin 3\theta/2 \cdot \sin (-\theta/2)}{\cos 3\theta/2 \cdot \sin \theta/2}$$

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

$$\frac{dy}{dx} = \frac{\sin 3\theta/2}{\cos 3\theta/2}$$

Eq. of the tangent at $(2a \cos \theta - a \cos 2\theta, 2a \sin \theta - a \sin 2\theta)$ is
 $y - 2a \sin \theta + a \sin 2\theta = \frac{\sin 3\theta/2}{\cos 3\theta/2} (x - 2a \cos \theta + a \cos 2\theta)$

$$\Rightarrow \cos \frac{3\theta}{2} y - 2a \sin \theta \cos \frac{3\theta}{2} + a \sin 2\theta \cos \frac{3\theta}{2} = x \sin \frac{3\theta}{2} - 2a \cos \theta \sin \frac{3\theta}{2} + a \cos 2\theta \sin \frac{3\theta}{2}$$

$$\Rightarrow x \sin \frac{3\theta}{2} - y \cos \frac{3\theta}{2} - 2a \left[\sin \frac{3\theta}{2} \cos \theta - \sin \theta \cos \frac{3\theta}{2} \right] + a \left[\sin \frac{3\theta}{2} \cos 2\theta - \sin 2\theta \cos \frac{3\theta}{2} \right] = 0$$

$$\Rightarrow x \sin \frac{3\theta}{2} - y \cos \frac{3\theta}{2} - 2a \left[\sin \left(\frac{3\theta}{2} - \theta \right) \right] + a \left[\sin \left(\frac{3\theta}{2} - 2\theta \right) \right] = 0$$

$$\Rightarrow x \sin \frac{3\theta}{2} - y \cos \frac{3\theta}{2} - 2a \sin \frac{\theta}{2} - a \sin \frac{\theta}{2} = 0$$

$$\Rightarrow x \sin \frac{3\theta}{2} - y \cos \frac{3\theta}{2} - 3a \sin \frac{\theta}{2} = 0$$

which is the equation of tangent.

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 - 0 - 3a \sin \frac{\theta}{2}|}{\sqrt{\sin^2 \frac{3\theta}{2} + \cos^2 \frac{3\theta}{2}}}$$

$$p = 3a \sin \frac{\theta}{2} \quad \text{--- III}$$

Now

$$r^2 = x^2 + y^2$$

$$r^2 = (2a \cos \theta - a \cos 2\theta)^2 + (2a \sin \theta - a \sin 2\theta)^2$$

$$r^2 = 4a^2 \cos^2 \theta + a^2 \cos^2 2\theta - 4a^2 \cos \theta \cos 2\theta + 4a^2 \sin^2 \theta + a^2 \sin^2 2\theta - 4a^2 \sin \theta \sin 2\theta$$

$$r^2 = 4a^2 (\cos^2 \theta + \sin^2 \theta) + a^2 (\cos^2 2\theta + \sin^2 2\theta) - 4a^2 [\cos 2\theta \cos \theta + \sin 2\theta \sin \theta]$$

$$r^2 = 4a^2 + a^2 - 4a^2 \cos(2\theta - \theta)$$

$$r^2 = 5a^2 - 4a^2 \cos \theta$$

$$r^2 = 5a^2 - 4a^2 \left[1 - 2 \sin^2 \frac{\theta}{2} \right]$$

$$r^2 = 5a^2 - 4a^2 + 8a^2 \sin^2 \frac{\theta}{2}$$

$$r^2 = a^2 + 8a^2 \sin^2 \frac{\theta}{2}$$

$$r^2 - a^2 = 8a^2 \sin^2 \frac{\theta}{2} \quad \text{--- IV}$$

Put (3) in (4)

$$\Rightarrow r^2 - a^2 = \frac{8c^2 b^2}{9a^2}$$

$$9(r^2 - a^2) = 8b^2$$

which is the required pedal Equation.

Q#16. Show that the pedal Equation of the curve
 $x = ae^\theta (\sin \theta - \cos \theta)$, $y = ae^\theta (\sin \theta + \cos \theta)$
 is $r = \sqrt{2} p$

Soln.

$$\text{Here } x = ae^\theta (\sin \theta - \cos \theta) \quad \text{--- (1)}$$

$$y = ae^\theta (\sin \theta + \cos \theta) \quad \text{--- (2)}$$

$$\begin{aligned} 1) \Rightarrow \frac{dx}{d\theta} &= ae^\theta (\sin \theta - \cos \theta) + ae^\theta (\cos \theta + \sin \theta) \\ &= ae^\theta \sin \theta - ae^\theta \cos \theta + ae^\theta \cos \theta + ae^\theta \sin \theta \\ &= 2ae^\theta \sin \theta \end{aligned}$$

$$\begin{aligned} 2) \Rightarrow \frac{dy}{d\theta} &= ae^\theta (\sin \theta + \cos \theta) + ae^\theta (\cos \theta - \sin \theta) \\ &= ae^\theta \sin \theta + ae^\theta \cos \theta + ae^\theta \cos \theta - ae^\theta \sin \theta \\ &= 2ae^\theta \cos \theta \end{aligned}$$

$$\begin{aligned} \text{But } \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^\theta \cos \theta}{2ae^\theta \sin \theta} \\ &= \frac{\cos \theta}{\sin \theta} \end{aligned}$$

Eq. of the tangent at $(ae^\theta (\sin \theta - \cos \theta), ae^\theta (\sin \theta + \cos \theta))$ is

$$y - ae^\theta \sin \theta - ae^\theta \cos \theta = \frac{\cos \theta}{\sin \theta} (x - ae^\theta \sin \theta + ae^\theta \cos \theta)$$

$$\begin{aligned} y \sin \theta - ae^\theta \sin^2 \theta - ae^\theta \sin \theta \cos \theta &= x \cos \theta - ae^\theta \sin \theta \cos \theta + ae^\theta \cos^2 \theta \\ x \cos \theta - y \sin \theta + ae^\theta (\cos^2 \theta + \sin^2 \theta) &= 0 \end{aligned}$$

$$\begin{aligned} 1 - \cos \theta &= 2 \sin^2 \frac{\theta}{2} \\ \Rightarrow 1 - 2 \sin^2 \frac{\theta}{2} &= \cos \theta \end{aligned}$$

$$x \cos \theta - y \sin \theta - a e^{\theta} = 0$$

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 - 0 - a e^{\theta}|}{\sqrt{\cos^2 \theta + \sin^2 \theta}}$$

$$p = a e^{\theta} \quad \text{--- III}$$

$$\therefore r^2 = x^2 + y^2$$

$$r^2 = a^2 e^{2\theta} \{ (\sin \theta - \cos \theta)^2 + (\sin \theta + \cos \theta)^2 \}$$

$$r^2 = a^2 e^{2\theta} [\sin^2 \theta + \cos^2 \theta - 2 \sin \theta \cos \theta + \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta]$$

$$r^2 = 2 a^2 e^{2\theta} \quad \text{--- IV}$$

Put III in IV

$$r^2 = 2 p^2$$

$$r = \sqrt{2} p$$

which is the required pedal Equation.

Q#17. Prove that the pedal Equation of the Curve

$$x = a(3 \cos \theta - \cos^3 \theta), \quad y = a(3 \sin \theta - \sin^3 \theta)$$

$$\text{is } 3p^2(7a^2 - r^2) = (10a^2 - r^2)^2.$$

Soln.

Here

$$x = a(3 \cos \theta - \cos^3 \theta) \quad \text{--- (1)}$$

$$y = a(3 \sin \theta - \sin^3 \theta) \quad \text{--- (2)}$$

$$\begin{aligned} 1) \Rightarrow \frac{dx}{d\theta} &= -3a \sin \theta + 3a \cos^2 \theta \sin \theta \\ &= 3a \sin \theta [\cos^2 \theta - 1] \\ &= 3a \sin \theta [-\sin^2 \theta] \\ &= -3a \sin^3 \theta \end{aligned}$$

$$\begin{aligned} 2) \Rightarrow \frac{dy}{d\theta} &= 3a \cos \theta - 3a \sin^2 \theta \cos \theta \\ &= 3a \cos \theta [1 - \sin^2 \theta] \\ &= 3a \cos \theta [\cos^2 \theta] \\ &= 3a \cos^3 \theta \end{aligned}$$

$$\text{But } \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{3a \cos^3 \theta}{-3a \sin^3 \theta}$$

$$\frac{dy}{dx} = - \frac{\cos^3 \theta}{\sin^3 \theta}$$

Eq. of the tangent at $(a(3\cos\theta - \cos^3\theta), a(3\sin\theta - \sin^3\theta))$

$$y - 3a\sin\theta + a\sin^3\theta = - \frac{\cos^3\theta}{\sin^3\theta} (x - 3a\cos\theta + a\cos^3\theta)$$

$$\begin{aligned} y\sin^3\theta - 3a\sin^4\theta + a\sin^6\theta &= -x\cos^3\theta + 3a\cos^4\theta - a\cos^6\theta \\ x\cos^3\theta + y\sin^3\theta + a[\sin^6\theta + \cos^6\theta] &= 3a[\sin^4\theta + \cos^4\theta] \\ x\cos^3\theta + y\sin^3\theta + a[(\sin^2\theta)^3 + (\cos^2\theta)^3] &= 3a[(\sin^2\theta)^2 + (\cos^2\theta)^2] \\ x\cos^3\theta + y\sin^3\theta + a[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] &= 3a[(\sin^2\theta + \cos^2\theta)^2 - 2\cos^2\theta\sin^2\theta] \\ x\cos^3\theta + y\sin^3\theta + a[1 - 3\sin^2\theta\cos^2\theta] &= 3a[1 - 2\cos^2\theta\sin^2\theta] \\ x\cos^3\theta + y\sin^3\theta + a - 3a\sin^2\theta\cos^2\theta &= 3a - 6a\sin^2\theta\cos^2\theta \\ x\cos^3\theta + y\sin^3\theta + a - 3a - 3a\sin^2\theta\cos^2\theta + 6a\sin^2\theta\cos^2\theta &= 0 \\ x\cos^3\theta + y\sin^3\theta - 2a + 3a\sin^2\theta\cos^2\theta &= 0 \\ x\cos^3\theta + y\sin^3\theta + 3a\sin^2\theta\cos^2\theta - 2a &= 0 \end{aligned}$$

$$\begin{aligned} a^2 + b^2 &= (a+b)^2 - 2ab \\ a^3 + b^3 &= (a+b)^3 - 3ab(a+b) \end{aligned}$$

which is the Eq. of the tangent.

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 + 0 + 3a\sin^2\theta\cos^2\theta - 2a|}{\sqrt{\cos^6\theta + \sin^6\theta}}$$

$$p = \frac{|3a\sin^2\theta\cos^2\theta - 2a|}{\sqrt{1 - 3\sin^2\theta\cos^2\theta}} \quad \text{--- (3)}$$

Now $r^2 = x^2 + y^2$

$$r^2 = a^2(3\cos\theta - \cos^3\theta)^2 + a^2(3\sin\theta - \sin^3\theta)^2$$

$$r^2 = a^2(9\cos^2\theta + \cos^6\theta - 6\cos^4\theta) + a^2(9\sin^2\theta + \sin^6\theta - 6\sin^4\theta)$$

$$r^2 = 9a^2(\cos^2\theta + \sin^2\theta) + a^2 6(-\cos^4\theta - \sin^4\theta) + a^2(\sin^6\theta + \cos^6\theta)$$

$$r^2 = a^2[9 - 6(\cos^4\theta + \sin^4\theta) + (\sin^6\theta + \cos^6\theta)]$$

$$r^2 = a^2[9 - 6\{(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta\} + (\sin^2\theta + \cos^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\sin^2\theta + \cos^2\theta)]$$

$$r^2 = a^2 [9 - 6 \{1 - 2 \sin^2 \theta \cos^2 \theta\} + 1 - 3 \sin^2 \theta \cos^2 \theta]$$

$$r^2 = 9a^2 - 6a^2 + 12a^2 \sin^2 \theta \cos^2 \theta + a^2 - 3a^2 \sin^2 \theta \cos^2 \theta$$

$$r^2 = 4a^2 + 9a^2 \cos^2 \theta \sin^2 \theta$$

$$r^2 = a^2 [4 + 9 \cos^2 \theta \sin^2 \theta]$$

$$r^2 - 4a^2 = 9a^2 \cos^2 \theta \sin^2 \theta$$

$$\Rightarrow \cos^2 \theta \sin^2 \theta = \frac{r^2 - 4a^2}{9a^2} \quad \text{--- (4)}$$

Put (4) in (3)

$$p = \frac{|3a \cdot \frac{r^2 - 4a^2}{9a^2} - 2a|}{\sqrt{1 - 3 \left[\frac{r^2 - 4a^2}{9a^2} \right]}}$$

$$p = \frac{\left| \frac{r^2 - 4a^2}{3a} - 2a \right|}{\sqrt{1 - \left(\frac{r^2 - 4a^2}{3a^2} \right)}}$$

$$p = \frac{r^2 - 4a^2 - 6a^2}{3a} \cdot \frac{\sqrt{3} a}{\sqrt{3a^2 - r^2 + 4a^2}}$$

$$p = \frac{r^2 - 10a^2}{\sqrt{3}} \times \frac{1}{\sqrt{7a^2 - r^2}}$$

$$p = \frac{r^2 - 10a^2}{\sqrt{3} \sqrt{7a^2 - r^2}}$$

$$\Rightarrow p^2 = \frac{(r^2 - 10a^2)^2}{3(7a^2 - r^2)}$$

$$3p^2(7a^2 - r^2) = (r^2 - 10a^2)^2$$

is the required pedal equation.

Q#18. If $x = a \cos g(t)$, $y = b \sin g(t)$, prove that

$$xy^2 \frac{d^2 y}{dx^2} = b^2 \frac{dy}{dx}$$

Soln:

$$\text{Here } x = a \cos g(t) \quad \text{--- I}$$

$$y = b \sin g(t) \quad \text{--- II}$$

$$\Rightarrow \frac{dx}{dt} = -a \sin g(t) \cdot g'(t)$$

$$\Rightarrow \frac{dy}{dt} = b \cos g(t) \cdot g'(t)$$

$$\frac{dy}{dx} = \frac{b \cos g(t) \cdot g'(t)}{-a \sin g(t) \cdot g'(t)}$$

$$= -\frac{b}{a} \cot g(t) \quad \text{--- (iii)}$$

Again) Diff. w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 g(t) g'(t) \frac{dt}{dx}$$

$$= \frac{b}{a} \operatorname{cosec}^2 g(t) g'(t) \frac{1}{-a \sin g(t) \cdot g'(t)}$$

$$= -\frac{b}{a^2 \sin^3 g(t)}$$

Consider

$$xy^2 \frac{d^2y}{dx^2} = a \cos g(t) \cdot b^2 \sin^4 g(t) \cdot -\frac{b}{a^2 \sin^3 g(t)}$$

$$= -\frac{b^3 \cos g(t)}{a \sin g(t)}$$

$$= b^2 \left(-\frac{b \cos g(t)}{a \sin g(t)} \right)$$

$$= b^2 \frac{dy}{dx}$$

from (iii)

$$\text{Hence } xy^2 \frac{d^2y}{dx^2} = b^2 \frac{dy}{dx}$$

is as required.