Mth601
Operations Research
Final Term
Lecture #23

4.10 (Artificial variable Technique;)

IN Simplex Method;

we Convert Inequility to Standard form;

≤ Add a slack variable.

≥ Subtract a slack variable.

= Nothing.

IN Big M Method;

≤ Add a slack variable.

≥ Subtract a slack variable and adde an Artificial variable.

= Add an Artificial variable.

Artifical variable;-

Artifical variable do not have any physical Meaning. It is merely a device to get the starting basic

Feasible Solution, so that the simplex procedure May be abopted as usuall untill optimal Solution is obtained. Cost of Artificial variable is (-M)

Example:-

Consider the problem;

Maximize $Z = 2x + 5y$

Subject to $x \leq 40$, $y \leq 30$

$x + y \geq 60$, $x, y \geq 0$

Solution:-

Convert the problem into standard form by including slack variables.

$$x + S_1 = 40$$

$$y + S_2 = 30$$

$$x + y - S_3 + A_1 = 60$$

$$\text{Max } Z = 2x + 5y + 0S_1 + 0S_2 + 0S_3 - MA_1$$

$$x, y, S_1, S_2, S_3, A_1 \geq 0$$

Initial basic Feasible Solution is given by;

$x = y = 0$

$S_1 = 40$, $S_2 = 30$; $A_1 = 60$

$$\begin{pmatrix} x & y & S_1 & S_2 & S_3 & A_1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ S_1 \\ S_2 \\ S_3 \\ A \end{pmatrix} = \begin{pmatrix} 40 \\ 30 \\ 60 \end{pmatrix}$$

(Initial table)

	C_j		2	5	0	0	0	-M	Ratio
C_B	B	x_B	x	y	S_1	S_2	S_3	A	x_B/y
0	S_1	40	1	0	1	0	0	0	∞
0	S_2	30	0	1	0	1	0	0	30
-M	A_1	60	1	1	0	0	-1	1	60
$Z_j - C_j$			-M-2	-M-5	0	0	M	0	

$$Z_j - C_j = C_B x_j - C_j$$

$$Z_1 - C_1 = C_B x - C_1$$
$$= (0 \times 1 + 0 \times 0 - M \times 1) - 2$$
$$= -M - 2$$

M / -M

$$Z_2 - C_2 = C_B y - C_2$$
$$= (0 \times 0 + 0 \times 1 - M \times 1) - 5$$
$$= -M - 5$$

$$Z_3 - C_3 = C_B S_1 - C_3$$
$$= (0 \times 1 + 0 \times 0 - M \times 0) - 0$$
$$= 0$$

y is Entering variable and S_2 is leaving variable.

1st Iteration :-

$R_3 \rightarrow R_3 - R_2$

	C_j		2	5	0	0	0	-M	Ratio
C_B	B	x_B	x ↙	y	S_1	S_2	S_3	A	x_B/x
0	S_1	40	1	0	1	0	0	0	40
5	y	30	0	1	0	1	0	0	∞
-M	A	30	(1)	0	0	-1	-1	1	30 ←
$Z_j - C_j$			-M-2 ↑	0	0	5+M	M	0	

x is Entering variable and A is leaving Variable.

2nd Iteration :-

$R_1 \rightarrow R_1 - R_3$

	C_j		2	5	0	0	0	-M	Ratio
C_B	B	x_B	x	y	S_1	S_2	S_3 ↙	A	x_B/S_3
0	S_1 ←	10	0	0	1	1	1	-1	10 ←
5	y	30	0	1	0	1	0	0	∞
2	x	30	1	0	0	-1	-1	1	-30
$Z_j - C_j$			0	0	0	3	-2 ↑	2+M	

S_3 is Entering variable and S_1 is leaving variable.

3rd Iteration :-

$\sim R_3 \rightarrow R_3 + R_1$

	C_j		2	5	0	0	0	-M
C_B	B	X_B	x	y	S_1	S_2	S_3	A
0	S_3	10	0	0	1	1	1	-1
5	y	30	0	1	0	1	0	0
2	x	40	1	0	1	0	0	0
$Z_j - C_j$			0	0	2	5	0	M

Now;

$x = 40, \quad y = 30$

$\text{Max } Z = 2(40) + 5(30)$

$= 80 + 150$

$= \underline{230}.$

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Lecture #24

The Big M-Method (Minimization problem)

Example:-

Minimize $Z = 4x_1 + x_2$

Subject to

$$3x_1 + 4x_2 \geq 20 \rightarrow (1)$$
$$-x_1 - 5x_2 \leq -15 \rightarrow (2)$$
$$x_1, x_2 \geq 0.$$

Solution:-

Change the Inequility of 2nd eq by Multiplying with (-1) we get;

$$x_1 + 5x_2 \geq 15$$

Now, By Introducing Slack & Artificial variables, equations are transformed into;

$$3x_1 + 4x_2 - S_1 + A_1 = 20$$
$$x_1 + 5x_2 - S_2 + A_2 = 15$$

$$\text{Max } Z \;/\; Z' = -4x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$
$$x_1, x_2, A_1, A_2, S_1, S_2 \geq 0$$

Initial Basic Feasible Solution;

$x_1 = x_2 = 0$

we get;

$A_1 = 20$

$A_2 = 15$

Initial Table:-

Matrix Form

$$Ax = B$$

$$\begin{pmatrix} x_1 & x_2 & S_1 & S_2 & A_1 & A_2 \\ 3 & 4 & -1 & 0 & 1 & 0 \\ 1 & 5 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 15 \end{pmatrix}$$

Initial Table:-

	C_j		-4	-1	0	0	-M	-M	Ratio
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	X_B/x_2
-M	A_1	20	3	4	-1	0	1	0	5
-M	A_2	15	1	5	0	-1	0	1	3
	$Z_j - C_j$		-4M+4	-9M+1	M	M	0	0	

x_2 is Entering variable

A_2 is leaving variable.

1st Iteration:

$R_2 \longrightarrow \frac{R_2}{5}$, $R_1 \longrightarrow R_1 - 4R_2$

	C_j		-4	-1	0	0	-M	-M	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	A_1	A_2	x_B/x_1
-M	A_1	8	11/5	0	-1	4/5	1	-4/5	40/11
-1	x_2	3	1/5	1	0	-1/5	0	1/5	15
	$Z_j - C_j$		$\frac{-11M+19}{5}$	0	M	$\frac{-4M+1}{5}$	0	$\frac{9M-1}{5}$	

x_1 Enter and A_1 leaves the Basis.

2nd Iteration. $R_1 \times \frac{5}{11}$ to Make key element '1'

$R_2 \longrightarrow 5R_2 - R_1$

	C_j		-4	1	0	0	-M	-M	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	A_1	A_2	~~x_B/x_1~~
-4	x_1	$\frac{40}{11}$	1	0	$-\frac{5}{11}$	$\frac{4}{11}$	$\frac{4}{11}$	-4/11	40/11
-1	x_2	$\frac{125}{11}$	0	5	$\frac{5}{11}$	$\frac{-15}{11}$	$-\frac{4}{11}$	$\frac{29}{11}$	15
	$Z_j - C_j$		0	-4	$\frac{5}{11}$	$\frac{-1}{11}$	$\frac{-12+M}{11}$	$\frac{42+M}{11}$	

$$R_2 \longrightarrow \frac{R_2}{5}$$

	C_j		-4	-1	0	0	-M	-M	Ratio
C_B	B	x_B	x_1	x_2	S_1	S_2	A_1	A_2	
-4	x_1	$\frac{40}{11}$	1	0	$-\frac{5}{11}$	$\frac{4}{11}$	$\frac{4}{11}$	$-\frac{4}{11}$	$40/11$
-1	x_2	$\frac{25}{11}$	0	1	$1/5$	$4/55$	$4/55$	$-4/55$	15
$Z_j - C_j$			0	-4	$-5/11$	$-1/11$	$\frac{-12+M}{11}$	$\frac{42+M}{11}$	

So Here we can End our problem and we can Find an optimum solution.

Because No Artificial variable is in Basic variable.

optimum solution ;

$$x_1 = 40/11$$

$$x_2 = 25/11$$

$$Z^* = 4x_1 + x_2$$

$$= 4\left(\frac{40}{11}\right) + \frac{25}{11}$$

$$= \frac{160 + 25}{11} = \frac{185}{11}$$

$$Z^* = \frac{185}{11}$$ o.

Lecture #24

All Calculations Rough Work.

For Initial Table;

$$Z_j - C_j = C_B x_j - C_j$$

$$Z_1 - C_1 = C_B x_1 - C_1$$
$$= (-3M - M) - (-4) = -4M + 4$$

$$Z_2 - C_2 = C_B x_2 - C_2$$
$$= (-4M - 5M) - (-1)$$
$$= -9M + 1$$

$$Z_3 - C_3 = C_B S_1 - C_3$$
$$= (-M)(-1) + 0(1) - 0 = M$$

$$Z_4 - C_4 = C_B S_2 - C_4$$
$$= 0 + (-M)(-1) - 0 = M$$

$$Z_5 - C_5 = C_B A_1 - C_5$$
$$= -M + 0 - (-M) = -M + M = 0$$

$$Z_6 - C_6 = C_B A_2 - C_6$$
$$= (0 - M) - (-M) = -M + M = 0$$

(For 1st Iteration):-

$R \rightarrow R_2/5$

$\frac{R_2}{5} = \frac{15}{5} \quad 1/5 \quad \frac{5}{5} \quad \frac{0}{5} \quad -\frac{1}{5} \quad \frac{0}{5} \quad \frac{1}{5}$

$R_2 = 3 \quad \frac{1}{5} \quad 1 \quad 0 \quad -\frac{1}{5} \quad 0 \quad \frac{1}{5}$

$R_1 \longrightarrow R_1 - 4R_2$

$R_1 =$	3	4	−1	0	1	0
$-4R_2 =$	$\frac{4}{5}$	4	0	$-\frac{4}{5}$	0	$\frac{4}{5}$
$R_1 - 4R_2 =$	$3-\frac{4}{5}$	0	−1	$\frac{4}{5}$	1	$-\frac{4}{5}$
$R_1 - 4R_2 =$	$\frac{15-4}{5}$	0	−1	$\frac{4}{5}$	1	$-\frac{4}{5}$
$R_1 =$	$11/5$	0	−1	$\frac{4}{5}$	1	$-\frac{4}{5}$

$Z_j - C_j$ of 1st Iteration :-

$$Z_j - C_j = C_B x_j - C_j$$

$$Z_1 - C_1 = C_B x_1 - C_1$$

$$= -\frac{11M}{5} - \frac{1}{5} - (-4)$$

$$= -\frac{11}{5}M - \frac{1+20}{5}$$

$$= -\frac{11M - 1 + 20}{5} = -\frac{11M+19}{5}$$

2nd Iteration :-

$R_1 \longrightarrow \frac{5}{11}R_1$:

$\frac{5}{11}R_1 =$	$\frac{5}{11}\times 8$	$\frac{8}{11}\times\frac{11}{8}$	0	$-\frac{5}{11}$	$\frac{8}{11}\times\frac{4}{5}$	$\frac{5}{11}$	$\frac{8}{11}\times -\frac{4}{8}$
$R_1 =$	$\frac{40}{11}$	1	0	$-\frac{5}{11}$	$\frac{4}{11}$	$\frac{5}{11}$	$-\frac{4}{11}$

$R_2 - 5R_2 - R_1$

$5R_2 =$	$\frac{5}{8}^{1}$	5	0	$\frac{-5}{8}^{-1}$	0	$\frac{5}{8}^{1}$
$-R_1 =$	1	0	$\frac{-5}{11}$	$\frac{4}{11}$	$\frac{4}{11}$	$\frac{-4}{11}$
$R_2 =$	0	5	$\frac{5}{11}$	$-1-\frac{4}{11}$	$-\frac{4}{11}$	$1+\frac{4}{11}$

$$R_2 = 0 \quad 5 \quad \frac{5}{11} \quad \frac{-15}{11} \quad -\frac{4}{11} \quad \frac{26}{11}$$

$$Z_j - C_j = B A_2 - C_6$$

$$= (-4)\left(-\frac{4}{11}\right) - \frac{26}{11} + M$$

$$= \frac{-16}{11} - \frac{26}{11} + \frac{M}{1}$$

$$= \frac{-42 + M}{11}$$

$5R_2 - R_1$

$$15 - \frac{40}{11} = \frac{165 - 40}{11} = \frac{125}{11}$$

(Last)

$$\frac{R_2}{5}$$

$$\frac{125}{11} \div 5 \Rightarrow \frac{125}{11 \times 5} = \frac{125}{55} = \frac{25}{11}$$

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Lecture # 25
(The Two Phase Method)
Basic Concepts

Phase I - Eliminate Artifical variables.

Phase II - Find optimum Solution.

Some Basic steps For Moving to Phase II.

Phase I:-

In this phase we Construct an auxilarily Lpp.

Step 1:, Assign a cost -A to each artifical Variable and cost 0 to other variable.

Step 2:- Write down the auxiliary Lpp in which the New objective function is to be Maximized.

Step 3:- Slove the auxilary Lpp by Simplex method until either of following three casses arise.

(i) Max $z^* < 0$ and at least one artificial variable appear in the optimum basis at positive level. and $z_j - c_j \geq 0$

(ii) Max $z^* = 0$ and at least one Artifical variable appear in basis at zero level and $z_j - c_j \geq 0$

(iii) Max $z^* = 0$ and No Artificial variable appear in Basis and $z_j - c_j \geq 0$

In case (i), given LPP does not posses any feasible solution, where in case (ii) and (iii) we go to (Phase II.)

Mth601
Lecture #25
(The Two Phase Method)

(Phase I) Eliminate Artifical variables.

(Phase II) Find optimum Solution.

(Example)

Maximize;

$$Z = 2x+5y$$

Subject to;

$$x \leq 40$$
$$y \leq 30$$
$$x+y \geq 60$$
$$x, y \geq 0$$

(Solution):-

we Convert the Lpp into into Standard form by using Slack variable S_1, S_2 and S_3 and an Artificial variable A_1.

$$\text{Max } Z = 2x + 5y + 0S_1 + 0S_2 + 0S_3 - A_1$$

Subject to

$$x + S_1 = 40$$
$$y + S_2 = 30$$
$$x + y - S_3 + A_1 = 60$$

$x_1, y, s_1, s_2, A_1 \geq 0$

An Initial Basic Feasible solution.

$x = y = 0$

$s_1 = 40, \quad s_2 = 30, \quad A_1 = 60$

(Phase I)

Auxiliary LPP.

$$\text{Max } z^* = 0x_1 + 0y + 0s_1 + 0s_2 + 0s_3 - A_1$$

Subject to ;

$$x + s_1 = 40$$
$$y + s_2 = 30$$
$$x + y - s_3 + A_1 = 60$$

(Matrix Form)

$$Ax = B$$

$$\begin{pmatrix} x & y & s_1 & s_2 & s_3 & A_1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \\ A_1 \end{pmatrix} = \begin{pmatrix} 40 \\ 30 \\ 60 \end{pmatrix}$$

	C_j		0	0	0	0	0	-1	Ratio
C_B	B	x_B	x	y	s_1	s_2	s_3	A	x_B/x
0	s_1	40	1	0	1	0	0	0	40 ←
0	s_2	30	0	1	0	1	0	0	∞
-1	A	60	1	1	0	0	-1	1	60
z_j	$-c_j$		-1 ↑	-1	0	0	1	0	

$-1 - (-1) = -1 + 1 = 0$

x or y can Enter the bais as they have same coefficients, so we can Chosse anyone. Say x. and Hence S_1 leaves the basis.

$R_3 \rightarrow R_3 - R_1$

	C_j		0	0	0	0	0	-1	Ratio
C_B	B	x_B	x	y	S_1	S_2	S_3	A	x_B/y
0	x	40	1	0	1	0	0	0	∞
0	S_2	30	0	1	0	1	0	0	30
-1	A	20	0	1	-1	0	-1	1	20
$Z_j - C_j$			0	-1	1	0	1	-1	

y Enter and A leave the Baises.

$R_2 \rightarrow R_2 - R_3$

	C_j		0	0	0	0	0	-1	
C_B	B	x_B	x	y	S_1	S_2	S_3	A	
0	x	40	1	0	1	0	0	0	
0	S_2	10	0	0	1	1	1	-1	
0	y	20	0	1	-1	0	-1	1	
$Z_j - C_j$			0	0	0	0	0	1	

In above table the objective function $Z = 0$ and A has left the Basis. Hence A becomes a Non-Basis variable. This indicates that the problem has a Feasible solution and we can processed to Phase II.

(Phase II)

	C_j		2	5	0	0	0	Ratio
C_B	B	x_B	x	y	S_1	S_2	S_3	x_B/S_3
2	x	40	1	0	1	0	0	∞
0	S_2	10	0	0	1	1	1	10
5	y	20	0	1	-1	0	-1	-20
$Z_j - C_j$			0	0	-3	0	-5	

S_3 Enter and S_2 leaves the Basis.

$$R_3 \longrightarrow R_3 + R_2$$

	C_j		2	5	0	0	0
C_B	B	x_B	x	y	S_1	S_2	S_3
2	x	40	1	0	1	0	0
5	S_3	10	0	0	1	1	1
0	y	30	0	1	0	1	0
$Z_j - C_j$			0	5	7	5	5

we get the optimum value of Z in the above table as no Negative coefficient is present in the objective Function.

$x = 40$, $y = 30$, $S_3 = 10$, $S_1 = S_2 = 0$

$$\text{Max } Z = 2x + 5y + 0S_1 + 0S_2 + 0S_3$$
$$= 2(40) + 5(30) + 0 + 0 + 0(\times 10)$$
$$\text{Max } Z = 80 + 150 + 0 = \underline{230}$$

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Lecture # 29, 30

Duality Theory
&
Applications of Duality

Example:-

Maximize $Z = 6x_1 + 7x_2 + 8x_3$

Subject to; $x_1 + 2x_2 + 7x_3 \leq 60$

$2x_1 + 3x_3 \leq 20$

$4x_2 + 5x_3 \leq 45$

$x_3 \leq 15$

$x_1, x_2, x_3 \geq 0$

Solution:-

Minimize $Z = 60y_1 + 20y_2 + 845y_3 + 15y_4$

Coefficient Matrix;

let;

$$M = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 0 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Now for dual;

$$M^t = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 7 & 3 & 5 & 1 \end{bmatrix}$$

Hence we have the dual for the problem ;

$$Z = 60y_1 + 30y_2 + 45y_3 + 15y_4$$

$$y_1 + 2y_2 \geq 6$$

$$2y_1 + 4y_3 \geq 7$$

$$7y_1 + 3y_2 + 5y_3 + y_4 \geq 8$$

$$y_1, y_2, y_3, y_4 \geq 0$$

Example:-

Maximize $Z = -6x_1 + 7x_2$

Subject to;

$$-x_1 + 2x_2 \leq -5$$

$$3x_1 + 4x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

Minimize $Z = -5y_1 + 7y_2$

$$-y_1 + 3y_2 \geq -6$$

$$2y_1 + 4y_2 \geq 7$$

$$y_1, y_2 \geq 0$$

=> Dual Formulation:-

Following steps to be followed to Convert the primal linear programming problem into its dual.

Step 1:-

For each Constraint in primal problem there is an associated variable in the dual problem.

Step 2:-

The elements of the RHS of the Constraints in the primal problem are equal to the Respective coefficients of the objective function in the dual problem.

Step #3:-

When primal problem seeks Maximization as the Measure of effectiveness the dual problem seeks Minimization, and vice versa.

Step 4:-

The Maximization problem has constraint of type ($\leq$) and the Minimization problem has Constraints of type ($\geq$).

Step # 5:-

The variables in both the problem are Non-Negative (Sometime unrestricted)

Step # 06:-

The Rows of the Primal problem are changed to Column in the dual problem.

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Chapter # 05 Lecture # 31
Transportation Models

(5.1) Introduction :-

The transportation problem is one of the subclass of Linear programing problems.

In a transportation problem, we have certain origins, which may represent Factories where we produce items and supply a required quantity of the products to a certain Number of destinations.

This Must be done in such a way as to Maximize the profit & Minimize the cost.

Thus we have places of production as origins and the places of supply as destinations.

-10-10-

=> Sometime the origins and destinations are also termed as sources and sinks

=> The Model of a transportation problem can be represented in a tabular form.

Origins \ Destination	1	2	3	4	⟶ n	Supply
1	c_{11} x_{11}	c_{12} x_{12}	c_{13} x_{13}	c_{14} x_{14}	c_{1n} x_{1n}	a_1
2	c_{21} x_{21}	c_{22} x_{22}	c_{23} x_{23}	c_{24} x_{24}	c_{2n} x_{2n}	a_2
3	c_{31} x_{31}	c_{32} x_{32}	c_{33} x_{33}	c_{34} x_{34}	c_{3n} x_{3n}	a_3
4	c_{41} x_{41}	c_{42} x_{42}	c_{43} x_{43}	c_{44} x_{44}	c_{4n} x_{4n}	a_4
m	c_{m1} x_{m1}	c_{m2} x_{m2}	c_{m3} x_{m3}	c_{m4} x_{m4}	c_{mn} x_{mn}	a_m
Demand	b_1	b_2	b_3	b_4	b_n	

c_{ij} → cost of transporting one unit of the product from i^{th} origin to j^{th} destination.

x_{ij} → Quantity transported from i^{th} origin to j^{th} destination.

Objective:-

is to transport various Quantities that are initially stored at various origin to different destination in such a way that the transportation cost is Minimum.

Transportation problem

- Unbalanced Transportation problem
  - Demand > Supply
    $\sum b_j > \sum a_i$
    ↓
    Add a dummy Row with cost zero and Supply equals to (Demand - Supply)
  - Demand < Supply
    $\sum b_j < \sum a_i$
    ↓
    Add a dummy Column with cost zero and demand equals to (Supply - demand)
- Blanced Transportation problem
  ↓
  Demand = Supply
  $\sum b_j = \sum a_i$

	D_1	D_2	D_3	D_4	Supply
O_1	5	4	2	6	30
O_2	8	3	5	7	40
O_3	10	11	3	1	20
O_4	9	12	6	3	10
Demand	10	50	30	20	

Demand = 110

Supply = 100

	D_1	D_2	D_3	D_4	Supply
O_1	1	9	14	8	50
O_2	13	2	7	12	20
O_3	15	6	3	11	10
O_4	5	10	16	04	30
Demand	10	20	30	40	

Demand = 100

Supply = 110

	D_1	D_2	D_3	D_4	Supply
O_1	5	4	2	6	30
O_2	8	3	5	7	40
O_3	10	11	3	1	20
O_4	9	12	6	3	10
O_5	0	0	0	0	10
Demand	10	50	30	20	110

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	1	9	14	8	0	50
O_2	13	2	7	11	0	20
O_3	15	6	3	4	0	10
O_4	5	10	16	20	0	30
Demand	10	20	30	40	10	110

Solutions;

The Solution of transportation problem can be obtained in two stages.

(a) Initial Solution

(b) optimal Solution

=) Initial solution can be obtained by using any one of three.



1. North - west Corner Rule (NWCR)
2. Least Cost Method (LCM)
3. Vogel's Approximation Method (VAM)

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Lecture # 31

Part – 2

Initial Basic Feasible Solution:-

(i) North West Corner Rule (NWC)
(ii) Least Cost Method (LCM)
(iii) Vogel's Approximation Method (VAM)

=> **North West Corner Rule;**

Example;

obtain the Initial Basic Feasible Solution of A transportation problem whose cost and rim requirement table is given below:-

origin	P	Q	R	Supply
A	5	7	8	70
B	4	4	6	30
C	6	7	7	50
Demand	65	42	43	150

Solution;

The problem is balanced TP as the total supply is equal to the total Demand.

we obtain the Initial Feasible Solution as follows;

Origin	P	Q	R	Supply
A	5 (65)	7 (5)	8	70
B	4	4 (30)	6	30
C	6	7 (7)	7 (43)	50
Demand	65	42	43	150

$= 5 \times 65 + 7 \times 5 + 4 \times 30 + 7 \times 7 + 7 \times 43$

$= 830$

2- Least Cost Method (LCM)

Example; Sowe the TP using least cost Method.

Origin	P	Q	R	Supply
A	5 [35]	7 [35]	8	70
B	4 [30]	4	6	30
C	6	7 [7]	7 [43]	50
Demand	65	42	43	150

$= 5 \times 35 + 4 \times 30 + 7 \times 35 + 7 \times 7 + 7 \times 43$

$= 890$

Lec# 32

Mth601 Operations Research

Chapter #05

Transportation Methods/Models

=> Initial Basic Feasible Solution:-

(i) North west Corner Rule (NWC)
(ii) Least Cost Method (LCM)
(iii) Vogel's Approximation Method (VAM)

=> Vogel's Approximation Method (VAM)

Example:-

Apply vogel's Approximation Method on the Following Table;

origin	Destination			Supply
	P	Q	R	
A	5 (65)	7 (5)	8	70
B	4	4 (30)	6	30
C	6	7. (7)	7 (43)	50
demand	65	42	43	

	P_1	P_2	P_3	P_4
	2	1	1	–
←	0	2	–	–
	0	0	0	0

P_1	1	0	1
P_2	–	0	1
P_3	–	0	1
P_4	–	7	7

Origin	P	Q	R	Supply
A	5 (65)	7 (5)	8	70
B	4	4 (30)	6	30
C	6	7 (7)	7 (43)	50
Demand	65	42	43	

$= 5\times65 + 7\times5 + 4\times30 + 7\times7 + 7\times43$

The cost of Allocation by Vogel's Approximation Method is;

$= 5\times65 + 7\times5 + 4\times30 + 7\times7 + 7\times43$

$=$ Rs-830.

Lecture # 33

Mth601 operations Research

Chapter # 05

Transportation Models

=> Optimality Test;

For Conducting the optimality test to any Feasible Solution of (m×n) transportation problem, the Following two Condition Must be satisfied:-

1. It Consists of exactly (m+n-1) individual cells being Allocated.
2. These Allocations are in independent positions.

Example:-

Mam enterprise has three Factories Located at A, B and C and Supplies to three Warehouses Located at D, E and F. Monthly factory Capacities are 10, 80 and 15 units Respectively. Monthly warehouse requirements are 45, 20 and 40 units Respectively. Unit transportation Costs in

rupees are given below;

Warehouses

Factory	D	E	F
A	5	1	7
B	6	4	6
C	3	2	5

Solution:-

Using NWC Rule we obtain the Initial basic Feasible Solution;

Warehouse

Factory	D	E	F	Supply
A	5 (10)	1	7	10
B	6 (35)	4 (20)	6 (25)	80
C	3	2	5 (15)	15
Demand	45	20	40	105

There are Five allotments. As per the optimality test condition there must be (3+3-1) Cells occupied and we see that this condition is satisfied. All the cells should be in Independent positions.

Having satisfied the two condition we proceed to the optimality test;

	D	E	F	
A	5 (10)	1	7	$u_1 = -1$
B	6 (35)	4 (20)	6 (25)	$u_2 = 0$
C	3	2	5 (15)	$u_3 = -1$
	$v_1 = 6$	$v_2 = 4$	$v_3 = 6$	

Calculate $\Delta ij = Cij - (Ui + Vj)$

$\Delta_{12} = C_{12} - (U_1 + V_2)$

$= 1 - (-1 + 4) = -2$

$\Delta_{13} = C_{13} - (U_1 + V_3)$

$= 7 - (-1 + 6)$

$= 7 - (5) = 2$

$\Delta_{31} = C_{31} - (U_3 + V_1)$

$= 3 - (-1 + 6)$

$= 3 - (5) = -2$

$\Delta_{32} = C_{32} - (U_3 + V_2)$

$= 2 - (-1 + 4)$

$= 2 - (3) = -1$

we Choose $\Delta_{31} = -2$ as Most Negative

	D	E	F	
A	5 (10)	1	7	$U_1 =$
B	6 (35)	4 (20)	6 (25)	$U_2 =$
C	3	2	5 (15)	$U_3 =$
	$V_1 =$	$V_2 =$	$V_3 =$	

	D	E	F	
A	5 (10)	1	7	$U_1 = -1$
B	6 (20)	4 (20)	6 (40)	$U_2 = 0$
C	3 (15)	2	5	$U_3 = -3$
	$V_1 = 6$	$V_2 = 4$	$V_3 = 6$	

$C_{12} = 1 - (-1 + 4) = 1 - 2 = -42$

$C_{13} = 7 - (-1 + 6) = 7 - 5 = -2$

$C_{32} = 2 - (-3 + 4) = 2 - 1 = 1$

$C_{33} = 5 - (-3 + 6) = 5 - 3 = 2$

we Choose $C_{12} = -4$ as Most Negative value :-

Rupees ... given below;

	D	E	F	
A	5 (10)	1	7	$U_1 =$
B	6 (35)	4 (20)	6 (25)	$U_2 =$
C	3	2	5 (15)	$U_3 =$
	$V_1 =$	$V_2 =$	$V_3 =$	

	D	E	E	
A	5 (10)	1	7	$U_1 = -1$
B	6 (20)	4 (20)	6 (10)	$U_2 = 0$
C	3 (15)	2	5	$U_3 = -3$
	$V_1 = 6$	$V_2 = 4$	$V_3 = 6$	

$C_{12} = 1 - (-1 + 4) = 1 - 2 = -42$

$C_{13} = 7 - (-1 + 6) = 7 - 5 = -2$

$C_{32} = 2 - (-3 + 4) = 2 - 1 = 1$

$C_{33} = 5 - (-3 + 6) = 5 - 3 = 2$

we Choose $C_{12} = -4$ as Most Negative value:-

	D	E	F	
A	5 (10)	1 (+)	7	$U_1 = -1$
B	6 (+) (30)	4 (-) (10)	6 (40)	$U_2 = 0$
C	3 (15)	2	5	$U_3 = -3$

$v_1 = 6 \quad v_2 = 4 \quad v_3 = 5$

$C_{12} = 1 - (-1+4) = 1 - 3 = -2$

$C_{13} = 7 - (-1+6) = 7 - 5 = 2$

$C_{32} = 2 - (-3+4) = 2 - 1 = 1$

$C_{33} = 5 - (-3+6) = 5 - 3 = 2$

We choose $a_{12} = -4$ as Most Negative value;

	D	E	F	
A	5	1 (10)	7	$U_1 = -3$
B	6 (30)	4 (10)	6 (40)	$U_2 = 0$
C	3 (15)	2	5	$U_3 = -3$

$v_1 = 6, v_2 = 4, v_3 = 6$

$a_{11} = 5 - (-3+6) = 5 - 3 = 2$

$a_{13} = 7 - (-3+4) = 7 - (1) = 6$

$a_{32} = 2 - (-3+4) = 2 - (1) = 1$

$a_{33} = 5 - (-3+6) = 5 - 3 = 2$

$= 1 \times 10 + 6 \times 30 + 4 \times 10 + 6 \times 40 + 3 \times 15$

$= Rs. 515$

Mth601 Operations Research

Lecture # 34

(VAM For Maximization Problem)

Here our objective is to Maximize the total profit for which the profit Matrix is given:-

First we have to Convert the Maximization problem into Minimization by Subtracting all the elements from the Highest element in the given table.

Example:-

Slove the Following transportations problem to Maximize profit and give Criteria for optimality.

	1	2	3	4	supply
A	40	25	22	33	100
B	44	35	30	30	30
C	38	38	28	30	70
Demand	40	20	60	30	

Solution:-

For converting it into Minimization type. we Subtract All the elements from the Highest element.

	1	2	3	4	Supply
A	4	19	22	11	$100
B	0	9	14	14	30
C	6	6	16	14	70
Demand	40	20	60	30	

Since Demand ≠ Supply

Here Demand < Supply

So we Add a dummy Column Here with Zero cost.

	1	2	3	4	5	supply
A	4	19	22	11	0	100
B	0	9	14	14	0	30
C	6	6	16	14	0	70
Demand	40	20	60	30	50	

Now we Apply the VAM Method:-

	1	2	3	4	5	Supply
A	4	19	22 (20)	11 (20)	0 (60)	100
B	0 (30)	9	14	14	0	300
C	6 (10)	6 (20)	16 (40)	14	0	70
Demand	40	20	60	30	50	

	P1	P2	P3	P4	P5	P6
Penalty	4	7	7	7	11	22
	0	9	-	-	-	-
	6	0	0	8	2	16

P1	4	3	2	3	0
P2	4	3	2	3	-
P3	2	13	6	3	-
P4	2	-	6	3	-
P5	-	-	6	3	-
P6	-	-	6	-	-

After Applying VAM Method;

	1	2	3	4	5	Supply
A	4	19	22 (20)	11 (20)	0 (60)	100
B	0 (30)	9	14	14	0	30
C	6 (10)	6 (20)	16 (40)	14	0	70
Demand	40	20	60	30	50	200

{ m+n-1
3+5-1=7
And it's Non-degenerate.

Now Applying optimality Test;

	1	2	3	4	5	
A	4	19	22	11	0	$u_1 = 6$
B	0	9	14	14	0	$u_2 = -6$
C	6	6	16	14	0	$u_3 = 0$

$v_1 = 6$ $v_2 = 6$ $v_3 = 16$ $v_4 = 5$ $v_5 = -6$

Calculations for un-Allocated cells;

$$\Delta_{ij} = c_{ij} - (u_i + v_j)$$

$$\Delta_{11} = c_{11} - (u_1 + v_1)$$
$$= 4 - (6+6) = -8$$

$$\Delta_{12} = 19 - (6+6) = 7$$

$$\Delta_{22} = 9 - (-6+6) = 9$$

$$\Delta_{23} = 14 - (-6+16) = 4$$

$$\Delta_{24} = 14 - (-6+5) = 15$$

$$\Delta_{25} = 0 - (-6-6) = 12$$

we choose $\Delta_{11} = -8$ as Most Negative value.

	1	2	3	4	5	
A	4	19	22	11	0	$u_1 = 0$
B	0	9	14	14	0	$u_2 = -4$
C	6	6	16	14	0	$u_3 = -6$

$v_1 = 4$ $v_2 = 12$, $v_3 = 22$, $v_4 = 11$, $v_5 = 0$

Calculations For un-Allocated cells;

$\Delta_{12} = 7$, $\Delta_{21} = 1$, $\Delta_{23} = -4$, $\Delta_{24} = 7$, $\Delta_{25} = 4$

$\Delta_{31} = 8$, $\Delta_{34} = 9$, $\Delta_{35} = 6$

We Choose $\Delta_{3,2} = -4$ As Most Negative value.

	1	2	3	4	5	
A	40 (20)	25	22	33 (20)	0 (50)	$u_1 = 0$
B	44 (20)	35	30 (10)	30	0	$u_2 = 4$
C	38	38 (20)	28 (50)	30	0	$u_3 = -2$

$v_1 = 4$, $v_2 = 8$ $v_3 = 12$, $v_4 = 11$, $v_5 = 0$

Calculation for un-Allocated cells;

$\Delta_{1,2} = 17$, $\Delta_{1,3} = 4$, $\Delta_{2,2} = 5$, $\Delta_{2,4} = 7$

$\Delta_{2,5} = 4$, $\Delta_{3,1} = 4$.

Hence All $C_{ij} - (u_i + v_j) \geq 0$ So, we obtain our optimal solution Here;

	1	2	3	4	5
A	40 (20)	25	22	33 (20)	0 (50)
B	44 (20)	35	30 (10)	30	0
C	38	38 (20)	28 (50)	30	0

$= (40 \times 20) + (33 \times 30) + (0 \times 50) + (44 \times 20) + (30 \times 10) + (38 \times 20) + (28 \times 50)$

$= 800 + 990 + 0 + 880 + 300 + 760 + 1400$.

$=$ Rs. 5130.



Lecture # 35

MTh 601

Operations Research

(Lecture # 25)

Q; What is Degeneracy In OR?

In Standard transportation problem with m Sources of Supply and n demand destinations, the test of optimality of any Feasible Solution required Allocations in (m+n-1) independent cells.

If the Number of Allocation is Short of the required Number, then the Solution is said to be <u>degenerate</u>.

Q; How Degeneracy in a transportation problem may be Resolved?

The Conventional Method is to Allocate an Small amount (ε) to one of the Independent cell. (Allocate a Small positive Quantity 'e' to one or More unoccupied cell that have lowest transportation Costs. so as to Make (m+n-1) Allocations.

⇒ (Degeneracy In transportation problem)

(Procedure For Degeneracy:-) m+n-1

1- Checking the Demand & Supply (blanced or unblanced condition)

2- Initial Basic Solution.

3- Applying the Condition (m+n-1) (problem is non-degenerate or degenerate)

4- Converting Degeneracy to non-degeneracy-

5- Apply optimality test.

Example;

Slove the Following transportation problem to Minimize the total cost of transportation:-

origin	1	2	3	4	supply
1	14	56	48	27	70
2	82	35	21	81	47
3	99	31	71	63	93
Demand	70	35	45	60	210

Apply Vogel's Approximation Method to obtain an Initial basic Feasible Solution:-

	1	2	3	4	Supply	
1	14 (70)	56	48	27	~~70~~	0
2	82	35 (2)	21 (45)	81	~~47~~	20
3	99	31 (33)	71	63 (60)	~~93~~	~~33~~ 0
Dem	~~70~~ 0	~~35~~ ~~33~~ 0	~~45~~ 0	~~60~~ 0		

P_1	P_2	P_3	P_4
13	–	–	–
14	14	46 ←	–
32	32	32	32

P_1	68 ↑	4	27	36
P_2	–	4	50 ↑	18
P_3	–	4	–	18
P_4	–	31	–	63 ↑

After Applying VAM Method:-

	1	2	3	4
1	14 (70)	56	48	27
2	82	35 (2)	21 (45)	81
3	99	31 (33)	71	63 (60)

Now Check Condition for Allocations;

$$m+n-1 = 3+4-1 = 6$$

But we have made Allotment to 5 cells only. Hence this is a degenerate Basic Feasible Solution.

To Resolve the case of Degeneracy, we Introduce a very small Quantity ε in a (lowest or Right corner Independent cell).

Degeneracy to Non-Degeneracy

	1	2	3	4
1	14 (70)	56	48	27 (3)
2	82	35 (2)	21 (45)	81
3	99	31 (33)	71	63 (60)

Now Applying optimality-Test.

	1	2	3	4	
1	14 (70)	56	48	27 (3)	$u_1 = 0$
2	82	35 (2)	21 (45)	81	$u_2 = 40$
3	99	31 (33)	71	63 (60)	$u_3 = 36$

$v_1 = 14$ $v_2 = -5$ $v_3 = -19$ $v_4 = 27$

Now Calculations for un-Allocated Cells;

$\Delta_{12} = 56 - (0 - 5) = 56 + 5 = 61$

$\Delta_{13} = 48 - (0 - 19) = 48 + 19 = 67$

$\Delta_{21} = 82 - (40 + 14) = 82 - 54 = 28$

$\Delta_{24} = 81 - (40 + 27) = 81 - 67 = 14$

$\Delta_{31} = 99 - (36 + 14) = 99 - 50 = 49$

$\Delta_{33} = 71 - (36 - 19) = 71 - 17 = 54$

Hence All $C_{ij} - (u_i + v_j) \geq 0$ So, Here we can obtain our optimal Solution.

$= (14 \times 70) + (35 \times 2) + (21 \times 45) + (31 \times 33) + (63 \times 60)$

$= 980 + 70 + 945 + 1023 + 3780$

$= \underline{6798}$ Rs.

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Operations Research

Chapter 6:-

Assignment Models (Hungarian Algorithm)

Introduction:

Suppose there are n Jobs to be performed and n persons are available for doing these Jobs. Assume that each Person can do each Job at a time. Through with varying degree of efficiency

Let C_{ij} be the cost if the i^{th} Person is assigned to J^{th} Job. The problem is to Find an assignment so that the total cost of Performing all the Jobs is Minimum.

Problem of this Kind are known as Assignment problem.

It can be stated in form of n×n Cost Matrix:-

=) Types of Assignment problem:-

1. Balanced (n×n)
2. un-Balanced (m×n)

Solution of Assignment Problem;

Example :-

A Worker's Manager has to allocate four different Jobs to four workmen. Depending on the efficiency and the capacity of the Individual the time taken by each differ. How should the tasks be Assigned one Job to a worker so as to Minimize the total Man Hours?

Jobs	Worker A	B	C	D
1	10	20	18	14
2	15	25	9	25
3	30	19	17	12
4	19	24	20	10

Solution ;

Step 1; Select the Minimum element in each Row. Subtract this smallest element from all the elements in that Row;

Jobs	worker A	B	C	D
1	0	10	8	4
2	6	16	0	16
3	18	7	5	0
4	9	14	10	0

Step #02 :-

we Subtract the Minimum element in each Column from All the elements in its Column.

	A	B	C	D
1	0	3	8	4
2	6	9	0	16
3	18	0	5	0
4	9	7	10	0

n x n
4

Step #03 :-

Draw Minimum No. of Horizontal and vertical lines (N) to Cover all zero.

(a) If N = n , n = order of Matrix.
then an optimal Solution can be Made.

(b) if N < n , then go to Next step.

∵ (Row Scaning)

Jobs	Workers A	B	C	D
1	[0]	3	8	4
2	6	9	[0]	16
3	18	[0]	5	0
4	9	7	10	[0]

Row → 1 Zero
- Marked
Row → Max Zero
- Skip

N = n
4 = 4

⇒ if $N < n$ then

(4) Determine the smallest uncovered element (x)

(a) write uncovered value = uncovered value − x

(b) Intersection value = Intersection value + x

(c) line value (other value) as same.

(5) Go to the step-3

Number of line draw to cover all Zeros = 4

and order of Matrix = 4

So we can form an Assignment.

Jobs	Workers
1	A
2	C
3	B
4	D

Minimum (Total cost) = 10 + 9 + 19 + 10 = 48

(Unblanceel Assignment Problem)

	A	B	C	D	E
1	4	3	6	2	7
2	6	7	8	9	10
3	4	3	2	1	5
4	0	0	6	7	8

Now; $m < n$

The problem is unblanced and to blanceel this we Add a dummy Row with Zero entries.

	A	B	C	D	E
1	4	3	6	2	7
2	6	7	8	9	10
3	4	3	2	1	5
4	0	0	6	7	8
5	0	0	0	0	0

All follow same step Now as we done for Balanceel Condition.

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Chapter #06
Assignment Models/problems

=> **Multiple Solutions;**

Solution 1

		M_1	M_2	M_3	M_4	M_5
		1	2	3	4	5
J_1	1	5	0	3	2	6
J_2	2	0	~~0~~	5	4	7
J_3	3	~~0~~	3	0	4	~~0~~
J_4	4	~~0~~	1	~~0~~	3	0
J_5	5	6	5	0	~~0~~	~~0~~

$J_1 \rightarrow M_2$
$J_2 \rightarrow M_1$
$J_3 \rightarrow M_3$
$J_4 \rightarrow M_5$
$J_5 \rightarrow M_4$

Solution 2 :-

		M_1	M_2	M_3	M_4	M_5
		1	2	3	4	5
J_1	1	5	0	3	2	6
J_2	2	0	~~0~~	5	4	7
J_3	3	~~0~~	3	~~0~~	4	0
J_4	4	~~0~~	1	0	3	~~0~~
J_5	5	6	5	~~0~~	0	~~0~~

$J_1 \rightarrow M_2$
$J_2 \rightarrow M_1$
$J_3 \rightarrow M_5$
$J_4 \rightarrow M_3$
$J_5 \rightarrow M_4$

(6.5) Maximization In Assignment Model :-

Example :-

Six salesman are to be allocated to Six Sales Regions. The Earning of each Salesman at each Region is given below. How can you find an Allocation, so that the Earnings will be Maximum?

Salesman	1	2	3	4	5	6
A	15	35	0	25	10	45
B	40	5	45	20	15	20
C	25	60	10	65	25	10
D	25	20	35	10	25	60
E	30	70	40	5	40	50
F	10	25	30	40	50	15

Solution ;

Convert ;

Maximization → Minimization

(By Subtracting Highest element from All elements of table / Matrix)

In given problem Highest element is (70.)

	1	2	3	4	5	6
A	55	35	70	45	60	25.
B	20	65	25	50	55	50
C	45	10	60	5	45	60
D	45	40	35	60	45	10
E	40	0	30	65	30	20
F	60	45	40	30	20	55

=> Row Reduction Matrix :-

	1	2	3	4	5	6
A	30	10	45	20	35	0
B	0	45	5	30	30	30
C	40	5	55	0	40	55
D	35	30	25	50	35	0
E	40	0	30	65	30	20
F	40	25	20	10	0	35

=> Column Reduction Matrix;

	1	2	3	4	5	6
A	30	10	40	20	35	0
B	0	45	0	30	30	30
C	40	5	45	0	40	55
D	35	30	20	50	35	0
E	40	0	25	65	30	20
F	40	25	15	10	0	35

n=6

N ≠ n

N < n

Hence the Number of line drawn to Cover All Zeros = 5, is less than Order of Matrix = 6

we Form 2nd Modified table.

	1	2	3	4	5	6
A	15	10	25	20	35	0
B	~~20~~	60	0	45	45	45
C	25	5	30	0	40	55
D	20	30	Min 5	50	35	0
E	25	0	10	65	30	20
F	25	25	0	10	0	35

N < n

Hence the Number of line drawn to Cover All Zeros = 5 is less than the order of Matrix = 6

we Form 3rd Modified table:-

	1	2	3	4	5	6
A	10	10	20	20	30	0
B	0	65	0	50	45	50
C	20	5	25	0	35	55
D	15	30	0	50	30	0
E	20	0	5	65	25	20
F	25	30	0	15	0	35

N = n

Hence No. of line to cover Zeros equal to order of Matrix = 6

So, we Make the Assignment.

	1	2	3	4	5	6
A	10	10	20	20	30	[0]
B	[0]	65	~~0~~	~~5~~0	45	50
C	20	5	25	[0]	35	55
D	15	30	[0]	50	30	~~0~~
E	20	[0]	5	65	25	20
F	25	30	~~0~~	15	[0]	30

The optimum Assignment is;

A ⟶ 6
B ⟶ 1
C ⟶ 4
D ⟶ 3
E ⟶ 2
f ⟶ 5

Total Earnings are:-

= 45 + 40 + 65 + 35 + 70 + 50 = Rs. 305