Mth621 for final

- Mth621 Quiz#2 All in 1 file 2021(pg51)
- Mth621 Quiz#3 All in 1 file 2021(pg38)
- MTH621 M$FINAL WIKI (pg227)
- Mth621 mega file final term (pg421)
- Mth621.Final Term Mega File..Solved By MS (pg421)
- Mth621 final term subjective solve (pg17)
- MTH621 PAST PAPER (pg52)
- MTH 621 important question by Waseem Gill (pg33)
- final subjective by Khalil sahir mth621(q1-q9)(pg3)
- Mth621.Final Solved Subjective. Make by Stylish 1paper (q1-q6)(pg6)
- Mth621.Final subjective file..MS 1paper(pg6)
- Mth621.Final subjective .Solved by MS(Q36,q32,q33,q38)(pg8)

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MTH621 Quiz#2 Module (126-151)

Due Date 2-4 August 2021

- Mid.Mth621 sol. Files preparation group 25feb2019 quiz#2 (pg10)
- MTH621-QUIZ-Unsolved-By-MTA quiz2 (pg10)
- MTH621 Quiz file current 02-21-2021 11.26 Quiz2 (pg21)
- (same)Mid.Mth621.sol.files preparation Group Quiz2 21feb2021 (pg21)
- MTH621-QUIZ-BY-KINAT-SHAFIQUE-2 (pg52)
- 621 final term (pg48)
- (Same) Personal-File(1)mth621 (Pg48)
- 621 final term (pg31)
- (Same) Personal-File(2)mth621 (Pg31)
- Mth621.Quiz#2.2 Aug2021(126to151)Made by stylish (pg61)
- Mth621.Quiz#2.(4Aug 2021).Solved by Stylish (pg18)
- Mth621.Final quiz Current(02-21-2021).Solved by MS (pg21)
- Mth621 2Aug Solved by billi (pg61) quiz2
- Mth621 4aug2021 (pg10)

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MTH621 Quiz 2

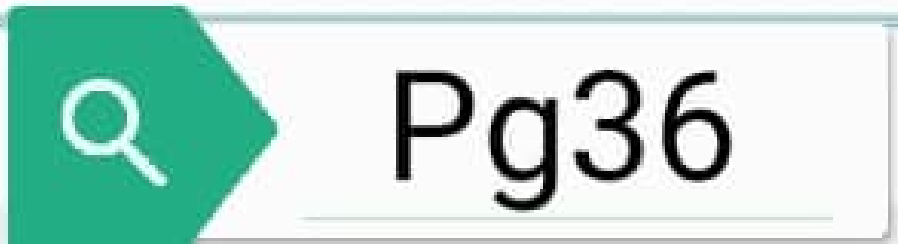
Question # 10 of 10 (Start time: 02:15:53 PM, 28 November 2017)

The limit of the sequence

$$s_n = \frac{1}{n} + \frac{2(1+3/n)}{1+1/n}.$$

Select the correct option

- 1
- 4
- 3
- 2



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MTH621:Quiz 2

Question # 6 of 10 (Start time: 02:11:19 PM, 28 November 2017)

$\limsup_{n \to +\infty} s_n = \infty$ if

Select the correct option

- (●) $\{s_n\}$ is not bounded above.
- ( ) $\{s_n\}$ has unique limit.
- ( ) $\{s_n\}$ is convergent.
- ( ) $\{s_n\}$ has limit inferior.

Pg 40

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MTH621:Quiz 2

Question # 9 of 10 (Start time: 02:14:42 PM, 28 November 2017)

Every Cauchy sequence has a

Select the correct option

- convergent subsequence. 
- increasing subsequence.
- decreasing subsequence.
- positive subsequence.

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MTH621:Quiz 2

Question # 3 of 10 (Start time: 02:08:24 PM, 28 November 2017)

Which of the following given sets is compact subset of $\mathbb{R}$.

Select the correct option

- ○ (1,5]
- ○ {1,2,4}
- ● [0,1] ✓
- ○ (2,3)

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MTH621:Quiz 2

Question # 5 of 10 (Start time: 02:10:35 PM, 28 November 2017)

Which statement(s) is(are) true about the following sequence $s_0 = 1$ and $s_n = 1 - e^{-s_{n-1}}$

Select the correct option

- ○ $0 < s_n \leq 1$ for all n.
- ○ $s_{n+1} - s_n = -(e^{-s_n} - e^{-s_{n-1}})$.
- ○ Sequence is convergent.
- ● All of the above.



Pg 33



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MTH621:Quiz 2

Question # 4 of 10 (Start time: 02:09:40 PM, 28 November 2017)

Identify the identical sequences.

Select the correct option

- $\left\{\frac{1}{n^2-2}\right\}_3^{\infty}$ and $\left\{\frac{1}{n}\right\}_1^{\infty}$.
- $\left\{\frac{1}{n}\right\}_3^{\infty}$ and $\left\{\frac{1}{n}\right\}_1^{\infty}$.
- $\left\{\frac{1}{n-2}\right\}_3^{\infty}$ and $\left\{\frac{1}{n^2}\right\}_1^{\infty}$.
- $\left\{\frac{1}{n-2}\right\}_3^{\infty}$ and $\left\{\frac{1}{n}\right\}_1^{\infty}$. ✓

Pg29

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MTH621:Quiz 2

Question # 2 of 10 (Start time: 02:07:17 PM, 28 November 2017)

If the sequence is increasing, then it

Select the correct option

- converges to its supremum.
- diverges.
- may converges to its supremum.
- is bounded.



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MTH621:Quiz 2

Question # 7 of 10 (Start time: 02:12:09 PM, 28 November 2017)

The series $\sum_{n=0}^{\infty} r^n, \quad -1 < r < 1$, is

Select the correct option

- divergent.
- convergent and converges to $\frac{1}{1+r^2}$.
- convergent and converges to 5.
- convergent and converges to $\frac{1}{1-r}$.



Pg 43

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MTH621:Quiz 2

Question # 8 of 10 (Start time: 02:13:27 PM, 28 November 2017)

The set of terms of $\{s_{n_k}\}$ is contained in the set of terms of $\{s_n\}$ implies

Select the correct option

- $\sup\{s_n\} = \sup\{s_{n_k}\}$.
- $\sup\{s_n\} \leq \sup\{s_{n_k}\}$.
- $\sup\{s_n\} \geq \sup\{s_{n_k}\}$. Pg38
- $\{s_n\}$ is convergent.

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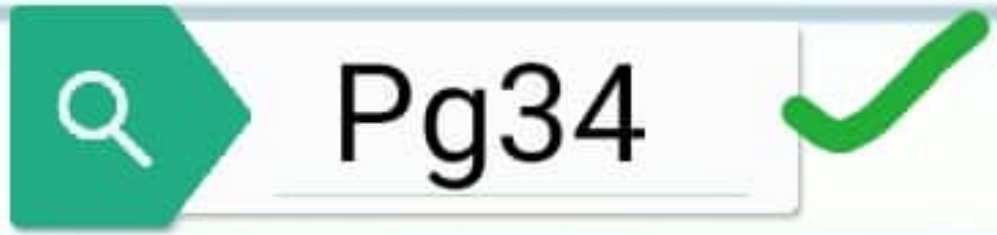
MTH621 Quiz 2

Question # 6 of 10 (Start time: 08:56:23 PM, 30 November 2017)

If $\{s_n\}$ converges to -1 and $\{t_n\}$ converges to 5, then the sequence $\{s_n + t_n\}$

Select the correct option

- will be divergent.
- will be convergent and its limit is 5.
- will be convergent and its limit is 4.
- may be convergent and may be divergent, it depends on the particular given sequences $\{s_n\}$ and $\{t_n\}$.



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MTH621 Quiz 2

Question # 2 of 10 (Start time: 08:54:27 PM, 30 November 2017)

Which statement(s) is(are) true for the following sequence $(\sqrt[n]{n})$

Select the correct option

○	The sequence is divergent.
●	The sequence is convergent and converges to 1.
○	The sequence is convergent and converges to 0.
○	The sequence is divergent to ∞.

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MTH621:Quiz 2

Question # 8 of 10 (Start time: 08:57:41 PM, 30 November 2017)

The value of the $\lim_{n \to +\infty} \frac{1}{n^p}$ is

Select the correct option

- $\lim_{n \to +\infty} \frac{1}{n^p} = 0$ for $p > 0$.
- $\lim_{n \to +\infty} \frac{1}{n^p} = 3$ for all p.
- $\lim_{n \to +\infty} \frac{1}{n^p} = 1$ for all p.
- $\lim_{n \to +\infty} \frac{1}{n^p} = 1$ for all $p > 1$.



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MTH621 Quiz 2

Question # 5 of 10 (Start time: 08:56:02 PM, 30 November 2017)

$$S = (2, 4) \cup (5, 6)$$

is

Select the correct option

- ○ compact
- ○ closed
- ● open
- ○ semi infinite

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MTH621 Quiz 2

Question # 4 of 10 (Start time: 08:55:16 PM, 30 November 2017)

If the sequence is convergent then

Select the correct option

- ○ it has two limits
- ● it is bounded 
- ○ it is bounded above but may not be bounded below
- ○ it is bounded below but may not be bounded above

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MTH621:Quiz 2 Quiz

Question # 6 of 10 (Start time: 10:21:25 PM, 14 February 2020)

Generalized Mean Value Theorem says, If f and g are continuous on the closed interval $[a, b]$ and differentiable on the

Select th

Pg112

- $[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$ for some c in (a, b).
- $[g(b) - g(a)] = [f(b) - f(a)]g'(c)$ for some c in (a, b).
- $g'(x) = 0$ for all x in (a, b).
- $g'(x) = f'(x)$ for all x in (a, b).

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The value of $x + \lim_{x \to 0+} \frac{|x|}{x} =$

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Select the correct option

- ○ $x - 1.$
- ● $x + 1.$
- ○ $x.$
- ○ $x + 1/x.$

Pg81

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The value of the limit

$$\lim_{x \to 0+} x \log x,$$

Select the correct option

●	0
○	1
○	2
○	3

Pg117

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If $\lim_{x \to 0} f(x) = 10$ and $\lim_{x \to 0} g(x) = -2$ then $\lim_{x \to 0}(\frac{f(x)}{g(x)}$ is

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Select the correct option

○	-10.
○	5.
○	0.
●	-5.

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If $f(x) = \log x$ and $g(x) = \frac{1}{1-x^2}$, then $f \circ g$ is

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Select the correct option

- (●) $(f \circ g)(x) = \log \frac{1}{1-x^2}$.
- ( ) $(f \circ g)(x) = \log \frac{x}{1-x^2}$.
- ( ) $(f \circ g)(x) = \log \frac{1}{x^2}$.
- ( ) $(f \circ g)(x) = \log \frac{-1}{1-x^2}$.

Pg99

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The $\lim_{x \to 2}(3x - 5) =$

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Select the correct option

○	0.
●	1.
○	-1.
○	-2.



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The limit

$$\lim_{x \to \infty} x^2 - x.$$

is

Select the correct option

●	does not exist.
○	2.
○	0.
○	6.

Pg118

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For the function defined as

$$f(x) = cx,$$

for every $\epsilon > 0$ the formal definition ensures

Select the correct option

○	$\lvert f(x) - cx_0 \rvert <,\quad \lvert x - x_0 \rvert < \delta,\ \text{where}\ 0 < \delta < \epsilon.$
●	$\lvert f(x) - cx_0 \rvert <,\quad \lvert x - x_0 \rvert < \delta, \text{where}\ 0 < \delta < \epsilon/\lvert c \rvert.$
○	$\lvert f(x) - cx_0 \rvert <,\quad \lvert x - x_0 \rvert < \delta, \text{where}\ -\epsilon < \delta < \epsilon.$
○	$\lvert f(x) - cx_0 \rvert <,\quad \lvert x - x_0 \rvert < \delta, \text{where}\ 0 < \delta < \epsilon.$

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Pg78

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MTH621:Quiz 2

Question # 8 of 10 (Start time: 10:19:34 PM, 14 February 2020)

$\lim_{x \to 0+} x \log x$, has the following indeterminate form

Select the correct option

- ○ $\infty \times \infty$.
- ◉ $(0)(\infty)$.
- ○ $\frac{0}{0}$.
- ○ 0^{∞}.

Pg117

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The $\lim_{x \to 0} \frac{4 - 4\cos x - 2\sin^2 x}{x^4}$ is

Select the correct option

○	$\frac{1}{4}$.
○	$-\frac{1}{2}$.
●	$\frac{1}{2}$.
○	2.

Pg116

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The limit

$$\lim_{x \to 1} \sqrt{x} - 1.$$

Select the correct option

○	1.
●	0.
○	does not exist.
○	-1.

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B

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For the function $f(x) = x \sin \frac{1}{x}, \quad x \neq 0$, which statement is true

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Select the correct option

- $\lim_{x \to 1/\pi} f(x) = 0.$
- The function is not defined at $x = 0.$
- $\lim_{x \to 0} f(x) = 1.$
- $\lim_{x \to 0} f(x) = 0.$

Pg78

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For the piecewise defined function $f(x) = \left\{ {

$$\begin{cases} x^3, & x \leqslant 0, \\ x^2 \sin \frac{1}{x}, & x > 0, \end{cases}$$

Select the correct option

- [x] 0
- [ ] 2
- [ ] 1
- [ ] -1

Pg108

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The value of

$$\lim_{x \to \infty} \frac{e^x}{x^2}$$

elect the correct option

- 0
- $-\infty$.
- ∞.
- 3

Pg116

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For the function $\frac{|x|}{x}$, identify the false statement(s)

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Select the correct option

○	$\lim_{x \to 0+} \frac{|x|}{x} = 1$
○	$\lim_{x \to 0-} \frac{|x|}{x} = -1.$
○	$\lim_{x \to 0} \frac{|x|}{x}$ doesn't exist.
●	$\lim_{x \to 0} \frac{|x|}{x} = 1.$

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MTH621:Quiz-2

Question # 3 of 10 (Start time: 06:37:57 PM, 04 August 2021)

$Investigate\ the\ \lim it\ of\ \lim_{x \to 0} \frac{\sin x}{x} = - - - - -.$

Select the correct option

●	1
○	0
○	$none\ of\ these$
○	x



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MTH621:Quiz-2

Question # 8 of 10 (Start time: 06:39:11 PM, 04 August 2021)

If f is differentiable at a local extreme point $x_0 \in D_f^0$, then

Select the correct option

- $x_0 = 0.$
- $f'(x_0) = 0.$ Pg110
- $|f'(x_0)| < 0.$
- $f'(x_0) \neq 0.$

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MTH621:Quiz-2

Question # 10 of 10 (Start time: 06:39:37 PM, 04 August 2021)

The inverse of the given function $f(x) = 2x + 4, \quad 0 \leqslant x \leqslant 2$, is

Select the correct option

○	$f^{-1}(y) = \frac{x-2}{4}$.
●	$f^{-1}(y) = \frac{y-4}{2}$. 
○	$f^{-1}(y) = \frac{y-2}{4}$.
○	$f^{-1}(y) = \frac{y-4}{4}$

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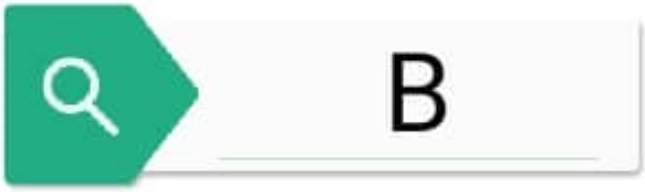
MTH621:Quiz-2

Question # 3 of 10 (Start time: 06:41:31 PM, 04 August 2021)

Which function has $0/0$ form

Select the correct option

- $f(x) = 2 \ln x$ at $x = 2$.
- $f(x) = \frac{\sin \pi x}{x-1}$ at $x = 1$.
- $f(x) = \frac{e^{-x}}{x}$ at $x = 2$
- $f(x) = \cos x + x$ at $x = 0$.



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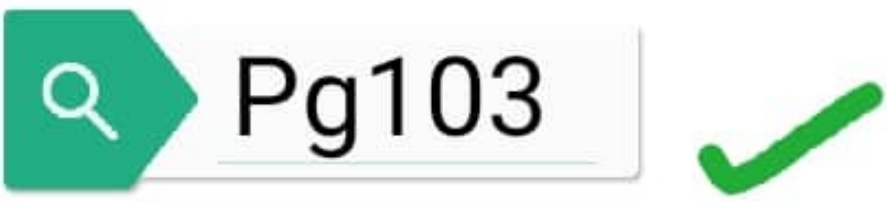
MTH621:Quiz-2

Question # 4 of 10 (Start time: 06:41:42 PM, 04 August 2021)

The derivative of the function x^n is

Select the correct option

○	nx^n.
○	x^{n-1}.
○	$\frac{n}{x^{n-1}}$.
●	nx^{n-1}.



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Question # 5 of 10 (Start time: 06:41:55 PM, 04 August 2021)

The Mean Value Theorem says, suppose that f is differentiable on $[a, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$.

Select the correct option

- (●) Then $f'(c) = \mu$ for some c in (a, b).
- ( ) Then $f'(c) \neq \mu$ for some c in (a, b).
- ( ) Then $f'(c)$ may be 0 or μ for some c in (a, b).
- ( ) Then $f'(c) = 0$ for some c in (a, b).

Pg111

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MTH621:Quiz-2

Question # 9 of 10 (Start time: 06:43:17 PM, 04 August 2021)

The inverse of the function $f(x) = x^2$, is

Select the correct option

- $f^{-1}(y) = y^2$
- $f^{-1}(y) = \frac{1}{x^2}$
- $f^{-1}(y) = \frac{1}{y^2}$
- $f^{-1}(y) = \sqrt{y}.$



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MTH621:Quiz-2

Question # 2 of 10 (Start time: 06:46:23 PM, 04 August 2021)

The function $g(x) = x^2$ is

Select the correct option

- undefined on $[0, \infty)$.
- decreasing on $[0, \infty)$.
- increasing on $[0, \infty)$. Pg97
- having negative values on $[0, \infty)$.

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MTH621:Quiz-2

Question # 3 of 10 (Start time: 06:46:37 PM, 04 August 2021)

$Investigate\ the\ \lim it\ of\ \lim_{x\to 0^-}\left(\frac{|x|}{x}+x\right)=-----.$

Select the correct option

○	1
○	*none of these*
●	-1 Pg82 ✓
○	0

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Question # 4 of 10 (Start time: 06:46:48 PM, 04 August 2021)

$The\ series\ \sum a_n b_n\ \ converges\ if\ a_{n+1} \leq a_n\ for\ n\ ---\ k,\ \lim_{x\to\infty}\ a_n = 0,\ and$

$|b_k + b_{k+1} + \ldots + b_n| \leq M,\ \ for\ some\ constant\ M.$

Select the correct option

- $<$
- $\geq$
- $\leq$
- $>$

Pg61

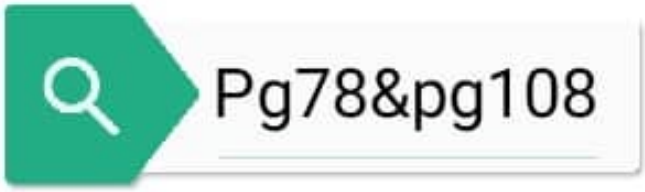
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MTH621:Quiz-2

Question # 1 of 10 (Start time: 08:14:09 PM, 04 August 2021)

$Investigate\ the\ \lim it\ of\ \lim_{x\to 0} x\sin\frac{1}{x} = ----.$

Select the correct option

○	*none of these*
●	0  
○	x
○	1

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MTH621:Quiz-2

Question # 9 of 10 (Start time: 08:15:58 PM, 04 August 2021)

$Investigate\ the\ \lim it\ of\ \lim_{x\to 0^-} \frac{1}{x} = ----.$

Select the correct option

○	$none\ of\ these$
●	$-\infty$ Pg84 ✓
○	∞
○	0

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MTH621:Quiz-2

Question # 6 of 10 (Start time: 06:47:13 PM, 04 August 2021)

Investigate the $\lim it\ of\ \lim_{x\to 0} 2x \sin\sqrt{x} = ----.$

Select the correct option

○	*none of these*
●	0
○	1
○	x

Topic 108, pg80_81

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MTH621:Quiz-2

Question # 2 of 10 (Start time: 08:14:26 PM, 04 August 2021)

If a function f is continuous on the closed interval then f attains its

Select the correct option

- ● extreme values in the closed interval.
- ○ may or may not attain extreme values in the closed interval.
- ○ derivative in the closed interval.
- ○ extreme values at one point in the closed interval.

Pg110

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MTH621:Quiz-2

Question # 3 of 10 (Start time: 08:14:39 PM, 04 August 2021)

If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is

Select the correct option

○	having one sided limit.
●	continuous at c.
○	discontinuous at c.
○	undefined at c.

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MTH621:Quiz-2

Question # 4 of 10 (Start time: 08:14:50 PM, 04 August 2021)

$Investigate\ the\ \lim it\ of\ \lim_{x\to 0^+}\left(\frac{|x|}{x}+x\right)=----.$

Select the correct option

○	0
○	$none\ of\ these$
○	-1
●	1

Pg82

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Question # 5 of 10 (Start time: 08:15:02 PM, 04 August 2021)

Total Marks: 1

Rolle's Theorem says, suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and $f(a) = f(b)$.

Select the correct option

Reload Math Equations

- Then $f'(c) = 0$ for some c in the open interval (a, ∞).
- Then $f'(c) \neq 0$ for some c in the open interval (a, b).
- Then $f'(c) = 0$ for some c in the open interval (a, b).
- Then $f'(c)$ exists for some c in the open interval (a, b).



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MTH621:Quiz-2

Question # 6 of 10 (Start time: 08:15:16 PM, 04 August 2021)

The function $h(x) = -x^3$ is

Select the correct option

○	is undefined on $(-\infty, \infty)$.
●	decreasing on $(-\infty, \infty)$.
○	increasing on $(-\infty, \infty)$.
○	positive on $(-\infty, \infty)$.

Pg 97

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MTH621:Quiz-2

Question # 7 of 10 (Start time: 08:15:30 PM, 04 August 2021)

The series $\sum a_n b_n$ *converges if* $a_{n+1} \leq a_n$ *for* $n \geq k$, $\lim_{x \to \infty} a_n = - - - - - -$, *and* $|b_k + b_{k+1} + \ldots + b_n| \leq M$, *for some constant* M.

Select the correct option

- ○ -1
- ● 0
- ○ *None of these*
- ○ 1



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MTH621:Quiz-2

Question # 8 of 10 (Start time: 08:15:43 PM, 04 August 2021)

The function $f(x) = \begin{cases} x, & 0 \leqslant x < 1, \\ 2, & 1 \leqslant x \leqslant 2, \end{cases}$ is

Select the correct option

- is nonincreasing on $I = [0, 2]$.
- is having negative values on $I = [0, 2]$.
- is nondecreasing on $I = [0, 2]$.
- is undefined on $I = [0, 2]$.

Pg97

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Question # 10 of 10 (Start time: 08:16:14 PM, 04 August 2021)

$The\ \ function\ g(x) = -x^3\ is\ decrea\sin g\ on\ ------.$

Select the correct option

- $none\ of\ these$
- $(-\infty, 0)$
- $(-\infty, \infty)$
- $[0, \infty)$

Pg 97

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Mth621 Quiz#3 Topic (152 to 175) page (112-134)

Due date 16-18 August 2021

- Mth621 real analysis 17feb2019 (pg4)
- Mth621-QUIZ#3-FALL-2019-BY-SHAZIA 16feb2019 (pg10)
- Mth621-QUIZ-3-POINTS-BY-MTA-117feb2019 (pg1)
- Mth621-QUIZ-BY-KINAT-SHAFIQUE-2 mix quiz (pg52)
- Mid.Mth621.sol.files preparation Group-5 2018,2017, quiz4,2,3 mix (pg52)
- Mth621 mega.quiz3 17feb2019 4files (pg68)
- 621 16Aug quiz.pdf (pg10)
- Mth 621 quiz..pdf (pg10)
- Mth621.Quiz#3(16Aug2021) made by stylish (pg10)
- Mth621 quiz#3(16 aug2021).current quiz. Solved by stylish(pg10)
- Mth621 quiz no 4-1 17aug2021 (pg21) exactly quiz3
- Mth621 quiz no 4 17aug2021 (pg17) exactly quiz3
- Mth621 verified by tajji Badsha 18aug2021 (pg10)
- Mth621 quizz#3 17august 2021 for final by @ TALHA (37pg)
- Mth621.Quiz#3(152to175)..(17Aug2021).Made by Stylish(pg23)
- 621{18AUG} solved quiz by billi (112 pg)
- Mth621..mega quiz 3.Solved By MS (pg68)

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MTH621:Quiz 3 - Lecture 23 to 27

Question # 9 of 10 (Start time: 09:48:57 PM, 18 January 2018)

For the function $f(x) = x \sin \frac{1}{x}, \quad x \neq 0$, which statement is true

Select the correct option

- $\lim_{x \to 1/\pi} f(x) = 0.$
- The function is not defined at $x = 0.$
- $\lim_{x \to 0} f(x) = 1.$
- $\lim_{x \to 0} f(x) = 0.$

Pg78

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MTH621:Quiz 3 - Lecture 23 to 27

Question # 7 of 10 (Start time: 07:52:40 PM, 16 January 2018)

For the function defined as

$$f(x) = cx,$$

for every $\epsilon > 0$ the formal definition ensures

Select the correct option

- $|f(x) - cx_0| <, \quad |x - x_0| < \delta, \text{ where } 0 < \delta < \epsilon.$
- $|f(x) - cx_0| <, \quad |x - x_0| < \delta, \text{where } 0 < \delta < \epsilon/|c|.$ ✓
- $|f(x) - cx_0| <, \quad |x - x_0| < \delta, \text{where } -\epsilon < \delta < \epsilon.$
- $|f(x) - cx_0| <, \quad |x - x_0| < \delta, \text{where } 0 < \delta < \epsilon.$

Pg78

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MTH621:Quiz 3 - Lecture 23 to 27

Question # 9 of 10 (Start time: 07:54:42 PM, 16 January 2018)

The limit

$$\lim_{x \to 1} \sqrt{x} - 1.$$

Select the correct option

- 1.
- 0.
- does not exist.
- -1.

B

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MTH621:Quiz 3 - Lecture 23 to 27

Question # 6 of 10 (Start time: 07:51:32 PM, 16 January 2018)

The $\lim_{x \to 0} \frac{4 - 4\cos x - 2\sin^2 x}{x^4}$ is

Select the correct option

- $\frac{1}{4}$
- $-\frac{1}{2}$
- $\frac{1}{2}$ Pg116 ✓
- 2

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MTH621:Quiz 3 - Lecture 23 to 27

Question # 6 of 10 (Start time: 09:47:44 PM, 18 January 2018)

If $\lim_{x \to 0} f(x) = 10$ and $\lim_{x \to 0} g(x) = -2$ then $\lim_{x \to 0} \frac{f(x)}{g(x)}$ is

Select the correct option

○	-10
○	5
○	0
●	-5

D

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MTH621:Quiz 3 - Lecture 23 to 27

Question # 8 of 10 (Start time: 09:48:22 PM, 18 January 2018)

For the piecewise defined function $f(x) = \left\{ {

$$\begin{array}{ll} x^3, & x \leq 0, \\ x^2 \sin \frac{1}{x}, & x > 0, \end{array}$$

Select the correct option

- (●) 0
- ( ) 2
- ( ) 1
- ( ) -1



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MTH621:Quiz No 3

Question # 3 of 10 (Start time: 03:28:42 PM, 16 February 2019)

If a function f is continuous on the closed interval then f attains its

Select the correct option

- extreme values in the closed interval.
- extreme values at one point in the closed interval.
- may or may not attain extreme values in the closed interval.
- derivative in the closed interval.

Pg110

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MTH621:Quiz No 3

Question # 10 of 10 (Start time: 03:33:30 PM, 16 February 2019)

The function $f(x) = \begin{cases} x, & 0 \leq x < 1, \\ 2, & 1 \leq x \leq 2, \end{cases}$ is

Select the correct option

- is nondecreasing on $I = [0, 2]$.
- is nonincreasing on $I = [0, 2]$.
- is having negative values on $I = [0, 2]$.
- is undefined on $I = [0, 2]$.

Pg 97

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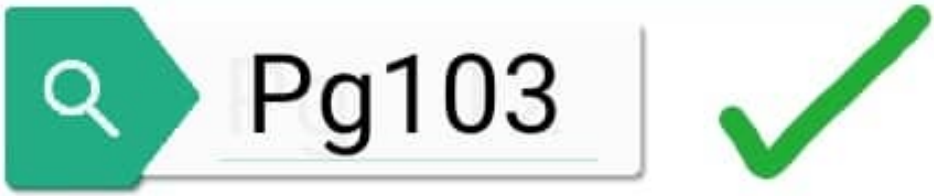
MTH621:Quiz No 3

Question # 4 of 10 (Start time: 03:29:43 PM, 16 February 2019)

The derivative of the function x^n is

Select the correct option

- x^{n-1}.
- nx^{n-1}.
- $\frac{n}{x^{n-1}}$.
- nx^n.



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Question # 5 of 10 (Start time: 03:30:23 PM, 16 February 2019)

The Mean Value Theorem says, suppose that f is differentiable on $[a, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$.

Select the correct option

- Then $f'(c) \neq \mu$ for some c in (a, b).
- Then $f'(c) = \mu$ for some c in (a, b).
- Then $f'(c) = 0$ for some c in (a, b).
- Then $f'(c)$ may be 0 or μ for some c in (a, b).

Pg111

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MTH621:Quiz No 3

Question # 2 of 10 (Start time: 03:28:07 PM, 16 February 2019)

If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is

Select the correct option

●	continuous at c.
○	discontinuous at c.
○	undefined at c.
○	having one sided limit.

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MTH621:Quiz No 3

Question # 10 of 10 (Start time: 03:33:30 PM, 16 February 2019)

The function $f(x) = \begin{cases} x, & 0 \leq x < 1, \\ 2, & 1 \leq x \leq 2, \end{cases}$ is

Select the correct option

- is nondecreasing on $I = [0, 2]$.
- is nonincreasing on $I = [0, 2]$.
- is having negative values on $I = [0, 2]$.
- is undefined on $I = [0, 2]$.

Pg97

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MTH621:Quiz No 3

Question # 4 of 10 (Start time: 01:01:38 AM, 17 February 2019)

The function $h(x) = -x^3$ is

Select the correct option

- ○ is undefined on $(-\infty, \infty)$.
- ● decreasing on $(-\infty, \infty)$. Pg 97 ✓
- ○ increasing on $(-\infty, \infty)$.
- ○ positive on $(-\infty, \infty)$.

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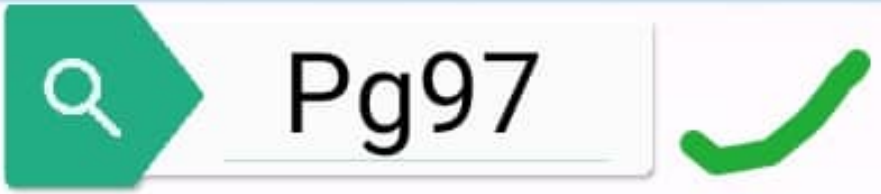
MTH621:Quiz No 3

Question # 1 of 10 (Start time: 12:59:51 AM, 17 February 2019)

The function $g(x) = x^2$ is

Select the correct option

- ○ decreasing on $[0, \infty)$.
- ● increasing on $[0, \infty)$.
- ○ having negative values on $[0, \infty)$.
- ○ undefined on $[0, \infty)$.



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MTH621:Quiz-3

Question # 1 of 10 (Start time: 05:37:55 PM, 17 August 2021)

The series $\sum(-1)^n a_n$ *converges if* $0 \leq a_{n+1} \leq a_n$ *and* $\lim_{x\to\infty} a_n =$ ------.

Select the correct option

- -1
- 0
- *None of these*
- 1

Pg62

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MTH621:Quiz-3

Quiz Start Time: 10:57 PM, 17 August 2021

Question # 2 of 10 (Start time: 10:58:26 PM, 17 August 2021)

Total Marks: 1

Let f be bounded on $[a,b]$, and let P be a partition on $[a,b]$. Then
The upper sum $s(P)$ of f over P is the $--------$ of the set of all Riemann sums of f over P.

Select the correct option

Reload Math Equations

- infimum
- supremum

Pg131

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sec(s)

MTH621:Quiz-3

Quiz Start Time: 10:57 PM, 17 August 2021

Question # 8 of 10 (Start time: 11:03:35 PM, 17 August 2021)

Total Marks: 1

Generalized Mean Value Theorem says, If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

Select the correct option

Reload Math Equations

- $g'(x) = 0$ for all x in (a, b).
- $[g(b) - g(a)] = [f(b) - f(a)]g'(c)$ for some c in (a, b).
- $[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$ for some c in (a, b).
- $g'(x) = f'(x)$ for all x in (a, b).

Pg121

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Question # 1 of 10 (Start time: 03:52:38 AM, 18 August 2021)

$If\ \ f\ is\ continuous\ at\ x_o\ ,\ then\ f\ is\ diffrentiable\ at\ x_o.$

Select the correct option

- ● $false$ Pg105 ✔
- ○ $true$

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MTH621:Quiz-3

Question # 10 of 10 (Start time: 01:19:07 AM, 18 August 2021)

The radius of convergence of $\sum a_n (x - x_n)^n$ *is given by* $- - - - - = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ *if the limit exists in the extended real system.*

Select the correct option

○	R
○	*none of these*
●	$\frac{1}{R}$
○	n

Pg70

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Question # 3 of 10 (Start time: 08:00:34 PM, 18 August 2021)

$If\ \ f\ is\ diffrentiable\ at\ x_o\ ,\ then\ f\ is\ continuous\ at\ x_o.$

Select the correct option

- ● $true$ ✓



- ○ $false$

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MTH621:Quiz-3

Question # 7 of 10 (Start time: 01:09:33 AM, 18 August 2021)

$The\ \ function\ g(x) = x^2\ is\ ------\ on\ [0, \infty).$

Select the correct option

- Pg 97 $inrea\sin g$
- $decrea\sin g$

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MTH621:Quiz-3

Question # 3 of 10 (Start time: 01:20:54 AM, 18 August 2021)

$The\ \ function\ g(x) = x^2\ \ is\ inrea\sin g\ on\ \ ------$

Select the correct option

- $(-\infty, -1)$
- $none\ of\ these$
- $(-\infty\,, 0)$
- $[0, \infty)$

Pg97

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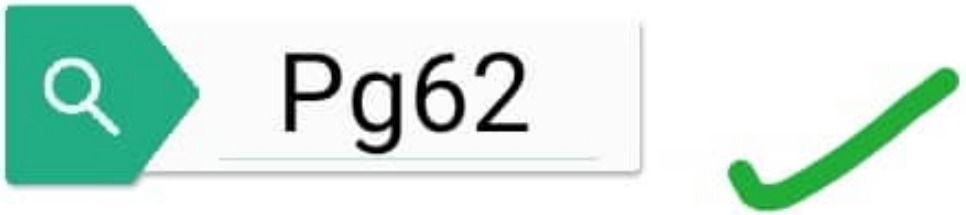
MTH621:Quiz-3

Question # 5 of 10 (Start time: 01:24:53 AM, 18 August 2021)

$The\ series\ \sum (-1)^n a_n \quad - - - - - - - \quad if\ 0 \leq a_{n+1} \leq a_n\ and\ \lim_{x \to \infty}\ a_n = 0$

Select the correct option

- (●) *converges*
- ( ) *diverges*



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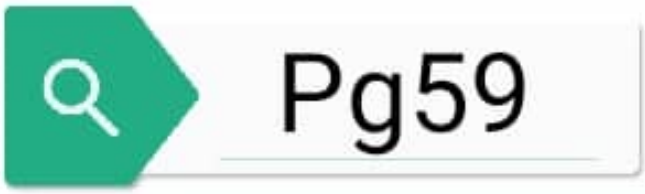
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Question # 3 of 10 (Start time: 03:53:08 AM, 18 August 2021)

$The\ series\ \sum a_n b_n\ \ converges\ if\ a_{n+1} \leq a_n\ for\ n \geq k,\ \lim_{x\to\infty}\ a_n = 0,\ and$

$|b_k + b_{k+1} + \ldots + b_n| - - - - M,\ for\ some\ constant\ M.$

Select the correct option

○	$<$
○	$None\ of\ these$
●	$\leq$ 
○	$\geq$

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Question # 6 of 10 (Start time: 03:53:57 AM, 18 August 2021)

Suppose that f has n derivative at x_o and n is the the smallest positive integer such that $f^n(x_o) \neq 0$. If n is even, x_o is ---------- if $f^{(n)}(x_o) >$

Select the correct option

- *None of these*
- *a local maximum of f.*
- *not a local extreme point of f.*
- *a local minimum of f.* ✓



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MTH621:Quiz-3

Question # 8 of 10 (Start time: 03:54:19 AM, 18 August 2021)

Investigate the value of $\lim_{x\to\infty} x^{\frac{1}{x}} = -----.$

Select the correct option

- ○ *None of these*
- ● Pg119 ✓ 1
- ○ 0
- ○ -1

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Question # 9 of 10 (Start time: 03:54:48 AM, 18 August 2021)

Suppose that f has n derivative at x_o and n is the the smallest positive integer such that $f^n(x_o) \neq 0$. If n is odd, x_o is ----------.

Select the correct option

- *a local maximum of f.*
- *None of these*
- *not a local extreme point of f.* ✓
- *a local minimum of f.*

Pg123

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MTH621:Quiz-3

Question # 10 of 10 (Start time: 03:55:01 AM, 18 August 2021)

$The\ radius\ of\ convergence\ of\ the\ given\ power\ series\ \sum n!x^n\ is\ -----$

Select the correct option

- (●) 0
- ( ) $none\ of\ these$
- ( ) ∞
- ( ) 1

Pg72

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MTH621:Quiz-3

Question # 2 of 10 (Start time: 08:00:21 PM, 18 August 2021)

$If \sum_{n=1}^{\infty} b_n \ is\ rearrangement\ of\ an\ absolutely\ convergernt\ series$

$\sum_{n=1}^{\infty} a_n,\ then \sum_{n=1}^{\infty} b_n\ also\ ------\ absolutely,\ and\ to\ the\ same\ sum.$

Select the correct option

○	*diverges*
●	*converges*

Pg64

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MTH621:Quiz-3

Question # 4 of 10 (Start time: 08:00:49 PM, 18 August 2021)

$In\ the\ Riemann\ \text{integral}\ \int_a^b f(x)\,dx\ ,\ if\ it\ exist\ ,\ is\ ---$

Select the correct option

- $different$
- $unique$

Pg127

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MTH621:Quiz-3

Question # 5 of 10 (Start time: 08:01:01 PM, 18 August 2021)

The radius of convergence of the given power series $\sum \frac{x^n}{n!}$ *is* $------$.

Select the correct option

○	0
●	∞
○	1
○	*None of these*

Pg71

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MTH621:Quiz-3

Quiz Start Time: 08:00 PM, 18 August 2021

Question # 6 of 10 (Start time: 08:01:15 PM, 18 August 2021)

Total Marks: 1

Suppose that f has n derivative at x_0 and n is the the smallest positive integer such that $f^n(x_0) \neq 0$. If n is even, x_0 is ---------- if $f^{(n)}(x_0) < 0$.

Select the correct option

Reload Math Equations

- *None of these*
- *a local maximum of f.*
- *not a local extreme point of f.*
- *a local minimum of f.*

Pg123

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MTH621:Quiz-3

Question # 7 of 10 (Start time: 08:01:27 PM, 18 August 2021)

$\lim_{x \to 0+} x \log x$, has the following indeterminate form

Select the correct option

- ○ $\frac{0}{0}$.
- ○ $\infty \times \infty$.
- ● $(0)(\infty)$.
- ○ 0^{∞}.

Pg117

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MTH621:Quiz-3

Quiz Start Time: 08:00 PM, 18 A

Question # 8 of 10 (Start time: 08:01:39 PM, 18 August 2021)

T

$If\ f\ is\ unbounded\ on\ [a,b]\ ,\ then\ f\ is\ ---------\ on\ [a,b].$

Select the correct option

Reload Ma

- $not\ integrable$ ✓
- $integrable$

Pg129

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MTH621:Quiz-3

Question # 9 of 10 (Start time: 08:01:52 PM, 18 August 2021)

Investigate the value of $\lim_{x \to 0+} x \log x = - - - - -.$

Select the correct option

- -1
- *None of these*
- 0
- 1

Pg117

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Question # 10 of 10 (Start time: 08:02:04 PM, 18 August 2021)

$If\ f\ is\ ------\ on\ [a,b]\ ,\ then\ f\ is\ not\ \text{integrable}\ on\ [a,b].$

Select the correct option

- ○ *bounded*
- ● *unbounded*

Pg129

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MTH621:Quiz-3

Quiz Start Time: 09:58 AM, 18 August 2021

Question # 1 of 10 (Start time: 09:58:31 AM, 18 August 2021)

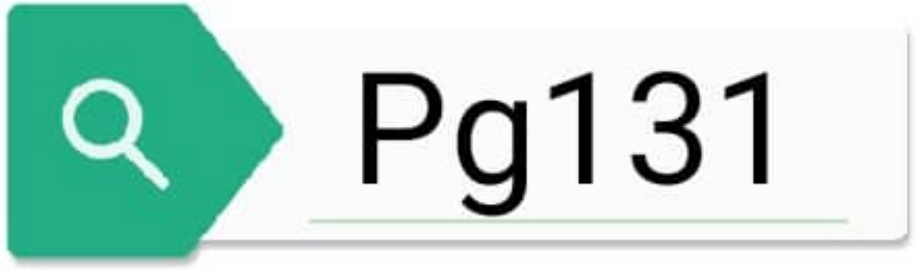
Total Marks: 1

Let f be bounded on $[a,b]$, and let P be a partition on $[a,b]$. Then
The lower sum $s(P)$ of f over P is the $--------$ of the set of all Riemann sums of f over P.

Select the correct option

Reload Math Equations

- sup *remum*
- inf *imum*





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MTH621 Real Analysis I

Question No : 1 of 41

The sum of a rational number and an irrational number is

page 16

Answer (Please select your correct option)

- may be a rational depends on the numbers
- always an irrational number
- always a rational number

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MTH621 Real Analysis I

Question No : 1 of 41

The intersection of set of odd numbers and the set of even numbers is

Answer (Please select your correct option)

- The set of rational numbers
- The empty set
- {1,2,3,4}
- The set of real numbers

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Question No : 2 of 41

Identify the identical sequences.

page 38

Answer (Please select your correct option)

- $\left\{\frac{1}{n^2-2}\right\}_3^\infty$ and $\left\{\frac{1}{n}\right\}_1^\infty$
- $\left\{\frac{1}{n}\right\}_3^\infty$ and $\left\{\frac{1}{n}\right\}_1^\infty$
- $\left\{\frac{1}{n-2}\right\}_3^\infty$ and $\left\{\frac{1}{n^2}\right\}_1^\infty$
- $\left\{\frac{1}{n-2}\right\}_3^\infty$ and $\left\{\frac{1}{n}\right\}_1^\infty$

Pg38

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MTH621 Real Analysis I

Question No : 2 of 41

Identify the false statement(s)

Answer (Please select your correct option)

- The set of natural numbers is not complete.
- The set of rational numbers is complete. ✓
- The set of rational numbers is an infinite set.
- The set of real numbers is complete.

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MTH621 Real Analysis I 188.

Question No : 3 of 41

$\liminf_{n\to\infty} s_n = -\infty$ if

Answer (Please select your correct option)

- (s_n) is not bounded above
- $\lim_{n\to\infty} s_n = 0$
- $\{s_n\}$ is not bounded below
- (s_n) is convergent

Pg40

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MTH621 Real Analysis I

Question No : 3 of 41

Which statement(s) is(are) true for the following sequence $\lim_{n\to\infty} \sqrt[n]{n}$

Answer (Please select your correct option)

- The sequence is oscillatory.
- The sequence converges to 1.
- The sequence converges to 0.
- The sequence is divergent to ∞

B

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111.9K/s

Question No : 4 of 41

If (s_n) is bounded and nonincreasing, then

Answer (Please select your correct option)

- $\lim_{n\to\infty} s_n = \sup(s_n)$
- $\lim_{n\to\infty} s_n = \inf(s_n)$
- (s_n) is divergent

Pg32

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MTH621 Real Analysis I

Question No : 4 of 41

$S = (2,4) \cup (5,6)$ is

Answer (Please select your correct option)

- compact.
- closed.
- open.
- semi infinite.

Pg34

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Question No : 5 of 41

Which of the following given sets is compact subset of ℝ

Answer (Please select your correct option)

- (1,5]
- {1,2,4}
- [0,1]
- (2,3)

Pg28

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MTH621 Real Analysis I

Question No : 5 of 41

A convergent series $\sum a_n$ of nonnegative terms is absolutely convergent, since

Answer (Please select your correct option)

- $\frac{\sum a_n}{\sum |a_n|} = -1$
- $\frac{\sum a_n}{\sum |a_n|} < \infty$
- $\sum a_n - \sum |a_n|$
- $\sum a_n - \sum |a_n|$

Pg58

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MTH621 Real Analysis I

Question No : 6 of 41

$\lim_{n\to\infty} s_n = s$ where s is finite. Then

Answer (Please select your correct option)

- $\lim_{n\to\infty}(cs_n) = s$
- $\lim_{n\to\infty}(cs_n) = c + s$
- $\lim_{n\to\infty}(cs_n) = cs$
- $\lim_{n\to\infty}(cs_n) = cs_n$

Pg34

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MTH621 Real Analysis I

Question No : 7 of 41

For the function $f(x) = x \sin \frac{1}{x}, \quad x \neq 0,$ which statement is true

Answer (Please select your correct option)

- $\lim_{x \to 1/\pi} f(x) = 0$
- The function is not defined at $x = 0$
- $\lim_{x \to 0} f(x) = 1$
- $\lim_{x \to 0} f(x) = 0$



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MTH621 Real Analysis I

Question No : 8 of 41

The value of $x + \lim_{x \to 0+} \frac{|x|}{x} =$

Answer (Please select your correct option)

- $x - 1$.
- $x + 1$.
- x.
- $x + 1/x$.

Pg81

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MTH621 Real Analysis I

Question No : 9 of 41

For the piecewise defined function

$$f(x) = \begin{cases} x^3, & x \leq 0, \\ x^2 \sin \frac{1}{x}, & x > 0, \end{cases}$$

The value of $\lim_{x \to 0^-} f(x)$ is

Answer (Please select your correct option)

- 2
- 0
- 1
- -1

Pg108

Start Time: 11:21 AM

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MTH621 Real Analysis I

Question No : 10 of 45

For f to be uniformly continuous on $[a,b]$.

Pg96

Answer (Please select your correct option)

- f is required to be continuous on a open and bounded interval (a,b)
- f is required to be continuous on a closed and bounded interval $[a,b]$
- f is required to be defined on a open and bounded interval $[a,b]$
- f is required to have limit on a open and bounded interval

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MTH621 Real Analysis I

Question No : 11 of 41

If $f(x) = \log x$ and $g(x) = \frac{1}{1-x^2}$, then $g \circ f$ is

Answer (Please select your correct option)

- $\log \frac{1}{1-x^2}$
- $\frac{1}{1-(\log x)^2}$
- $\log \frac{1}{x^2}$
- $\log \frac{x}{1-x^2}$

Pg90

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MTH621 Real Analysis I

Question No : 12 of 41

The function $g(x) = x^3$ is

Answer (Please select your correct option)

- decreasing on $[0,\infty)$
- increasing on $[0,\infty)$
- having negative values on $[0,\infty)$
- undefined on $[0,\infty)$

Pg97

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MTH621 Real Analysis I

Question No : 12 of 41

The function $f(x) = \begin{cases} x, & 0 \le x < 1, \\ 2, & 1 \le x \le 2, \end{cases}$ is

Answer (Please select your correct option)

- is nondecreasing on $I = [0,2]$.
- is nonincreasing on $I = [0,2]$
- is having negative values on $I = [0,2]$
- is undefined on $I = [0,2]$

Pg97

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MTH621 Real Analysis I

Question No : 13 of 41

The inverse of the given function $f(x) = 2x + 4, \quad 0 \leq x \leq 2,$ is

Answer (Please select your correct option)

- $f^{-1}(y) = \frac{y-4}{2}$
- $f^{-1}(y) = \frac{y-2}{4}$
- $f^{-1}(y) = \frac{x-2}{4}$
- $f^{-1}(y) = \frac{y-4}{4}$

Pg101

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MTH621 Real Analysis I

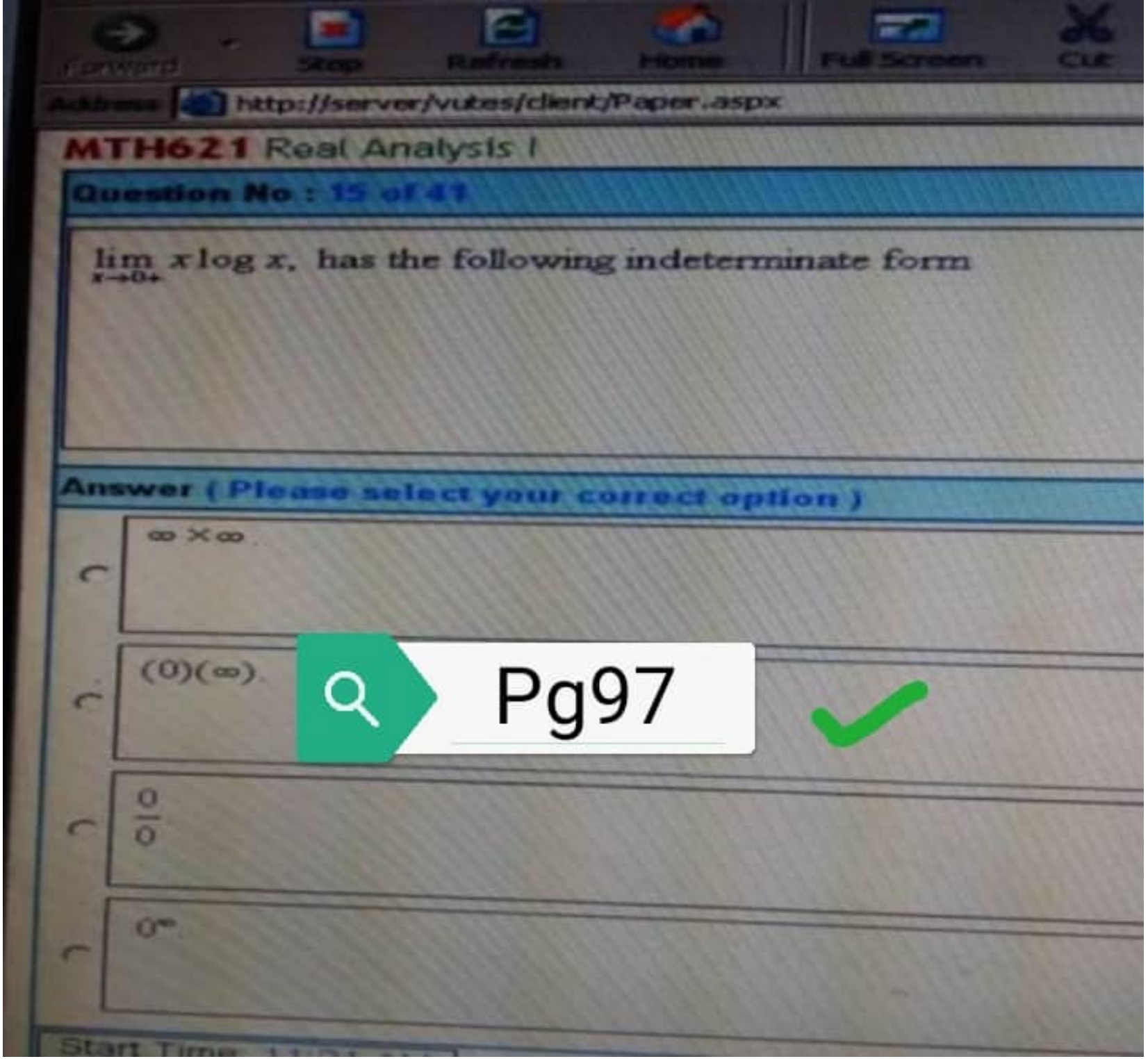
Question No : 14 of 41

If $f : I \longrightarrow \square$ has a derivative at $c \in I$, then f is

Answer (Please select your correct option)

- continuous at c.
- discontinuous at c.
- undefined at c.
- having one sided limit.

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MTH6621 Real Analysis I

Question No : 16 of 41

If f is differentiable at a local extreme point $x_0 \in D_f^0$, then

Answer (Please select your correct option)

- $f'(x_0) = 0$.
- $f'(x_0) = 0$.
- $x_0 = 0$.
- $|f'(x_0)| < 0$.

Pg110

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Question No : 17 of 45

If P and P' are partitions of $[a,b]$, then P' is a refinement of P

Answer [Please select your correct option]

- P has exactly four points
- if every partition point of P' is also a partition point of P
- if every partition point of P is also a partition point of P'
- None of the given choices

Pg126

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MTH621 Real Analysis I

Marks: 1 (Budgeted Time Min)

Question No: 12 of 41

Generalized Mean Value Theorem says, If f and g are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then

Answer (Please select your correct option)

- $f'(c) = g'(c)$ for some c in (a,b)
- $[g(b)-g(a)]=[f(b)-f(a)]$ for some c in (a,b)
- $[g(b)-g(a)]f'(c)=[f(b)-f(a)]g'(c)$ for some c in (a,b)
- $f'(c) = g'(c) = 0$ for some c in (a,b)

Pg150

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MTH621 Real Analysis I

Question No : 18 of 41

The function $f(x) = x$ over the interval $[2,3]$ is

Answer (Please select your correct option)

- Riemann integrable function
- Riemann integrable and equals to $\frac{5}{2}$ ✓
- Riemann integrable and equals to $\frac{2}{5}$
- is not Riemann integrable

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MTH621 Real Analysis I

Question No : 19 of 41

If f is unbounded on the finite closed interval $[a,b]$ then

Answer (Please select your correct option)

- f must be continuous.
- f must have extreme values inside the interval $[a,b]$
- f is Riemann integrable
- f is not Riemann integrable ✓

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MTH621 Real Analysis I

Question No : 20 of 41

If f and g are integrable functions on $[a,b]$ then

Answer (Please select your correct option)

- $\sqrt{f}$ is also integrable on $[a,b]$.
- fg is anti-derivative of f and g on $[a,b]$.
- f^2g is also integrable on $[a,b]$.
- None of the given choices.

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Question No : 21 of 41

If f is integrable on $[a,b]$ and $a \le c \le b$, then the function F defined by $F(x) = \int_a^x f(t)dt$, then which statement is true

Answer (Please select your correct option)

- $F(x)$ is a differentiable function on $[a,b]$.
- $F(x)$ is a constant function on $[a,b]$.
- $F(x)$ is monotone function on $[a,b]$.
- $F(x)$ is Lipschitz continuous on $[a,b]$.

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Pg146



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MTH621 Real Analysis I

Question No : 21 of 41

If a function f is Riemann integrable on $[a,b]$, then it is

Answer (Please select your correct option)

- differentiable on $[a,b]$.
- Riemann integrable on the any subinterval of $[a,b]$.
- Lipschitz continuous on $[a,b]$.
- None of the given choices.

Pg146

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MTH621 Real Analysis I

Question No : 22 of 41

$\sin 2t \Big|_{-\pi/4}^{\pi/4} =$

Answer (Please select your correct optio

- 1
- 2 ✓
- 4
- 0

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Question No : 22 of 41

If $f(x) = e^x$, then $f^{(n)}(x) =$

page 131

Answer (Please select your correct option)

- e^x
- ne^x
- e^{nx}
- $n!$

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MTH621 Real Analysis I

Question No : 23 of 41

By putting $x = \sin t$, in $1 - 2x^2 = \ldots$.

Answer (Please select your correct option)

- $\cos 2t$
- $\sin 2t$
- $\cos t$
- $-\cos 2t$

Pg152

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MTH621 Real Analysis I

Question No : 24 of 41

If $f(x) = 1 - \frac{1}{x^2}$, then $\lim_{x \to \infty} f(x) =$

Answer (Please select your correct option)

- 0
- -1
- 1
- ∞

Pg 83

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MTH621 Real Analysis I

Question No : 25 of 41

If f is a differentiable function at a local extreme point x_0 then $f'(x_0) =$

Answer (Please select your correct option)

- 0 ✔
- positive
- negative
- does not exist

Pg110

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MTH621 Real Analysis I

Question No : 26 of 41

Upper sum of f over P

Answer (Please select your correct opti

- $S(P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1})$ ✓
- $S(P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1})$
- $\overline{\int_a^b} f(x)dx$
- $\underline{\int_a^b} f(x)dx$

Pg130

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MTH621 Real Analysis I

Question No : 27 of 45 | Marks: 1 (Budgeted Time Min)

If f is continuous on $[a,b]$, then f has an antiderivative on $[a,b]$ & Moreover, if f is any antiderivative of f on $[a,b]$, then

Answers (Please select your correct option)

- $\int_a^b f(x)dx = 0$
- $\int_a^b f(x)dx = F(b) - F(x)$
- $\int_a^b f(x)dx = F(x) - F(a)$
- $\int_a^b f(x)dx = F(b) - F(a)$

Pg149

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Identify the anti-derivative of the function $f(x) = x^2 + \sin x$

Answer (Please select your correct option)

- $F(x) = \frac{x^3}{3} + \cos x$
- $F(x) = \frac{x^3}{2} - \sin x$
- $F(x) = \frac{x^3}{3} - \cos x$
- $F(x) = \frac{x^4}{4} - \cos x$

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MTH621 Real Analysis I

Question No : 28 of 41 Marks: 1 (Budgeted Time

For the function $F(x) = \int_0^x f(t)dt = \frac{x^2}{2}$, $0 < x \leq 1$ Identify the function such that $F(x)$ is anti-derivative of the function

Answer (Please select your correct option)

- $f(x) = x^3 + 1$
- $f(x) = x, \quad 0 \leq x \leq 1$
- $f(x) = 2x, \quad 0 \leq x \leq 1$
- $f(x) = x - 2, \quad 0 \leq x \leq 1$

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MTH621 Real Analysis I

Question No : 30 of 41

$\lim_{x \to 0} (\cot x)^{\sin 2x}$, has the following indeterminate form

Answer (Please select your correct option)

- 1^{∞}
- ∞^{0}
- 0^{0}
- 0^{∞}

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MTH621 Real Analysis I

Question No : 31 of 41

Check the continuity of $g(x) = \begin{cases} x+1, & x > 0, \\ x-1, & x \leq 0, \end{cases}$ at $x = 0$

Answer (Please click here to Add Answer)

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(9) $g(x) = \begin{cases} x+1 & x>0 \\ x-1 & x\leq 0 \end{cases}$ at $x=0$

Sol: $g(0) = 0-1 = -1$

LHL Now $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (x-1)$

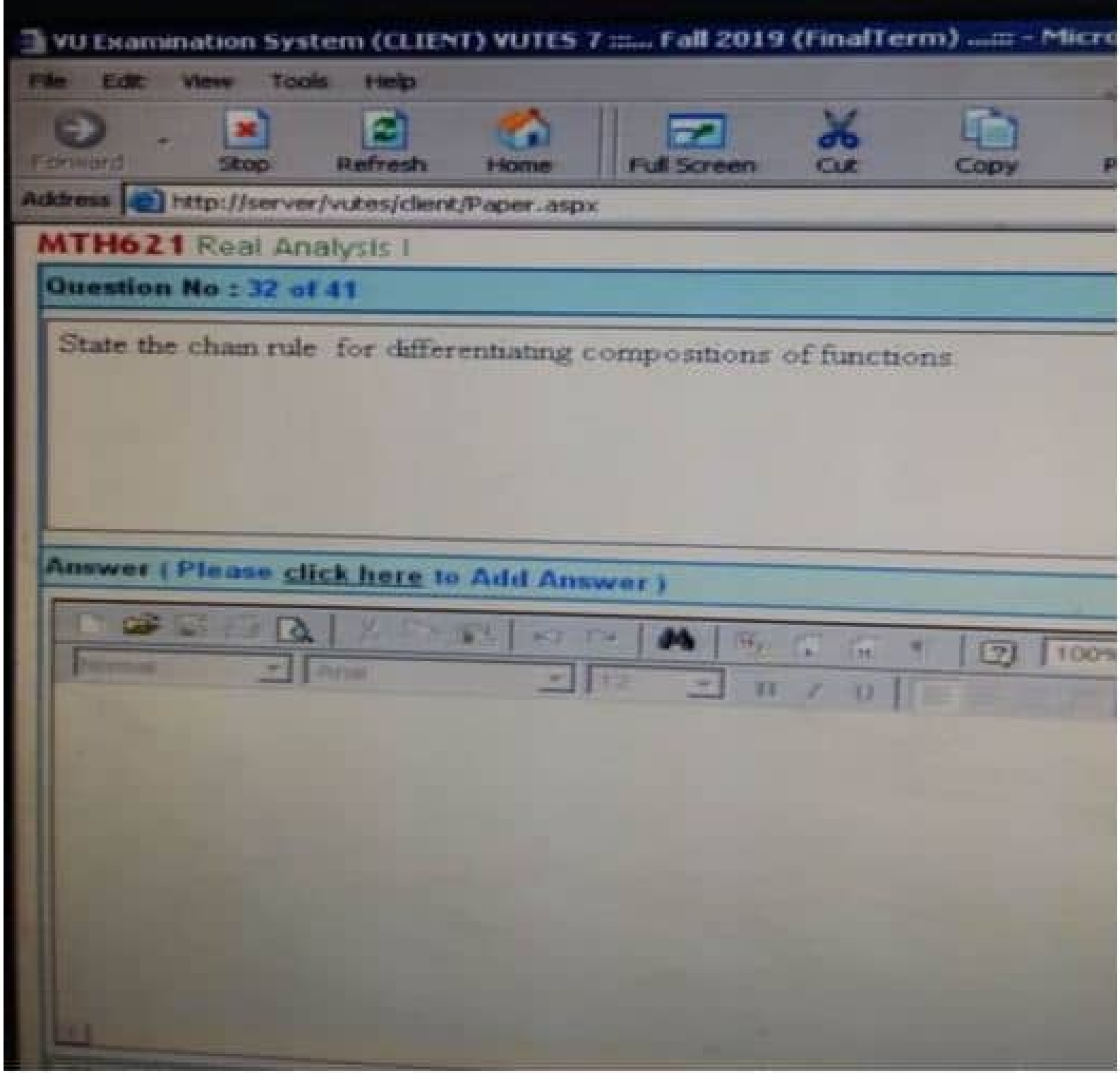
$\lim_{x\to 0^-} g(x) = 0-1 = -1$

RHL $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x+1$

$\lim_{x\to 0^+} g(x) = 0+1 = 1$

As $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$ → So $g(x)$ is not continous at $x=0$

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to differentiate a composition fun. we apply the chain rule.

Let y is a fun of u and u is a fun of x. i.e $(y \circ u)x = y(u(x))$

Then the derivative is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dt} \quad (32)$$

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MTH621 Real Analysis I

Question No : 33 of 41

State Rolle's Theorem.

Answer (Please click here to Add Answer

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(33)

Rolle's Theorem

If a fn f defined on $[a,b]$ is

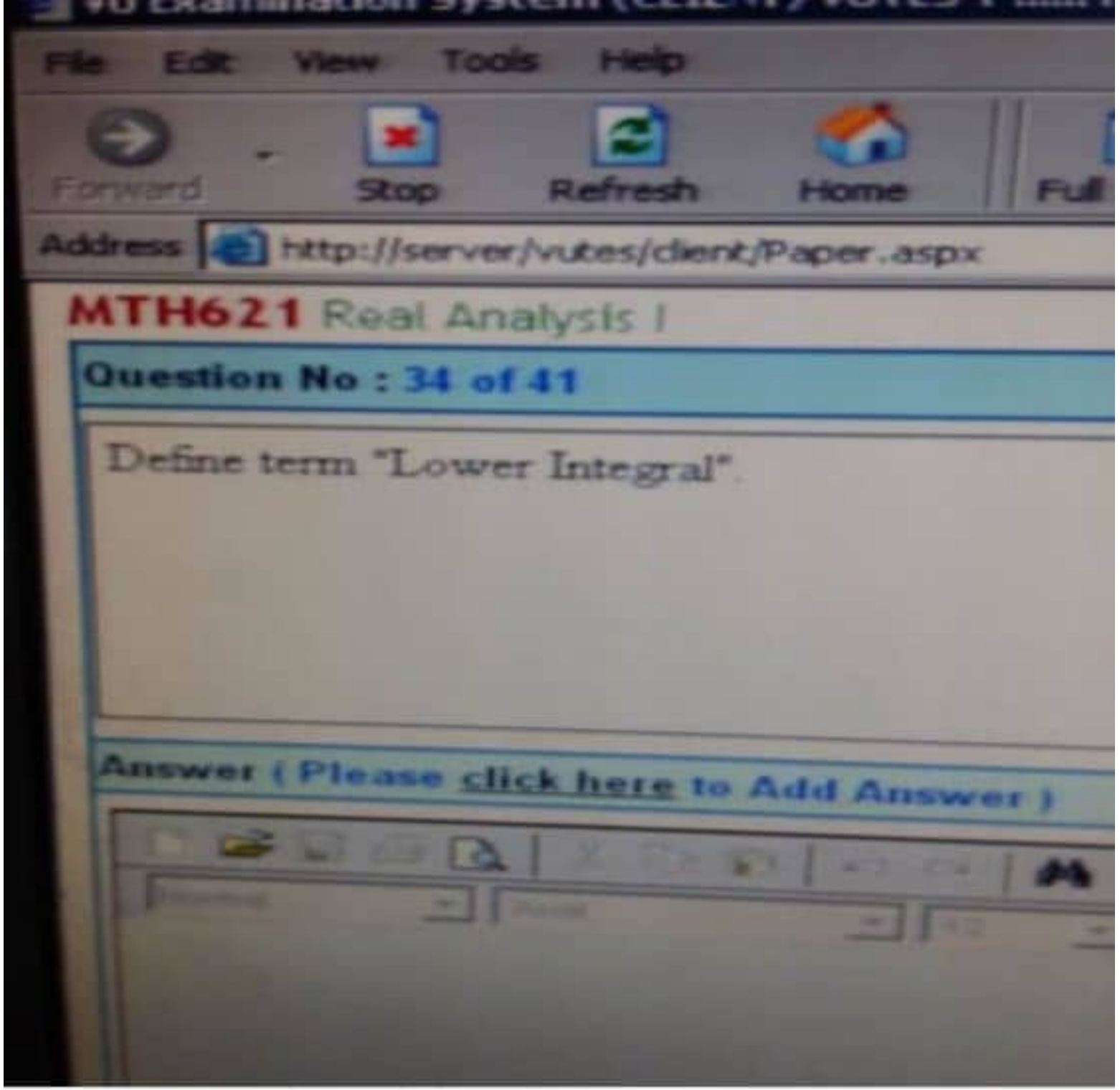
(i) continuous on $[a,b]$

(ii) derivable on (a,b) and

(iii) $f(a) = f(b)$

Then there exist one real number c b/w a and b $(a<c<b)$ such that $f'(c)=0$

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(39) Lower Integral

Let $[a,b]$ be a closed interval.

Let $f:[a,b] \to \mathbb{R}$ be a bounded real valued fn

The lower integral of f over $[a,b]$ is defined as:

$$\int_a^b f(x)\,dx = \sup_P L(P)$$

where supremum is Taken over all subdivision P of $[a,b]$. $L(P)$ denotes the lower sum of f on $[a,b]$.

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MTH621 Real Analysis I

Question No : 35 of 41 **Marks: 3 (Budgeted**

Give an example of the function which is not defined at a certain point but has got the limit at that point.

Answer (Please click here to Add Answer)

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35

$f(x) = \frac{\sin x}{x}$

The fn is undefined at $x = 0$ but limit exist at This point because

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

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MTH621 Real Analysis I

Question No : 36 of 41

If f is differentiable at x_0, then prove that f is continuous at x_0.

Answer (Please click here to Add Answer)

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Proof) Let f is differentiable at x_0. Then

$$D = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

We have to show that it is continuous at x_0.

$$\lim_{x \to x_0} f(x) - f(x_0) = 0$$

So $\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} (x - x_0)$

$$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0)$$

$$\lim_{x \to x_0} f(x) \cdot f(x_0) = (D)(0) = 0$$

$$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

So f is continous at $x = x_0$.

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MTH621 Real Analysis I

Question No : 37 of 41

Let $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases}$

Find $\int_a^b f(x)dx$

Answer (Please click here to Edit Answer)

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sol,
for the given function

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Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since every interval contains both rational and irrational numbers

$$m_j = 0 \quad \text{and} \quad M_j = 1, \quad 1 \leq j \leq n.$$

Hence,

$$\begin{aligned} S(P) &= \sum_{j=1}^{n} 1 \cdot (x_j - x_{j-1}) = b - a \\ s(P) &= \sum_{j=1}^{n} 0 \cdot (x_j - x_{j-1}) = 0. \end{aligned}$$

Since all upper sums equal $b - a$ and all lower sums equal 0, from definition we have

$$\overline{\int_a^b} f(x)\,dx = b - a \quad \text{and} \quad \underline{\int_a^b} f(x)\,dx = 0.$$

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MTH621 Real Analysis I

Question No : 38 of 41

Determine whether the function $f(x) = 1 + x$ is integrable on $[a, b]$?

Answer (Please click here to Add Answer)

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$f(x) = 1 + x$

$$\int_a^b f(x)\,dx = \int_a^b (1+x)\,dx$$

$$\int_a^b f(x)\,dx = x\Big|_a^b + \frac{x^2}{2}\Big|_a^b$$

$$= (b-a) + \frac{b^2}{2} - \frac{a^2}{2}$$

$$= b - a + \frac{b^2}{2} - \frac{a^2}{2}$$

is an integrable on $[a, b]$.

38

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MTH621 Real Analysis I

Question No : 39 of 41

Discuss the continuity of $f(x) = \sqrt{x}, \quad 0 \leq x < \infty$ using formal definition of continuity.

Answer (Please click here to Add Answer)

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Let $\epsilon > 0$, $a = 0$ choose $\delta = \epsilon^2$

then $|x| < \delta \Rightarrow |\sqrt{x}| < \epsilon$

as desired

Let $\epsilon > 0$, $a \in (0, \infty)$. choose

$\delta = \sqrt{a}\,\epsilon$. Then $|x - a| < \delta$

$$\Rightarrow |\sqrt{x} - \sqrt{a}| < \frac{\delta}{|\sqrt{x} + \sqrt{a}|} \leq \epsilon$$

So f is continous at $x = a$

$\forall\ a \in (0, \infty)$

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MTH621 Real Analysis I

Question No : 40 of 41

Prove that if f is integrable on $[a,b]$, then so is $|f|$.

Answer (Please click here to Add Answer)

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We have

$$U(P,f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \quad (40)$$

$$L(P,f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

and hence $U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1})$

it is easy to prove that

$$M_i(f) - m_i(f) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in [x_{i-1}, x_i]\}$$

Now by triangle inequality

$$||f(x_1)| - |f(x_2)|| \leq |f(x_1) - f(x_2)|$$

Therefore

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$$

$$U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f)$$

so $|f|$ is integrable on $[a,b]$

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MTH621 Real Analysis I

Question No : 41 of 41

Show that if f is Lipschitz continuous at $x = x_0$, then f is uniformly continuous at $x = x_0$.

Answer (Please click here to Add Answer)

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Q 41

→ **Def:** (Lipschitz continuity):

Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is Lipschitz continuous if there exists $M>0$ such that

$$|f(x)-f(y)| \leq M \cdot |x-y|, \quad \forall x,y \in A.$$

→ **Prop:** Every Lipschitz continuous function is uniformly continuous.

Proof: Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous.

Then, there exists $M>0$ such that:

$$|f(x)-f(y)| \leq M \cdot |x-y|, \quad \forall x,y \in A.$$

Let $\varepsilon>0$. We choose $\delta = \frac{\varepsilon}{M}$ (depends only on ε).

If $x,y \in A$, and $|x-y|<\delta$, then

$$|f(x)-f(y)| \leq M \cdot |x-y| < M \cdot \delta = M \cdot \frac{\varepsilon}{M} = \varepsilon,$$

i.e. $|f(x)-f(y)|<\varepsilon$.

Since $\varepsilon>0$ was arbitrary, f is uniformly continuous. ■

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Question 1

Define $(f+g)(x), (f-g)(x), (fg)(x)$ and (f/g), for the functions $f(x) = \sqrt{4-x^2}$ and $g(x) = \sqrt{x-1}$.

Solution:

First of all observe that $D_f = [-2, 2]$ and $D_g = [1, \infty)$.
Then $f+g$, $f-g$, and fg are defined on $D_f \cap D_g = [1, 2]$ by

$$\begin{aligned}(f+g)(x) &= \sqrt{4-x^2} + \sqrt{x-1}, \\ (f-g)(x) &= \sqrt{4-x^2} - \sqrt{x-1},\end{aligned}$$

and

$$(fg)(x) = (\sqrt{4-x^2})(\sqrt{x-1}) = \sqrt{(4-x^2)(x-1)}.$$

The quotient f/g is defined on $(1, 2]$ by

$$\left(\frac{f}{g}\right)(x) = \sqrt{\frac{4-x^2}{x-1}}.$$

the last expression in is also defined for $-\infty < x < -2$, it does not represent fg for such x, since f and g are not defined on $(-\infty, -2]$.

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Question 2

Find $\lim_{x\to 2} \dfrac{9-x^2}{x+1}$ and $\lim_{x\to 2}(9-x^2)(x+1)$.

Solution:

If c is a constant, then $\lim_{x\to x_0} c = c$, and $\lim_{x\to x_0} x = x_0$. Therefore

$$\begin{aligned}\lim_{x\to 2}(9-x^2) &= \lim_{x\to 2} 9 - \lim_{x\to 2} x^2 \\ &= \lim_{x\to 2} 9 - (\lim_{x\to 2} x)^2 \\ &= 9 - 2^2 = 5,\end{aligned}$$

and

$$\lim_{x\to 2}(x+1) = \lim_{x\to 2} x + \lim_{x\to 2} 1 = 2 + 1 = 3.$$

Therefore,

$$\lim_{x\to 2} \frac{9-x^2}{x+1} = \frac{\lim_{x\to 2}(9-x^2)}{\lim_{x\to 2}(x+1)} = \frac{5}{3}$$

and

$$\lim_{x\to 2}(9-x^2)(x+1) = \lim_{x\to 2}(9-x^2)\lim_{x\to 2}(x+1) = 5\cdot 3 = 15.$$

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Question 3

Check the continuity of $g(x) = \begin{cases} x+1, & x > 0, \\ x-1, & x < 0; \end{cases}$ **at x=0**

Solution:

every open interval containing $x_0 = 0$ also contains points x_1 and x_2 such that $|g(x_1) - g(x_2)|$ is as close to 2 · Therefore, $\lim_{x \to x_0} g(x)$ does not exist.

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Question 4

Discuss the continuity of f defined on $[0,2]$ by

$$f(x) = \begin{cases} x^2, & 0 \le x < 1, \\ x+1, & 1 \le x \le 2 \end{cases}$$

Solution: : If $0 < x, x_0 < 1$, then

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| = |x - x_0|\,|x + x_0| \\ &\le 2|x - x_0| < \varepsilon \quad \text{if} \quad |x - x_0| < \varepsilon/2. \end{aligned}$$

Hence, f is continuous at each x_0 in $(0,1)$. If $1 < x, x_0 < 2$, then

$$\begin{aligned} |f(x) - f(x_0)| &= |(x+1) - (x_0+1) = |x - x_0| \\ &< \varepsilon \quad \text{if} \quad |x - x_0| < \varepsilon. \end{aligned}$$

Hence, f is continuous at each x_0 in $(1,2)$.

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Question 5

Discuss the continuity of $f(x) = \sqrt{x}, \quad 0 \leq x < \infty.$

Solution: Consider

$$|f(x) - f(0)| = \sqrt{x} < \varepsilon \quad \text{if} \quad 0 \leq x < \varepsilon^2,$$

so $\lim_{x \to 0_+} f(x) = f(0)$. If $x_0 > 0$ and $x \geq 0$, then

$$|f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon \quad \text{if} \quad |x - x_0| < \varepsilon\sqrt{x_0}$$

so $\lim_{x \to x_0} f(x) = f(x_0)$. Hence, f is continuous on $[0, \infty)$.

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Question 6

Piecewise Continuous Functions

Piecewise continuous function: A function f is *piecewise continuous* on $[a, b]$ if

- $\lim_{x \to x_{0_+}} f(x)$ exists for all x_0 in $[a, b)$;
- $\lim_{x \to x_{0_-}} f(x)$ exists for all x_0 in $(a, b]$;
- $\lim_{x \to x_{0_+}} f(x) = \lim_{x \to x_{0_-}} f(x) = f(x_0)$ for all but finitely many points x_0 in (a, b).

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Question 7

the function $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$

Solution:

The function is continuous at all x_0 except $x_0 = 0$.

As x approaches 0 from either side, $f(x)$ oscillates between -1 and 1 with ever-increasing frequency, so neither $\lim_{x\to 0+} f(x)$ nor $\lim_{x\to 0-} f(x)$ exists.

Therefore, the discontinuity of f at 0 is not a jump discontinuity, and if $\rho > 0$, then f is not piecewise continuous on any interval of the form $[-\rho, 0]$, $[-\rho, \rho]$, or $[0, \rho]$.

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Question 8

Removable discontinuity: Let f be defined on a deleted neighborhood of x_0 and discontinuous (perhaps even undefined) at x_0. We say that f has a removable discontinuity at x_0 if $\lim_{x \to x_0} f(x)$ exists.

In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0, \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0, \end{cases}$$

is continuous at x_0.

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Question 9

If $f(x) = \log x$ and $g(x) = \dfrac{1}{1-x^2}$, we define $f \circ g$? and $g \circ f$?

Solution:

The domain of the given functions are

$$D_f = (0, \infty) \quad \text{and} \quad D_g = x \in \mathbb{R} / \, x \neq \pm 1.$$

Since $g(x) > 0$ if $x \in T = (-1, 1)$, the composite function $f \circ g$ is defined on $(-1, 1)$ by

$$(f \circ g)(x) = \log \frac{1}{1-x^2}.$$

$g \circ f$

The function $g \circ f$ is defined on $(0, 1/e) \cup (1/e, e) \cup (e, \infty)$ by

$$(g \circ f)(x) = \frac{1}{1 - (\log x)^2}.$$

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Question 11

Define Intermediate theorem?

Suppose that f is continuous on $[a,b]$, $f(a) \neq f(b)$, and μ is between $f(a)$ and $f(b)$. Then $f(c) = \mu$ for some c in (a,b).

Define Uniform Continuity?

A function f is *uniformly continuous* on a subset S of its domain if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon \text{ whenever } |x - x'| < \delta \text{ and } x, x' \in S.$$

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Question 12

Check the uniform continuity of the function $f(x) = 2x$.

Solution:

For the function $f(x)$, we have

$$|f(x) - f(x')| = 2|x - x'| < \varepsilon \quad \text{if} \quad |x - x'| < \varepsilon/2.$$

The function $f(x)$ is uniformly continuous on $(-\infty, \infty)$,

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Question 14

Show that the function $g(x) = x^2$ is uniformly continuous on $[-r, r]$ for any finite r.

Solution:

$$|g(x) - g(x') = |x^2 - (x')^2|$$

$$= |x - x'|\,|x + x'| \leq 2r|x - x'|$$

so

$$|g(x) - g(x')| < \varepsilon \quad \text{if} \quad |x - x'| < \delta = \frac{\varepsilon}{2r} \quad \text{and} \quad -r \leq x, x' \leq r.$$

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Question 15

The function $g(x) = x^2$ is not uniformly continuous on $(-\infty, \infty)$.

Solution:

we will show that if $\delta > 0$ there are real numbers x and x' such that

$$|x - x'| = \delta/2 \quad \text{and} \quad |g(x) - g(x')| \geq 1.$$

To this end, we write

$$|g(x) - g(x')| = |x^2 - (x')^2| = |x - x'|\,|x + x'|.$$

If $|x - x'| = \delta/2$ and $x, x' > 1/\delta$, then

$$|x - x'|\,|x + x'| > \frac{\delta}{2}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) = 1.$$

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Question 16

Monotonic functions: A function f is *nondecreasing* on an interval I if

$$f(x_1) \leq f(x_2) \quad \text{whenever} \quad x_1, x_2 \in I, \quad x_1 < x_2.$$

or *nonincreasing* on I if

$$f(x_1) \geq f(x_2) \quad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2.$$

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Question 17

If f is nondecreasing, then $\lim_{x \to a+} f(x) = \alpha$

Proof:

We first show that $\lim_{x \to a+} f(x) = \alpha$.

If $M > \alpha$, there is an x_0 in (a, b) such that $f(x_0) < M$. Since f is nondecreasing, $f(x) < M$ if $a < x < x_0$. Therefore, if $\alpha = -\infty$, then $\lim_{x \to a+} f(x) = -\infty$.

If $\alpha > -\infty$, let $M = \alpha + \varepsilon$, where $\varepsilon > 0$.

Then $\alpha \leq f(x) < \alpha + \varepsilon$, so

$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < x_0.$$

If $a = -\infty$, this implies that $f(-\infty) = \alpha$. If $a > -\infty$, let $\delta = x_0 - a$. Then is equivalent to

$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < a + \delta,$$

which implies that $f(a+) = \alpha$.

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Question 19

the function $f(x) = |x|$.

Solution:

The functions can be written as

$$f(x) = \begin{cases} x, & x > 0, \\ -x, & x \leq 0, \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} 1, & x > 0, \\ -1, & x \leq 0, \end{cases}$$

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x}{x}$$

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

are different,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist thus, f is not differentiable at 0, even though it is continuous at 0

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Question 20

Calculate the derivative of $h(x) = \sin\frac{1}{x}, \quad x \neq 0.$

Solution:

If

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{1}{x}, \quad x \neq 0,$$

then

$$h(x) = f(g(x))$$

$$= \sin\frac{1}{x}, \quad x \neq 0,$$

and

$$h'(x) = f'(g(x))g(x)$$

$$= \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \quad x \neq 0.$$

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Question 21

For the piecewise defined function

$$f(x) = \begin{cases} x^3, & x \le 0, \\ x^2 \sin \frac{1}{x}, & x > 0, \end{cases}$$

Investigate the one sided derivatives

Solution:

Solution: For the given function, we have

$$f'(x) = \begin{cases} 3x^2, & x < 0, \\ 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x > 0. \end{cases}$$

We will calculate the one sided derivatives

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0+} x \sin \frac{1}{x} = 0,$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x^3 - 0}{x - 0} = \lim_{x \to 0-} x^2 = 0;$$

hence, $f'(0) = f'_+(0) = f'_-(0) = 0$

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Question 22

Differentiable function: A function f is *differentiable on the closed interval* $[a, b]$ if f is differentiable on the open interval (a, b) and $f'_+(a)$ and $f'_-(b)$ both exist.

Continuously differentiable We say that f is *continuously differentiable on* $[a, b]$ if f is differentiable on $[a, b]$, f' is continuous on (a, b),

$$f'_+(a) = \lim_{x \to a+} f'(x)$$

, and

$$f'_-(b) = \lim_{x \to b-} f'(x).$$

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Question 23

Rolle's Theorem

Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some c in the open interval (a, b).

Proof:

Since f is continuous on $[a, b]$, f attains a maximum and a minimum value on $[a, b]$. If these two extreme values are the same, then f is constant on (a, b), so $f'(x) = 0$ for all x in (a, b).

If the extreme values differ, then at least one must be attained at some point c in the open interval (a, b), and $f'(c) = 0$.

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Question 24

Define

The Mean Value Theorem

Theorem: Suppose that f is differentiable on $[a, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c) = \mu$ for some c in (a, b).

Generalized Mean Value Theorem

Theorem: If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c) \qquad \text{for some } c \text{ in } (a, b).$$

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Question 25

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{for some } c \text{ in } (a,b).$$

Proof:

The function

$$h(x) = [b - a]f(x) - [f(b) - f(a)]x$$

is continuous on $[a,b]$ and differentiable on (a,b).

Furthermore

$$h(a) = h(b) = bf(a) - f(b)a.$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a,b).

Since $h'(c) = [b - a]f'(c) - [f(b) - f(a)]$.

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Question26

Define

Lipschitz Continuity

A function that satisfies inequality

$$|f(x) - f(x')| \leq M|x - x'|, \quad x, x' \in (a, b),$$

for all x and x' in an interval, where $M > 0$ is some real number is said to satisfy a *Lipschitz condition* on the interval.

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Question 27

Give an example of a function which is continuous but not Lipchitz continuous.

Solution:

Example: The function $f(x) = \sqrt{|x|}$ is continuous at $x = 0$, but not Lipschits continuous at $x = 0$.

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Question 28

Define Upper Integral

If f is bounded on $[a,b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, let

$$M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$$

The *upper sum of* f *over* P is

$$S(P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}),$$

and the *upper integral of* f *over*, $[a,b]$, denoted by

$$\overline{\int_a^b} f(x)\,dx,$$

is the infimum of all upper sums.

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Question 30

Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases} \qquad \text{Find} \qquad \overline{\int_a^b} f(x)\,dx$$

Solution:

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since every interval contains both rational and irrational numbers

$$M_j = 1, \quad 1 \le j \le n.$$

Hence,

$$S(P) = \sum_{j=1}^{n} 1 \cdot (x_j - x_{j-1}) = b - a$$

Since all upper sums equal $b - a$ we have

$$\overline{\int_a^b} f(x)\,dx = b - a$$

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Question 31

Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases} \qquad \text{Find} \qquad \int_a^b f(x)\,dx$$

Solution:

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since every interval contains both rational and irrational numbers

$$m_j = 0 \qquad 1 \le j \le n.$$

Hence,

$$s(P) = \sum_{j=1}^{n} 0 \cdot (x_j - x_{j-1}) = 0.$$

Since all lower sums equal 0, from definition we have

$$\int_a^b f(x)\,dx = 0.$$

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Question 32

If f is bounded on $[a,b]$, then $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$.

Proof:

Suppose that P_1 and P_2 are partitions of $[a,b]$ and P' is a refinement of both.

We have proved

$$S(P) \ge S(P') \ge S(P) - 2Mr\|P\| \quad (*)$$

and

$$s(P) \le s(P') \le s(P) + 2Mr\|P\| \quad (**).$$

Letting $P = P_1$ in $(**)$ and $P = P_2$ in $(*)$ shows that

$$s(P_1) \le s(P') \quad \text{and} \quad S(P') \le S(P_2).$$

Since $s(P') \le S(P')$, this implies that $s(P_1) \le S(P_2)$. Thus, every lower sum is a lower bound for the set of all upper sums.

Since $\overline{\int_a^b} f(x)\,dx$ is the infimum of this set, it follows that

$$s(P_1) \le \overline{\int_a^b} f(x)\,dx$$

for every partition P_1 of $[a,b]$.

This means that $\overline{\int_a^b} f(x)\,dx$ is an upper bound for the set of all lower sums.

Since $\underline{\int_a^b} f(x)\,dx$ is the supremum of this set

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Question 33

If f is monotonic on $[a,b]$, then f is integrable on $[a,b]$.

Proof:

Let $P=\{x_0,x_1,\ldots,x_n\}$ be a partition of $[a,b]$. Since f is nondecreasing,

$$\begin{aligned} f(x_j) &= M_j = \sup_{x_{j-1}\le x\le x_j} f(x) \\ f(x_{j-1}) &= m_j = \inf_{x_{j-1}\le x\le x_j} f(x). \end{aligned}$$

Hence,

$$S(P)-s(P)=\sum_{j=1}^{n}(f(x_j)-f(x_{j-1}))(x_j-x_{j-1}).$$

Since $0<x_j-x_{j-1}\le\|P\|$ and $f(x_j)-f(x_{j-1})\ge 0$,

$$\begin{aligned} S(P)-s(P) &\le \|P\|\sum_{j=1}^{n}(f(x_j)-f(x_{j-1})) \\ &= \|P\|(f(b)-f(a)). \end{aligned}$$

Therefore,

$$S(P)-s(P)<\varepsilon \quad \text{if} \quad \|P\|(f(b)-f(a))<\varepsilon,$$

so f is integrable on $[a,b]$.

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Question 34:

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Proof:

Let $P=\{x_0,x_1,\ldots,x_n\}$ be a partition of $[a,b]$

Since f is continuous on $[a,b]$, there are points c_j and c_j' in $[x_{j-1},x_j]$ such that

$$f(c_j)=M_j=\sup_{x_{j-1}\le x\le x_j} f(x)$$

and

$$f(c_j')=m_j=\inf_{x_{j-1}\le x\le x_j} f(x).$$

Therefore,

$$S(P)-s(P)=\sum_{j=1}^{n}\left[f(c_j)-f(c_j')\right](x_j-x_{j-1}).$$

Since f is uniformly continuous on $[a,b]$, there is for each $\varepsilon>0$ a $\delta>0$ such that

$$|f(x')-f(x)|<\frac{\varepsilon}{b-a},$$

If x and x' are in $[a,b]$ and $|x-x'|<\delta$ If $\|P\|<\delta$, then $|c_j-c_j'|<\delta$ and, from

$$S(P)-s(P)<\frac{\varepsilon}{b-a}\sum_{j=1}^{n}(x_j-x_{j-1})=\varepsilon.$$

Hence, f is integrable on $[a,b]$.

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Question 35

Determine whether the function $f(x) = 1 + x$ is integrable on $[a, b]$?

Solution:

If

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{1}{x}, \quad x \neq 0,$$

then

$$h(x) = f(g(x)) = \sin \frac{1}{x}, \quad x \neq 0,$$

and

$$h'(x) = f'(g(x))g(x) = \left(\cos \frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \quad x \neq 0.$$

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Question 36

If f is integrable on $[a, b]$ and $a \leq a_1 < b_1 \leq b$, then f is integrable on $[a_1, b_1]$.

Proof:

Thus $\overline{P} = \{x_r, x_{r+1}, \ldots, x_s\}$ is a partition of $[a_1, b_1]$ over which the upper and lower sums of f satisfy

$$S(\overline{P}) - s(\overline{P}) < \varepsilon.$$

Therefore, f is integrable on $[a_1, b_1]$.

If f is integrable on $[a, b]$ and $a \leq a_1 < b_1 \leq b$, then f is integrable on $[a_1, b_1]$.

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Q1 Prove that: If f is integrable on $[a,b]$, then so is $|f|$

Proof: Let P be a partition of $[a,b]$ and define

$$\begin{aligned} M_j &= \sup\{f(x) : x_{j-1} \le x \le x_j\}, \\ m_j &= \inf\{f(x) : x_{j-1} \le x \le x_j\}, \\ \overline{M}_j &= \sup\{|f(x)| : x_{j-1} \le x \le x_j\}, \\ \overline{m}_j &= \inf\{|f(x)| : x_{j-1} \le x \le x_j\}. \end{aligned}$$

Then
$$\begin{aligned} \overline{M}_j - \overline{m}_j &= \sup\{|f(x)| - |f(x')| : x_{j-1} \le x, x' \le x_j\} \\ &\le \sup\{|f(x) - f(x')| : x_{j-1} \le x, x' \le x_j\} \qquad (5.25) \\ &= M_j - m_j. \end{aligned}$$

Therefore, $$\overline{S}(P) - \overline{s}(P) \le S(P) - s(P),$$

where the upper and lower sums on the left are associated with $|f|$ and those on the right are associated with f.

Suppose that $\varepsilon > 0$. Since f is integrable on $[a,b]$, there is a partition P of $[a,b]$ such that $S(P) - s(P) < \varepsilon$.

This inequality and (5.25) imply that $\overline{S}(P) - \overline{s}(P) < \varepsilon$. Therefore, $|f|$ is integrable on $[a,b]$.

Q2 Is continuity implies differentiability?

Counter example: Consider the function $f(x) = |x|$.

The functions can be written as

$$f(x) = \begin{cases} x, & x > 0, \\ -x, & x \le 0, \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} 1, & x > 0, \\ -1, & x \le 0, \end{cases}$$

$$\lim_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0+} \frac{x}{x} \qquad (4.1)$$

$$\lim_{x\to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0-} \frac{-x}{x} = -1 \qquad (4.2)$$

are different,

$$- \lim_{x\to 0} \frac{f(x) - f(0)}{x \quad 0}$$

does not exist thus, f is not differentiable at 0, even though it is continuous at 0.

Q3 Theorem: If f is continuous on a finite closed interval $[a,b]$, then f is bounded on $[a,b]$.

Proof: Suppose that $t \in [a,b]$. Since f is continuous at t, there is an open interval I_t containing t such that

$$|f(x) - f(t)| < 1 \quad \text{if} \quad x \in I_t \cap [a,b].$$

The collection $\mathcal{H} = \{I_t : a \le t \le b\}$ is an open covering of $[a,b]$.

Since $[a,b]$ is compact, the Heine-Borel theorem implies that there are finitely many points $t_1, t_2, \ldots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b]$.

According to continuity of the function and with $t = t_i$,

$$|f(x) - f(t_i)| < 1 \quad \text{if} \quad x \in I_{t_i} \cap [a,b].$$

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Therefore,

$$\begin{aligned} |f(x)| &= |(f(x)-f(t_i))+f(t_i)| \le |f(x)-f(t_i)|+|f(t_i)| \\ &\le 1+|f(t_i)| \quad \text{if} \quad x \in I_{t_i} \cap [a,b]. \end{aligned}$$

Let

$$M = 1 + \max_{1\le i\le n} |f(t_i)|.$$

Since $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b])$, and we have $|f(x)| \le M$ if $x \in [a,b]$.

.

Q4 **Example**: Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Since every interval contains both rational and irrational numbers

$$m_j = 0 \quad \text{and} \quad M_j = 1, \quad 1 \le j \le n.$$

Hence,

$$\begin{aligned} S(P) &= \sum_{j=1}^{n} 1 \cdot (x_j - x_{j-1}) = b-a \\ s(P) &= \sum_{j=1}^{n} 0 \cdot (x_j - x_{j-1}) = 0. \end{aligned}$$

Since all upper sums equal $b-a$ and all lower sums equal 0, from definition we have

$$\overline{\int_a^b} f(x)\,dx = b-a \quad \text{and} \quad \underline{\int_a^b} f(x)\,dx = 0.$$

Q5 **Example**: Calculate the derivative of

$$h(x) = \sin\frac{1}{x}, \quad x \neq 0.$$

If

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{1}{x}, \quad x \neq 0,$$

then

$$h(x) = f(g(x)) = \sin\frac{1}{x}, \quad x \neq 0,$$

and

$$h'(x) = f'(g(x))g(x) = \cos\frac{1}{x} \quad -\frac{1}{x^2} \quad , \quad x \neq 0.$$

Q6 **Example**: Show that the function $g(x) = x^2$ is uniformly continuous on $[-r, r]$ for any finite r.

Solution: To see this, note that

$$|g(x) - g(x')| = |x^2 - (x')^2| = |x - x'|\,|x + x'| \le 2r|x - x'|,$$

so

$$|g(x) - g(x')| < \varepsilon \quad \text{if} \quad |x - x'| < \delta = \frac{\varepsilon}{2r} \quad \text{and} \quad -r \le x, x' \le r.$$

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Q7 **Lower integrals**: The *lower sum of* f *over* P is $s(P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1})$, and the *lower integral of* f *over* $[a, b]$, denoted by $\underline{\int_a^b} f(x)\,dx$, is the supremum of all lower sums.

Q8 ## 4.7 Generalized Mean Value Theorem

Theorem: If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c) \qquad \text{for some } c \text{ in } (a, b).$$

Q9 **Example**: If

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ x + 1, & 1 < x \le 2. \end{cases}$$

Then the function

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \frac{x^2}{2}, & 0 < x \le 1, \\ \frac{x^2}{2} + x - 1, & 1 < x \le 2, \end{cases}$$

$$F'(x) = \begin{cases} x = f(x), & 0 < x < 1, \\ x + 1 = f(x), & 1 < x < 2. \end{cases}$$

$$\begin{aligned} F'_-(2) &= \lim_{x \to 2-} \frac{F(x) - F(2)}{x - 2} = \lim_{x \to 2-} \frac{(x^2/2) + x - 1 - 3}{x - 2} \\ &= \lim_{x \to 2-} \frac{x + 4}{2} = 3 = f(2). \end{aligned}$$

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5- A theorem was from Monotonic Function? 5 marks
6-evaluate lim->0 ln sinx/ln x??
7- and Rest two was from the last 5th chp of about integration on closed interval.My Yesterday paper;
Mth621 👇

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Mcqs were based on concepts and 6,7 was from mcqs file of quizz.
1-Define Removeavle continuity? 2 marks
2-Deifine chain rule for composition of functions? 2 marks
3-Define Second mean theorem of Integration?
4- Find uniform continuity of 2x?
5- A theorem was from Monotonic Function? 5 marks
6-evaluate lim->0 ln sinx/ln x??
7- and Rest two was from the

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Removable discontinuity: Let f be defined on a deleted neighborhood of x_0 and discontinuous (perhaps even undefined) at x_0. We say that f has a removable discontinuity at x_0 if $\lim_{x \to x_0} f(x)$ exists.

In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0, \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0, \end{cases}$$

is continuous at x_0.

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$$= f(a)\int_a^b g(x)\,dx + f(b)\left(\int_a^b g(x)\,dx - \int_c^a g(x)\,dx\right)$$
$$= f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx.$$

5.6 Integration by Substitution

Theorem: Suppose that the transformation $x = \phi(t)$ maps the interval $c \le t \le d$ into the interval $a \le x \le b$, with $\phi(c) = \alpha$ and $\phi(d) = \beta$, and let f be continuous on $[a, b]$.

Let ϕ' be integrable on $[c, d]$. Then

$$\int_\alpha^\beta f(x)\,dx = \int_c^d f(\phi(t))\phi'(t)\,dt. \tag{5.45}$$

Proof: Both integrals in (5.45) exist: the one on the left by the fact that if f is continuous on $[a, b]$, then f is integrable on $[a, b]$, the one on the right by the continuity of $f(\phi(t))$.

The function

$$F(x) = \int_a^x f(y)\,dy$$

is an antiderivative of f on $[a, b]$ and, therefore, also on the closed interval with endpoints α and β.

Hence, by fundamental theorem of calculus,

$$\int_\alpha^\beta f(x)\,dx = F(\beta) - F(\alpha). \tag{5.46}$$

By the chain rule, the function

$$G(t) = F(\phi(t))$$

is an antiderivative of $f(\phi(t))\phi'(t)$ on $[c, d]$. Therefore, we have

$$\int_c^d f(\phi(t))\phi'(t)\,dt = G(d) - G(c) = F(\phi(d)) - F(\phi(c))$$
$$= F(\beta) - F(\alpha).$$

Comparing this with (5.46) yields (5.45).

Example: Evaluate the integral

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - 2x^2)(1 - x^2)^{-1/2}dx.$$

Solution: We let

$$f(x) = (1 - 2x^2)(1 - x^2)^{-1/2}, \quad -1/\sqrt{2} \le x \le 1/\sqrt{2},$$

and

$$x = \phi(t) = \sin t, \quad -\pi/4 \le t \le \pi/4.$$

Then $\phi'(t) = \cos t$ and

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} f(x)\,dx = \int_{-\pi/4}^{\pi/4} f(\sin t)\cos t\,dt$$
$$= \int_{-\pi/4}^{\pi/4} (1 - 2\sin^2 t)(1 - \sin^2 t)^{-1/2}\cos t\,dt. \tag{5.47}$$

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4.7 Generalized Mean Value Theorem

Theorem: If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$$

for some c in (a, b).

Proof: The function

$$h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$$

is continuous on $[a, b]$ and differentiable on (a, b).

Furthermore

$$h(a) = h(b) = g(b)f(a) - f(b)g(a).$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a, b).

$$\text{Since} \quad h'(c) = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c).$$

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Example: Check the uniform continuity of the function

$$f(x) = 2x.$$

Solution: For the function $f(x)$, we have

$$|f(x) - f(x')| = 2|x - x'| < \varepsilon \quad \text{if} \quad |x - x'| < \varepsilon/2.$$

The function $f(x)$ is uniformly continuous on $(-\infty, \infty)$,

Remark: A function f is *not* uniformly continuous on S if there is an $\varepsilon_0 > 0$ such that if δ is any positive number, there are points x and x' in S such that

$$|x - x'| < \delta$$

but

$$|f(x) - f(x')| \geq \varepsilon_0.$$

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Theorem: Suppose that f is monotonic on (a, b) and define

$$\alpha = \inf_{a<x<b} f(x) \quad \text{and} \quad \beta = \sup_{a<x<b} f(x).$$

1. If f is nondecreasing, then $\lim_{x\to a_+} f(x) = \alpha$ and $\lim_{x\to b_-} f(x) = \beta$.
2. If f is nonincreasing, then $\lim_{x\to a_+} f(x) = \beta$ and $\lim_{x\to b_-} f(x) = \alpha$. (Here $a+ = -\infty$ if $a = -\infty$ and $b- = \infty$ if $b = \infty$.)

3. If $a < x_0 < b$, then $\lim_{x\to x_{0_+}} f(x)$ and $\lim_{x\to x_{0_-}} f(x)$ exist and are finite; moreover,

$$\lim_{x\to x_{0_-}} f(x) \le f(x_0) \le \lim_{x\to x_{0_+}} f(x)$$

if f is nondecreasing, and

$$\lim_{x\to x_{0_-}} f(x) \ge f(x_0) \ge \lim_{x\to x_{0_+}} f(x)$$

if f is nonincreasing.

Proof: We first show that $\lim_{x\to a_+} f(x) = \alpha$.

If $M > \alpha$, there is an x_0 in (a, b) such that $f(x_0) < M$. Since f is nondecreasing, $f(x) < M$ if $a < x < x_0$. Therefore, if $\alpha = -\infty$, then $\lim_{x\to a_+} f(x) = -\infty$.

If $\alpha > -\infty$, let $M = \alpha + \varepsilon$, where $\varepsilon > 0$.

Then $\alpha \le f(x) < \alpha + \varepsilon$, so

$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < x_0. \tag{3.14}$$

If $a = -\infty$, this implies that $f(-\infty) = \alpha$. If $a > -\infty$, let $\delta = x_0 - a$. Then (3.14) is equivalent to

$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < a + \delta,$$

which implies that $f(a+) = \alpha$.

We now show that $\lim_{x\to b_-} f(x) = \beta$.

If $M < \beta$, there is an x_0 in (a, b) such that $f(x_0) > M$.

Since f is nondecreasing, $f(x) > M$ if $x_0 < x < b$. Therefore, if $\beta = \infty$, then $\lim_{x\to b_-} f(x) = \infty$.

If $\beta < \infty$, let $M = \beta - \varepsilon$, where $\varepsilon > 0$. Then $\beta - \varepsilon < f(x) \le \beta$, so

$$|f(x) - \beta| < \varepsilon \quad \text{if} \quad x_0 < x < b. \tag{3.15}$$

If $b = \infty$, this implies that $f(\infty) = \beta$. If $b < \infty$, let $\delta = b - x_0$.

Then (3.15) is equivalent to

$$|f(x) - \beta| < \varepsilon \quad \text{if} \quad b - \delta < x < b,$$

which implies that $f(b-) = \beta$.

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MTH621 Real Analysis I

Question No : 36 of 41

If f is differentiable at x_0, then prove that f is continuous at x_0

Answer (Please click here to Add Answer)

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4.1.1 Differentiability Implies Continuity

Theorem: If f is differentiable at x_0, then f is continuous at x_0.

Proof:

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

where

$$E(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0), & x \in D_f \text{ and } x \neq x_0, \\ 0, & x = x_0. \end{cases}$$

Examples:

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MTH621 Real Analysis I

Question No : 32 of 41

State the chain rule for differentiating compositions of functions

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Theorem (The chain rule): Suppose that g is differentiable at x_0 and f is differentiable at $g(x_0)$.

Then the composite function $h = f \circ g$, defined by

$$h(x) = f(g(x)),$$

is differentiable at x_0, with

$$h'(x_0) = f'(g(x_0))g'(x_0).$$

Proof: Since f is differentiable at $g(x_0)$, we can write

$$f(t) - f(g(x_0)) = [f'(g(x_0)) + E(t)][t - g(x_0)],$$

where

$$\lim_{t \to g(x_0)} E(t) = E(g(x_0)) = 0. \tag{4.3}$$

Letting $t = g(x)$ yields

$$f(g(x)) - f(g(x_0)) = [f'(g(x_0)) + E(g(x))][g(x) - g(x_0)].$$

Since $h(x) = f(g(x))$, this implies that

$$\frac{h(x) - h(x_0)}{x - x_0} = [f'(g(x_0)) + E(g(x))]\frac{g(x) - g(x_0)}{x - x_0}. \tag{4.4}$$

Since g is continuous at x_0 we have

$$\lim_{x \to x_0} E(g(x)) = E(g(x_0)) = 0.$$

Therefore, (4.4) implies that

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = f'(g(x_0))g'(x_0),$$

as stated.

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Question No : 33 of 41

State Rolle's Theorem.

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4.5 Rolle's Theorem

Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some c in the open interval (a, b).

Proof: Since f is continuous on $[a, b]$, f attains a maximum and a minimum value on $[a, b]$. If these two extreme values are the same, then f is constant on (a, b), so $f'(x) = 0$ for all x in (a, b).

If the extreme values differ, then at least one must be attained at some point c in the open interval (a, b), and $f'(c) = 0$.

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MTH621 Real Analysis I

Question No : 38 of 41

Determine whether the function $f(x) = 1 + x$ is integrable on $[a, b]$?

Answer (Please click here to Add Answer)

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Example: Determine whether the function $f(x) = 1 + x$ is integrable on $[a, b]$?

Solution: If

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{1}{x}, \quad x \neq 0,$$

then

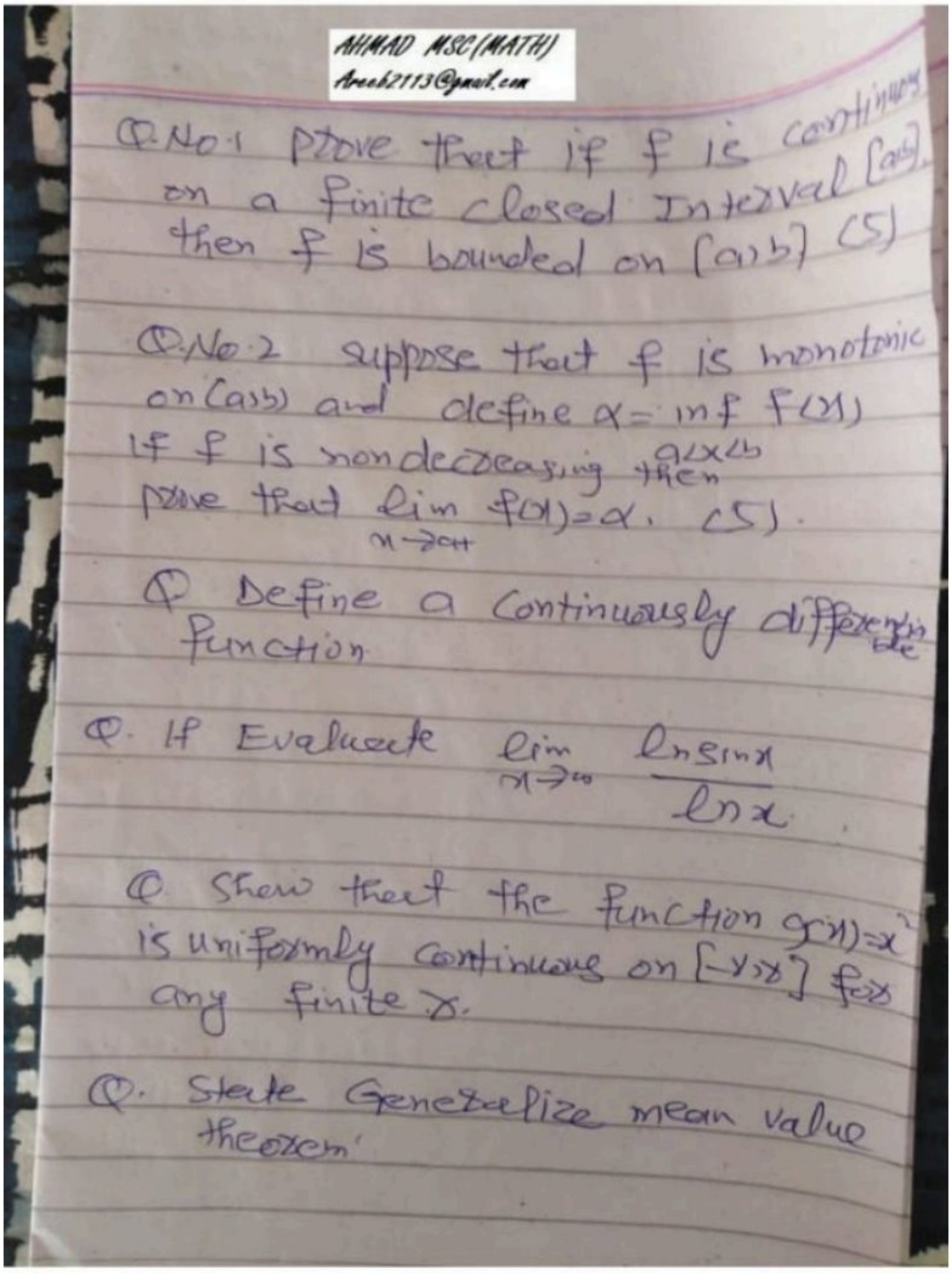
$$h(x) = f(g(x)) = \sin \frac{1}{x}, \quad x \neq 0,$$

and

$$h'(x) = f'(g(x))g(x) = \left(\cos \frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \quad x \neq 0.$$

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50
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Q.No.1 Prove that if f is continuous on a finite closed Interval [a,b] then f is bounded on [a,b] (5)

Q.No.2 suppose that f is monotonic on (a,b) and define $\alpha = \inf_{a<x<b} f(x)$ If f is non decreasing then prove that $\lim_{x \to a+} f(x) = \alpha$. (5).

Q Define a Continuously differentiable function

Q. If Evaluate $\lim_{x \to 0} \frac{\ln \sin x}{\ln x}$.

Q Show that the function $g(x) = x^2$ is uniformly continuous on $[-\gamma, \gamma]$ for any finite γ.

Q. State Generalize mean value theorem

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Theorem: If f is continuous on a finite closed interval $[a, b]$, then f is bounded on $[a, b]$.

Proof: Suppose that $t \in [a, b]$. Since f is continuous at t, there is an open interval I_t containing t such that

$$|f(x) - f(t)| < 1 \quad \text{if} \quad x \in I_t \cap [a, b].$$

The collection $\mathcal{H} = \{I_t : a \leq t \leq b\}$ is an open covering of $[a, b]$.

Since $[a, b]$ is compact, the Heine-Borel theorem implies that there are finitely many points $t_1, t_2, \ldots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a, b]$.

According to continuity of the function and with $t = t_i$,

$$|f(x) - f(t_i)| < 1 \quad \text{if} \quad x \in I_{t_i} \cap [a, b].$$

Therefore,

$$\begin{aligned} |f(x)| &= |(f(x) - f(t_i)) + f(t_i)| \leq |f(x) - f(t_i)| + |f(t_i)| \\ &\leq 1 + |f(t_i)| \quad \text{if} \quad x \in I_{t_i} \cap [a, b]. \end{aligned}$$

Let

$$M = 1 + \max_{1 \leq i \leq n} |f(t_i)|.$$

Since $[a, b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a, b])$, and we have $|f(x)| \leq M$ if $x \in [a, b]$.

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Theorem: Suppose that f is monotonic on (a, b) and define

$$\alpha = \inf_{a<x<b} f(x) \quad \text{and} \quad \beta = \sup_{a<x<b} f(x).$$

1. If f is nondecreasing, then $\lim_{x\to a+} f(x) = \alpha$ and $\lim_{x\to b-} f(x) = \beta$.
2. If f is nonincreasing, then $\lim_{x\to a+} f(x) = \beta$ and $\lim_{x\to b-} f(x) = \alpha$. (Here $a+ = -\infty$ if $a = -\infty$ and $b- = \infty$ if $b = \infty$.)

3. If $a < x_0 < b$, then $\lim_{x\to x_{0+}} f(x)$ and $\lim_{x\to x_{0-}} f(x)$ exist and are finite; moreover,

$$\lim_{x\to x_{0-}} f(x) \le f(x_0) \le \lim_{x\to x_{0+}} f(x)$$

if f is nondecreasing, and

$$\lim_{x\to x_{0-}} f(x) \ge f(x_0) \ge \lim_{x\to x_{0+}} f(x)$$

if f is nonincreasing.

Proof: We first show that $\lim_{x\to a+} f(x) = \alpha$.

If $M > \alpha$, there is an x_0 in (a, b) such that $f(x_0) < M$. Since f is nondecreasing, $f(x) < M$ if $a < x < x_0$. Therefore, if $\alpha = -\infty$, then $\lim_{x\to a+} f(x) = -\infty$.

If $\alpha > -\infty$, let $M = \alpha + \varepsilon$, where $\varepsilon > 0$.

Then $\alpha \le f(x) < \alpha + \varepsilon$, so

$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < x_0. \tag{3.14}$$

If $a = -\infty$, this implies that $f(-\infty) = \alpha$. If $a > -\infty$, let $\delta = x_0 - a$. Then (3.14) is equivalent to

$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < a + \delta,$$

which implies that $f(a+) = \alpha$.

We now show that $\lim_{x\to b-} f(x) = \beta$.

If $M < \beta$, there is an x_0 in (a, b) such that $f(x_0) > M$.

Since f is nondecreasing, $f(x) > M$ if $x_0 < x < b$. Therefore, if $\beta = \infty$, then $\lim_{x\to b-} f(x) = \infty$.

If $\beta < \infty$, let $M = \beta - \varepsilon$, where $\varepsilon > 0$. Then $\beta - \varepsilon < f(x) \le \beta$, so

$$|f(x) - \beta| < \varepsilon \quad \text{if} \quad x_0 < x < b. \tag{3.15}$$

If $b = \infty$, this implies that $f(\infty) = \beta$. If $b < \infty$, let $\delta = b - x_0$.

Then (3.15) is equivalent to

$$|f(x) - \beta| < \varepsilon \quad \text{if} \quad b - \delta < x < b,$$

which implies that $f(b-) = \beta$.

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Q3

Continuously differentiable We say that f is *continuously differentiable on* $[a, b]$ if f is differentiable on $[a, b]$, f' is continuous on (a, b),

$$f'_+(a) = \lim_{x \to a+} f'(x)$$

, and

$$f'_-(b) = \lim_{x \to b-} f'(x).$$

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25

Example: Show that the function $g(x) = x^2$ is uniformly continuous on $[-r, r]$ for any finite r.

Solution: To see this, note that

$$|g(x) - g(x') = |x^2 - (x')^2| = |x - x'|\,|x + x'| \leq 2r|x - x'|,$$

so

$$|g(x) - g(x')| < \varepsilon \quad \text{if} \quad |x - x'| < \delta = \frac{\varepsilon}{2r} \quad \text{and} \quad -r \leq x, x' \leq r.$$

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26

4.7 Generalized Mean Value Theorem

Theorem: If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$$

for some c in (a, b).

Proof: The function

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MTH 621 (Final 2nd)

Long Questions:-

* Prove if f is integrable on $[a,b]$ then so is $|f|$.
* Is continuity implies differentiability? explain with example.
* If f is continous on $[a,b]$, then it is banded on $[a,b]$.
* Find $\int_a^b f x\,dx$?

If $f(x) = \begin{cases} 0 & x \text{ is rational} \\ 1 & x \text{ is irrational.} \end{cases}$

* $h(x) = \sin 1/x$ find derivative by chain rule.
* $g(x) = x^2$ show uniform continuity.
* Define lower integral
* Define GMVT.
* check $\lim_{x \to 2^-} f(x) = f_2$

$f(x) = \begin{cases} x^2 & 1 \le x < 2 \\ x+1 & 1 < x \le 2 \end{cases}$

MCQ's:-

→ $\lim_{n \to \infty} 1/n^p = ?$

→ $\sin x \Big|_{-\pi/4}^{\pi/4} = ?$

→ lower sum ?

→ upper sum formula.

→ Comparison test

→ $\frac{1}{n} r^n = ?$

→ $\sup_{n \to 0} S_n = \infty$?

→ $e^x \longrightarrow$ in form of

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$\int_a^b f(x)\,dx = ?$

Good luck!
:)

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Q. prove continuity for the given function. 2 questions

Q. second mean value theorem statement

Q using mean value theorem prove that $x-1/x < \ln x < x-1$ for $x>1$

Q if $F'(x)=f(x)$ prove integral $f(x)=F(a)-F(b)$ for $[a,b]$

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Q if g is continuous at x0 and f is continuous on g(x0) then show that fog is continuous at x0.

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Theorem: If f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{5.37}$$

Proof: See Lecture.

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5.4 Fundamental Theorem of Calculus

Theorem: If f is integrable on $[a, b]$ and $a \le c \le b$, then the function F defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition on $[a, b]$, and is therefore continuous on $[a, b]$.

Proof: If x and x' are in $[a, b]$, then

$$F(x) - F(x') = \int_c^x f(t)\,dt - \int_c^{x'} f(t)\,dt = \int_{x'}^x f(t)\,dt.$$

Since $|f(t)| \le K$ $(a \le t \le b)$ for some constant K,

$$\left|\int_{x'}^x f(t)\,dt\right| \le K|x - x'|, \quad a \le x, x' \le b.$$

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Example: Find

$$\lim_{x\to\infty}\frac{2x^2-x+1}{3x^2+2x-1}.$$

Example: Find

$$\lim_{x\to 1}\frac{x^8-1}{x^4-1}.$$

$$f(x)=\begin{cases}\frac{1}{x+2}, & x<-2,\\ x^2-5, & -2<x\leq 3,\\ \sqrt{x+13}, & x>3.\end{cases}$$

3.5 Continuity

Continuity (Informal):

- We say that f is *continuous at* x_0 if f is defined on an open interval (a,b) containing x_0 and
$$\lim_{x\to x_0} f(x)=f(x_0).$$

- We say that f is *continuous from the left at* x_0 if f is defined on an open interval (a,x_0) and
$$\lim_{x\to x_0^-} f(x)=f(x_0).$$

- We say that f is *continuous from the right at* x_0 if f is defined on an open interval (x_0,b) and
$$\lim_{x\to x_0^+} f(x)=f(x_0).$$

Continuity (Formal):

- A function f is continuous at x_0 if and only if f is defined on an open interval (a,b) containing x_0 and for each $\varepsilon>0$ there is a $\delta>0$ such that
$$|f(x)-f(x_0)|<\varepsilon \quad \text{whenever} \quad |x-x_0|<\delta.$$

- A function f is continuous from the right at x_0 if and only if f is defined on an interval $[x_0,b)$ and for each $\varepsilon>0$ there is a $\delta>0$ such that
$$|f(x)-f(x_0)|<\varepsilon \quad \text{whenever} \quad x_0\leq x<x_0+\delta.$$

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First observe that if $L_2 \neq 0$, there is a $\delta_3 > 0$ such that

$$|g(x) - L_2| < \frac{|L_2|}{2}, \qquad |g(x)| > \frac{|L_2|}{2} \quad \text{if} \quad 0 < |x - x_0| < \delta_3.$$

Now suppose that $0 < |x - x_0| < \min(\delta_1, \delta_2, \delta_3)$. Then

$$\begin{aligned}
\left|\left(\frac{f}{g}\right)(x) - \frac{L_1}{L_2}\right| &= \left|\frac{f(x)}{g(x)} - \frac{L_1}{L_2}\right| = \frac{|L_2 f(x) - L_1 g(x)|}{|g(x)L_2|} \\
&\leq \frac{2}{|L_2|^2}|L_2 f(x) - L_1 g(x)| \\
&= \frac{2}{|L_2|^2}|L_2[f(x) - L_1] + L_1[L_2 - g(x)]| \\
&\leq \frac{2}{|L_2|^2}[|L_2||f(x) - L_1| + |L_1||L_2 - g(x)|] \\
&\leq \frac{2}{|L_2|^2}(|L_2| + |L_1|)\varepsilon.
\end{aligned}$$

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Example: Find

$$\lim_{x\to 2}\frac{9 - x^2}{x + 1} \quad \text{and} \quad \lim_{x\to 2}(9 - x^2)(x + 1).$$

Solution: If c is a constant, then $\lim_{x\to x_0} c = c$, and $\lim_{x\to x_0} x = x_0$. Therefore

$$\begin{aligned}
\lim_{x\to 2}(9 - x^2) &= \lim_{x\to 2} 9 - \lim_{x\to 2} x^2 \\
&= \lim_{x\to 2} 9 - (\lim_{x\to 2} x)^2 \\
&= 9 - 2^2 = 5,
\end{aligned}$$

and

$$\lim_{x\to 2}(x + 1) = \lim_{x\to 2} x + \lim_{x\to 2} 1 = 2 + 1 = 3.$$

Therefore,

$$\lim_{x\to 2}\frac{9 - x^2}{x + 1} = \frac{\lim_{x\to 2}(9 - x^2)}{\lim_{x\to 2}(x + 1)} = \frac{5}{3}$$

and

$$\lim_{x\to 2}(9 - x^2)(x + 1) = \lim_{x\to 2}(9 - x^2)\lim_{x\to 2}(x + 1) = 5 \cdot 3 = 15.$$

Example: The function

$$f(x) = 2x \sin\sqrt{x}$$

satisfies the inequality

$$|f(x)| < \varepsilon, \qquad \text{whenever} \quad 0 < x < \delta = \varepsilon/2.$$

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Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then the function F defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition on $[a,b]$, and is therefore continuous on $[a,b]$.

Proof: Consequently,

$$|F(x) - F(x')| \le K|x - x'|, \quad a \le x,\, x' \le b.$$

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then $F(x) = \int_c^x f(t)\,dt$ is differentiable at any point x_0 in (a,b) where f is continuous, with $F'(x_0) = f(x_0)$.

If f is continuous from the right at a, then $F'_+(a) = f(a)$. If f is continuous from the left at b, then $F'_-(b) = f(b)$.

Proof: Since

$$\frac{1}{x - x_0}\int_{x_0}^x f(x_0)\,dt = f(x_0).$$

We can write

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x [f(t) - f(x_0)]\,dt.$$

From this

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \le \frac{1}{|x - x_0|}\left|\int_{x_0}^x |f(t) - f(x_0)|\,dt\right|.$$

Since f is continuous at x_0, there is for each $\varepsilon > 0$ a $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta$$

and t is between x and x_0.

Therefore, we have

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4.5 Rolle's Theorem

Theorem: Suppose that f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), and $f(a)=f(b)$. Then $f'(c)=0$ for some c in the open interval (a,b).

Proof: Since f is continuous on $[a,b]$, f attains a maximum and a minimum value on $[a,b]$. If these two extreme values are the same, then f is constant on (a,b), so $f'(x)=0$ for all x in (a,b).

If the extreme values differ, then at least one must be attained at some point c in the open interval (a,b), and $f'(c)=0$.

4.6 The Mean Value Theorem

Theorem: Suppose that f is differentiable on $[a,b]$, $f'(a)\neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c)=\mu$ for some c in (a,b).

Proof: Suppose first that

$$f'(a)<\mu<f'(b)$$

and define

$$g(x)=f(x)-\mu x.$$

Then

$$g'(x)=f'(x)-\mu,\quad a\leq x\leq b,$$

and by our supposition $f'(a)<\mu<f'(b)$, we have

$$g'(a)<0\quad\text{and}\quad g'(b)>0.$$

Since g is continuous on $[a,b]$, g attains a minimum at some point c in $[a,b]$.

Due to Lemma and $g'(a)<0\quad$ and $\quad g'(b)>0$, imply that there is a $\delta>0$ such that

$$g(x)<g(a),\quad a<x<a+\delta,\quad\text{and}\quad g(x)<g(b),\quad b-\delta<x<b.$$

Therefore $c\neq a$ and $c\neq b$. Hence, $a<c<b$, and therefore $g'(c)=0$, by the fact that if f is differentiable at a local extreme point x_0 then $f'(x_0)=0$. From , $f'(c)=\mu$. The proof for the case where $f'(b)<\mu<f'(a)$ can be obtained by applying this result to $-f$.

Theorem: If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

for some c in (a,b).

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3.3 Limits at $\pm\infty$

If f is defined on an interval (a, ∞), then $f(x)$ *approaches the limit* L *as* x *approaches* ∞, and write

$$\lim_{x\to\infty} f(x) = L.$$

For each $\varepsilon > 0$, there is a number β such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad x > \beta.$$

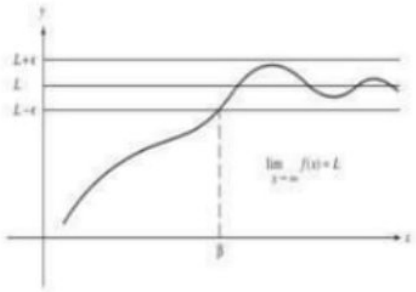


Figure 3.7: Limits at $\pm\infty$

Examples: Let

$$f(x) = 1 - \frac{1}{x^2}, \quad g(x) = \frac{2|x|}{1+x}, \quad \text{and} \quad h(x) = \sin x.$$

Find limits when $x \to \pm\infty$.

Solution: We have

$$\lim_{x\to\infty} f(x) = 1,$$

since

$$|f(x) - 1| = \frac{1}{x^2} < \varepsilon \quad \text{if} \quad x > \frac{1}{\sqrt{\varepsilon}}.$$

We have

$$\lim_{x\to\infty} g(x) = 2,$$

since

$$|g(x) - 2| = \left|\frac{2x}{1+x} - 2\right| = \frac{2}{1+x} < \frac{2}{x} < \varepsilon \quad \text{if} \quad x > \frac{2}{\varepsilon}.$$

However, $\lim_{x\to\infty} h(x)$ does not exist, since h assumes all values between -1 and 1 in any semi-infinite interval (τ, ∞). Discuss the limits at $x \to -\infty$.

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x^2},$$

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4.11 Taylor's Theorem

Recall the following lemma: If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

To generalize this result, we first restate it: the polynomial

$$T_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

which is of degree ≤ 1 and satisfies

$$T_1(x_0) = f(x_0), \quad T_1'(x_0) = f'(x_0).$$

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$T_1(x)$ approximates f so well near x_0 that

$$\lim_{x \to x_0} \frac{f(x) - T_1(x)}{x - x_0} = 0.$$

Taylor polynomial: Suppose that f has n derivatives at x_0 and T_n is the polynomial of degree $\leq n$ such that

$$T_n^{(r)}(x_0) = f^{(r)}(x_0), \quad 0 \leq r \leq n.$$

How well does T_n approximate f near x_0?
Since T_n is a polynomial of degree $\leq n$, it can be written as

$$T_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n,$$

where $a_0, \ldots, a_n$ are constants.
Differentiating yields

$$T_n^{(r)}(x_0) = r!a_r, \quad 0 \leq r \leq n,$$

We obtained a_r uniquely

$$a_r = \frac{f^{(r)}(x_0)}{r!}, \quad 0 \leq r \leq n.$$

Therefore,

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$\sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}(x - x_0)^r.$$

Special case of Taylor series: I picked power series, I think it's right

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Although each y_j occurs as a term of $\{x_n\}$, $\{y_j\}$ is not necessarily a subsequence of $\{x_n\}$, because if we write

$$y_j = x_{n_j}.$$

There is no reason to expect that $\{n_j\}$ is an increasing sequence as required in definition of subsequence.

However, it is always possible to pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ that is increasing, and then the [illegible] is a subsequence of both $\{y_j\}$ and $\{x_n\}$.

2.6 Limit Superior and Limit Inferior

Theorem:

- [illegible] diverge to $-\infty$, then there is a unique real number $\overline{s}$ such that, if $\varepsilon > 0$,

$$s_n < \overline{s} + \varepsilon \quad \text{for large } n \tag{2.4}$$

and

$$s_n > \overline{s} - \varepsilon \quad \text{for infinitely many } n. \tag{2.5}$$

- If $\{s_n\}$ is bounded below and does not diverge to ∞, then there is a unique real number $\underline{s}$ such that, if $\varepsilon > 0$,

$$s_n > \underline{s} - \varepsilon \quad \text{for large } n \tag{2.6}$$

and

$$s_n < \underline{s} + \varepsilon \quad \text{for infinitely many } n. \tag{2.7}$$

Proof: Proof of the first part. Since $\{s_n\}$ is bounded above, there is a number β such that $s_n < \beta$ for all n. Since $\{s_n\}$ does not diverge to $-\infty$, there is a number α such that $s_n > \alpha$ for infinitely many n. If we define

$$M_k = \sup\{s_k, s_{k+1}, \ldots, s_{k+r}, \ldots\},$$

then $\alpha \le M_k \le \beta$, so $\{M_k\}$ is bounded. Since $\{M_k\}$ is nonincreasing (why?), it converges. Let

$$\overline{s} = \lim_{k \to +\infty} M_k. \tag{2.8}$$

If $\varepsilon > 0$, then $M_k < \overline{s} + \varepsilon$ for large k, and since $s_n \le M_k$ for $n \ge k$, $\overline{s}$ satisfies (2.4).

If (2.5) were false for some positive ε, there would be an integer K such that

$$s_n \le \overline{s} - \varepsilon \quad \text{if} \quad n \ge K.$$

However, this implies that

$$M_k \le \overline{s} - \varepsilon \quad \text{if} \quad k \ge K,$$

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Example: If

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ x+1, & 1 < x \le 2. \end{cases}$$

Then the function

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \frac{x^2}{2}, & 0 < x \le 1, \\ \frac{x^2}{2} + x - 1, & 1 < x \le 2, \end{cases}$$

is continuous on $[0, 2]$.

We can conclude

$$F'(x) = \begin{cases} x = f(x), & 0 < x < 1, \\ x + 1 = f(x), & 1 < x < 2. \end{cases}$$

$$F'_+(0) = \lim_{x \to 0+} \frac{F(x) - F(0)}{x} = \lim_{x \to 0+} \frac{(x^2/2) - 0}{x} = 0 = f(0),$$

$$F'_-(2) = \lim_{x \to 2-} \frac{F(x) - F(2)}{x - 2} = \lim_{x \to 2-} \frac{(x^2/2) + x - 1 - 3}{x - 2}$$

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Example: If $s_n = [1+(-1)^n]n$, then $\{s_n\}$ is bounded below ($s_n \geq 0$) but unbounded above.

$\{-s_n\}$ is bounded above ($-s_n \leq 0$) but unbounded below.

If $s_n = (-1)^n$, then $\{s_n\}$ is bounded.

If $s_n = (-1)^n n$, then $\{s_n\}$ is not bounded above or below.

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Theorem: A convergent sequence is bounded.
Proof: By taking $\varepsilon = 1$, we see that if $\lim_{n\to\infty} s_n = s$, then there is an integer N such that

$$|s_n - s| < 1 \quad \text{if} \quad n \geq N.$$

Therefore,

$$|s_n| = |(s_n - s) + s| \leq |s_n - s| + |s| < 1 + |s| \quad \text{if} \quad n \geq N,$$

and

$$|s_n| \leq \max\{|s_0|, |s_1|, \ldots, |s_{N-1}|, 1 + |s|\}$$

for all n, so $\{s_n\}$ is bounded.

Theorem: A sequence $\{s_n\}$ converges to s if and only if every neighborhood of s contains s_n for all but finitely many n.

2.3 Monotonic Sequences

A sequence $\{s_n\}$ is *nondecreasing* if $s_n \geq s_{n-1}$ for all n, or *nonincreasing* if $s_n \leq s_{n-1}$ for all n.

A *monotonic sequence* is a sequence that is either nonincreasing or nondecreasing. If $s_n > s_{n-1}$ for all n, then $\{s_n\}$ is *increasing*, while if $s_n < s_{n-1}$ for all n, $\{s_n\}$ is *decreasing*.

Theorem:

- If $\{s_n\}$ is nondecreasing, then $\lim_{n\to\infty} s_n = \sup\{s_n\}$.
- If $\{s_n\}$ is nonincreasing, then $\lim_{n\to\infty} s_n = \inf\{s_n\}$.

Recall: The definition of supremum and infimum of a set.
Proof: Let $\beta = \sup\{s_n\}$. If $\beta < \infty$, then by definition if $\varepsilon > 0$ then

$$\beta - \varepsilon < s_N \leq \beta$$

for some integer N.

Since $s_N \leq s_n \leq \beta$ if $n \geq N$, it follows that

$$\beta - \varepsilon < s_n \leq \beta \quad \text{if} \quad n \geq N.$$

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5 marks prove from this one

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then the function F defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition on $[a,b]$, and is therefore continuous on $[a,b]$.

Proof: Consequently,

$$|F(x) - F(x')| \le K|x - x'|, \quad a \le x, x' \le b.$$

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then $F(x) = \int_c^x f(t)\,dt$ is differentiable at any point x_0 in (a,b) where f is continuous, with $F'(x_0) = f(x_0)$.

If f is continuous from the right at a, then $F'_+(a) = f(a)$. If f is continuous from the left at b, then $F'_-(b) = f(b)$.

Proof: Since

$$\frac{1}{x - x_0}\int_{x_0}^x f(x_0)\,dt = f(x_0).$$

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We can write

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x [f(t) - f(x_0)]\,dt.$$

From this

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \le \frac{1}{|x - x_0|}\left|\int_{x_0}^x |f(t) - f(x_0)|\,dt\right|.$$

Since f is continuous at x_0, there is for each $\varepsilon > 0$ a $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta$$

and t is between x and x_0.

Therefore, we have

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| < \varepsilon\frac{|x - x_0|}{|x - x_0|} = \varepsilon \quad \text{if} \quad 0 < |x - x_0| < \delta.$$

Hence, $F'(x_0) = f(x_0)$.

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then $F(x) = \int_c^x f(t)\,dt$ is differentiable at any point x_0 in (a,b) where f is continuous, with $F'(x_0) = f(x_0)$.

If f is continuous from the right at a, then $F'_+(a) = f(a)$. If f is continuous from the left at b, then $F'_-(b) = f(b)$.

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Example: Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

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Solution: See Lecture.

Example: Show that

$$\lim_{x \to 0+} \left(\frac{|x|}{x} + x \right) = 1, \qquad \lim_{x \to 0-} \left(\frac{|x|}{x} + x \right) = -1,$$

Solution: See Lecture

Theorem: A function f has a limit at x_0 if and only if it has left- and right-hand limits at x_0, and they are equal. More specifically,

$$\lim_{x \to x_0} f(x) = L$$

if and only if

$$\lim_{x \to x_0+} f(x) = \lim_{x \to x_0-} f(x) = L.$$

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3.1 Monotone Sequences

3.1.1 Definition:

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(i) A sequence $(a_n)_{n\geq 1}$ said to be **monotonically increasing**, if $(a_n)_{n\geq 1} \geq a_n \ \forall\, n \geq 1$.

(ii) A sequence $(a_n)_{n\geq 1}$ said to be **monotonically decreasing**, if $(a_n)_{n\geq 1} \leq a_n \ \forall\, n \geq 1$.

We can describe now the completeness property of the real numbers.

3.1.2 Completeness property
Every monotonically increasing sequence which is bounded above is convergent.

3.1.3 Theorem:

If $(a_n)_{n\geq 1}$ is monotonically decreasing and is bounded below, it is convergent.

Proof:
Follows from the following facts

(i) $(a_n)_{n\geq 1}$ monotonically decreasing if and only if $(a_n)_{n\geq 1}$ monotonically increasing.
(ii) $(a_n)_{n\geq 1}$ bounded below, if and only if $(a_n)_{n\geq 1}$ is bounded above.
(iii) $(a_n)_{n\geq 1}$ is convergent if and only if $(a_n)_{n\geq 1}$ is convergent.

3.1.4 Examples:

(i) Sequence $(2n)_{n\geq 1}$ is monotonically increasing and is not bounded above.

(ii) Sequence $(\frac{1}{n})_{n\geq 1}$ is monotonically decreasing and is not bounded below, say by 0.

(iii) Sequence $((-1)^n)_{n\geq 1}$ is neither monotonically increasing nor decreasing.

(iv) Let $a_1 = 1$, $a_n = \dfrac{3a_{n-1}+2}{6}$ $\forall\, n \geq 2$. As shown in problem 2, module 0, $a_n \geq \dfrac{2}{3}$ $\forall\ n$ and

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Lecture 25:

Uniform Continuity:

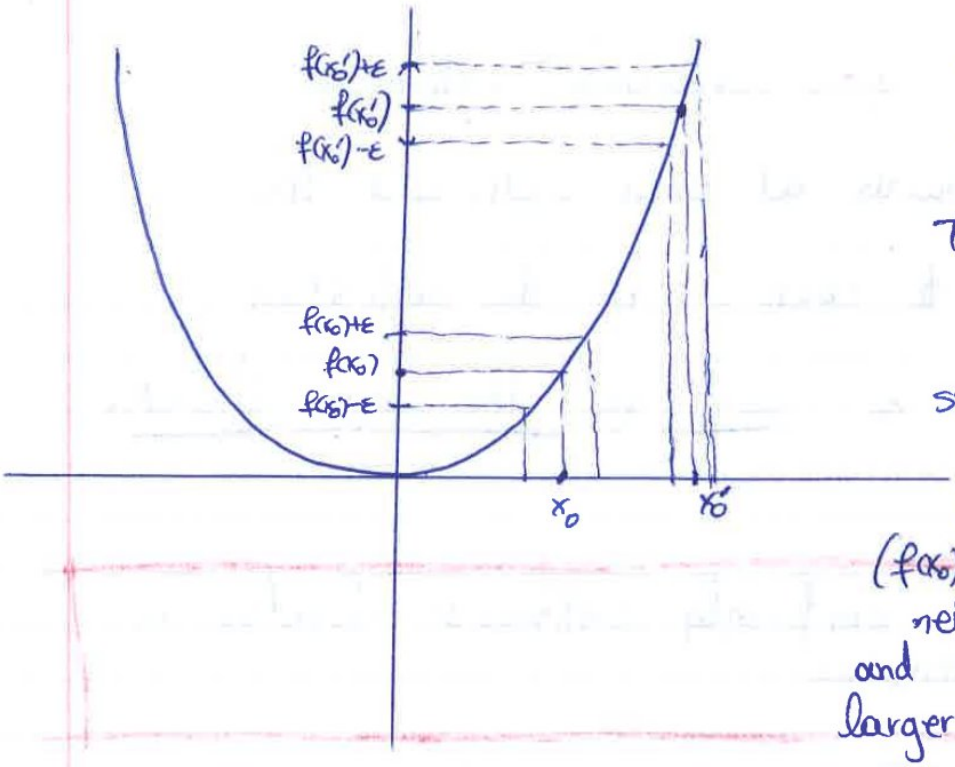
Let $f: A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. Let $x_0 \in A$.

Let us remember what it means that f is continuous at x_0: it means that, $\forall \varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ (i.e., $\delta > 0$ potentially depending on x_0 and ε), such that: if $x \in A$ with $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

There are of course examples of functions (such as $f(x) = x \ \forall x \in \mathbb{R}$, or $f(x) = c \ \forall x \in \mathbb{R}$, see old notes on continuity) where $\forall \varepsilon > 0$ we can pick a $\delta = \delta(\varepsilon) > 0$, i.e. a δ that only depends on ε, not on x_0, to satisfy the definition of continuity at x_0.

However, other functions (such as $f(x) = x^2 \ \forall x \in \mathbb{R}$) don't satisfy this:

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The larger x_0 is, the faster x_0^2 grows.

Thus, for fixed $\varepsilon > 0$,

while, when x_0 is small, quite a large neighbourhood of x_0 is sent inside $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$, this neighbourhood shrinks more and more as x_0 gets larger and larger.

So, for such a function, when we test for continuity at x_0, the $\delta > 0$ that corresponds to each $\varepsilon > 0$ has to depend on x_0, not just on ε.

→ **Def: (Uniform Continuity):**

Let $f: A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. We say that f is uniformly continuous if:

for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$,

s.t. if $x, y \in A$, with $|x-y| < \delta$,

then $|f(x)-f(y)| < \varepsilon$.

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In other words, f is uniformly continuous if it is continuous at each $x_0 \in A$, and the δ corresponding to each ε in the definition of continuity at x_0 can be the same $\forall x_0 \in A$.

Observation: Each uniformly continuous function is continuous.

Proof: Let $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly continuous.

Let $x_0 \in A$. Let $\varepsilon > 0$.

Since f is uniformly continuous, for this $\varepsilon > 0$ there exists $\delta (= \delta(\varepsilon)) > 0$, such that:

If $x, y \in A$, and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

In particular, if $x \in A$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, f is continuous at x_0.

Since $x_0 \in A$ was arbitrary, f is continuous on A.

■

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Lecture 26

Examples:

- $f(x)=x, \forall x\in\mathbb{R}$: uniformly continuous.
- $f(x)=x^2, \forall x\in\mathbb{R}$: not uniformly continuous.
- $f(x)=x^2, \forall x\in[-M,M]$: uniformly continuous. (M fixed, >0)

Let's prove the third bullet point:

Let $x,y\in[-M,M]$. Then,

$$|f(x)-f(y)| = |x^2-y^2| = |(x-y)\cdot(x+y)| =$$
$$= |x-y|\cdot \underbrace{|x+y|}_{\leq |x|+|y|\leq 2M} \leq 2M\cdot|x-y|.$$

Let $\varepsilon>0$. If $x,y\in[-M,M]$, with $|x-y|<\frac{\varepsilon}{2M}$, then, by the above,

$$|f(x)-f(y)| < 2M\cdot\frac{\varepsilon}{2M} = \varepsilon.$$

Since $\delta(\varepsilon)=\frac{\varepsilon}{2M}$ only depends on ε, f is uniformly continuous.

We generalise this in <u>both</u> the Propositions that follow:

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→ **Def:** (Lipschitz continuity):

Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is Lipschitz continuous if there exists $M>0$ such that

$$|f(x)-f(y)| \leq M \cdot |x-y|, \quad \forall x,y \in A.$$

→ **Prop:** Every Lipschitz continuous function is uniformly continuous.

Proof: Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous.

Then, there exists $M>0$ such that:

$$|f(x)-f(y)| \leq M \cdot |x-y|, \quad \forall x,y \in A.$$

Let $\varepsilon>0$. We choose $\delta = \frac{\varepsilon}{M}$ (depends only on ε).

If $x,y \in A$, and $|x-y|<\delta$, then

$$|f(x)-f(y)| \leq M \cdot |x-y| < M \cdot \delta = M \cdot \frac{\varepsilon}{M} = \varepsilon,$$

i.e. $|f(x)-f(y)|<\varepsilon$.

Since $\varepsilon>0$ was arbitrary, f is uniformly continuous. ∎

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↳ works for open intervals (as long as f' bounded).

Prop: Let $f: I \to \mathbb{R}$, f continuous on I, differentiable in the interior of I.

(I an interval)

If f' is bounded, then f is Lipschitz continuous (and thus uniformly continuous).

Proof: By our assumptions, there exists $M>0$ s.t.

$$|f'(y)| \leq M, \ \forall y \text{ in the interior of } I. \quad (*)$$

(WLOG) Let $x, y \in I$, with $x<y$. Since f is continuous on $[x,y]$ and differentiable on (x,y), we can apply the mean value theorem on $[x,y]$:

$$f(y) - f(x) = f'(\xi) \cdot (y-x),$$

thus $|f(y) - f(x)| = |f'(\xi)| \cdot |y-x|$, for some ξ between x and y, thus in the interior of I.

By $(*)$, $|f(y) - f(x)| \leq M \cdot |y-x|$.

Since $x, y \in I$ were arbitrary, f is Lipschitz continuous.

Therefore, f is uniformly continuous. ■

→ **ex:** $f(x) = \sin x$, $\forall x \in \mathbb{R}$: $|f'(x)| = |\cos x| \leq 1 \ \forall x \in \mathbb{R}$, thus f Lipschitz continuous (an interval), thus f uniformly continuous.

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Thm: Characterisation of uniform continuity via sequences:

Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$.

f is uniformly continuous

$\iff$

for any pair of sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in A,

with $x_n - y_n \to 0$,

we have $f(x_n) - f(y_n) \to 0$.

Proof: ($\Longrightarrow$) Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be in A, with $x_n - y_n \to 0$. We want to show that $f(x_n) - f(y_n) \to 0$:

Let $\varepsilon > 0$. Since f is uniformly continuous, we have that: for this $\varepsilon > 0$, there exists $\delta (= \delta(\varepsilon)) > 0$, such that: if $|x_n - y_n| < \delta$, then $|f(x_n) - f(y_n)| < \varepsilon$. (*)

Since $x_n - y_n \to 0$, there exists $n_0 \in \mathbb{N}$ s.t.:

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$\forall n \geq n_0, \quad |x_n - y_n| < \delta.$

By (*): $\forall n \geq n_0, \quad |f(x_n) - f(y_n)| < \varepsilon.$

Since $\varepsilon > 0$ was arbitrary, $f(x_n) - f(y_n) \longrightarrow 0.$

($\Longleftarrow$) Suppose that f is not uniformly continuous.

We will construct sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in A, with $x_n - y_n \longrightarrow 0$, but $f(x_n) - f(y_n) \not\longrightarrow 0$.

even better: $|f(x_n) - f(y_n)|$ be, for some fixed $\varepsilon > 0$

Indeed: Since f doesn't satisfy the definition of uniform continuity:

there exists $\varepsilon > 0$ s.t. : $\forall \delta > 0,$

there exist $x_\delta, y_\delta \in A$, with $|x_\delta - y_\delta| < \delta$, (*')

yet $|f(x_\delta) - f(y_\delta)| \geq \varepsilon.$

We apply (*') for $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$:

for $\delta = 1$: $\exists x_1, y_1 \in A$, with $|x_1 - y_1| < 1$, but $|f(x_1) - f(y_1)| \geq \varepsilon$.

for $\delta = 1/2$: $\exists x_2, y_2 \in A$, with $|x_2 - y_2| < \frac{1}{2}$, but $|f(x_2) - f(y_2)| \geq \varepsilon$.

$\vdots$

for $\delta = 1/n$: $\exists x_n, y_n \in A$, with $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \varepsilon$

$\vdots$

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Since $|x_n - y_n| < \frac{1}{n} \quad \forall n \in \mathbb{N}$,

we have $x_n - y_n \longrightarrow 0$ (by the sandwich lemma).

Thus, by our assumption, we should have

$$f(x_n) - f(y_n) \longrightarrow 0.$$

However, $|f(x_n) - f(y_n)| \geq \underbrace{\varepsilon}_{\text{fixed}} > 0, \quad \forall n \in \mathbb{N},$

thus $f(x_n) - f(y_n) \not\longrightarrow 0.$

This is a contradiction. Therefore, f is uniformly continuous.

■

→ This characterisation of uniform continuity can prove particularly useful when we want to show that a function f is not uniformly continuous.

→ Example: Show that $f: (0,1) \longrightarrow \mathbb{R}$, with $f(x) = \frac{1}{x}, \forall x \in (0,1)$, is not uniformly continuous.

! METHOD !

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Solution: We will find $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in $(0,1)$, with $x_n - y_n \longrightarrow 0$, but $f(x_n) - f(y_n) \not\longrightarrow 0$.

Indeed: Let $x_n = \frac{1}{n}$ $\forall n\in\mathbb{N}$,

and $y_n = \frac{1}{2n}$ $\forall n\in\mathbb{N}$.

The sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are both in $(0,1)$.

And: $x_n - y_n = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} \longrightarrow 0$.

But: $f(x_n) - f(y_n) = \frac{1}{x_n} - \frac{1}{y_n} = n - 2n = -n \longrightarrow -\infty \neq 0$

Thus, f is not uniformly continuous.

→ You can try a similar trick to show that

$f(x) = x^2$ $\forall x\in\mathbb{R}$ and $g(x) = \cos(x^2)$ $\forall x\in\mathbb{R}$

are not uniformly continuous.

(Exercise).

⚠ A bounded and continuous function is not necessarily uniformly continuous (ex.: $g(x) = \cos(x^2)$, $\forall x\in\mathbb{R}$).

- $f: I \longrightarrow \mathbb{R}$ continuous is not necessarily uniformly continuous. (ex.: $f(x) = \frac{1}{x}$, $\forall x\in(0,1)$). (I: open interval)

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→ works only for closed intervals.

Thm: Every continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous.

Proof: Suppose that f is not uniformly continuous.

Then, there exists $\varepsilon > 0$ and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in $[a,b]$, such that $x_n - y_n \to 0$, but $|f(x_n) - f(y_n)| \geq \varepsilon, \forall n \in \mathbb{N}$.

(follows from proof of characterisation via sequences. Stronger than just $f(x_n) - f(y_n) \not\to 0$!)

We will show that this is a contradiction.

Indeed:

- $(x_n)_{n\in\mathbb{N}}$ is bounded, thus, by Bolzano-Weierstrass, it has a convergent subsequence $(x_{k_n})_{n\in\mathbb{N}}$.
- That is, $x_{k_n} \to x_0 \in \mathbb{R}$, for this $(x_{k_n})_{n\in\mathbb{N}}$.

  Since $a \leq x_{k_n} \leq b \quad \forall n \in \mathbb{N}$,

  we have $x_0 \in [a,b]$. Therefore, f continuous at x_0 $\Longrightarrow f(x_{k_n}) \to f(x_0)$.
- Since $x_n - y_n \to 0$, we have $x_{k_n} - y_{k_n} \to 0$ as well.

  And since in addition $x_{k_n} \to x_0$, we have $y_{k_n} \to x_0$.

  $\left(y_{k_n} = x_{k_n} - (x_{k_n} - y_{k_n}) \to x_0 - x_0 = 0. \right)$

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Since f is continuous at x_0, it follows that

$$f(y_{k_n}) \longrightarrow f(x_0).$$

Therefore:

$$\left.\begin{array}{l} f(x_{k_n}) \longrightarrow f(x_0) \\ f(y_{k_n}) \longrightarrow f(x_0) \end{array}\right\} \longrightarrow f(x_{k_n}) - f(y_{k_n}) \longrightarrow f(x_0) - f(x_0) = 0.$$

However, this is a contradiction: we picked $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ such that $\left|f(x_{k_n}) - f(y_{k_n})\right| \geq \underbrace{\varepsilon}_{\text{fixed}} > 0$, $\forall n \in \mathbb{N}$, thus

$$f(x_{k_n}) - f(y_{k_n}) \not\longrightarrow 0.$$

Therefore, f is uniformly continuous.

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1.5 Field

A field is a set F, together with two laws of composition, addition $(+)$ and multiplication $(.)$ such that for all $a, b, c \in F$ the following properties holds

- $a + b = b + a$ and $ab = ba$ (commutative laws).
- $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ (associative laws).
- $a(b + c) = ab + ac$ (distributive law).
- There are distinct members 0 and 1 such that $a + 0 = a$ and $a1 = a$ for all a.
- For each $a \in F$ there is an element $-a \in F$ such that $a + (-a) = 0$. and if $a \neq 0$, there is an element $1/a$ such that $a(1/a) = 1$.

Remark: The left distributive law also holds.
Examples:

- Let $z_2 = \{0, 1\}$ such that

$$0 + 0 = 1 + 1 = 0; 0 + 1 = 1 + 0 = 1$$

and

$$0.0 = 0.1 = 1.0 = 0, 1.1 = 1.$$

- Is the set of integers a field?

1.6 Dedekind Infinite Set

Dedekind defined a set A to be infinite if there is an injective map $j : A \to A$ wh is not onto (surjective); such sets are now called Dedekind infinite.

Example: Show that $\mathbb{N}$ is Dedekind infinite.
The mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$ is injective, and is not surjec

Corollary: The set of natural numbers $\mathbb{N}$ is infinite set.

Countable sets: A set A is countable if it is finite or if there is a bijection $c : \mathbb{N}$ otherwise it is uncountable.

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Remark: A set is countable if it is empty or if there is a bijection from an initial segment of $\mathbb{N}$ onto A. The function c is called an enumeration of A. A set is countably infinite if it is infinite and countable.

Thus A is countably infinite if and only if the elements of A can be listed, or enumerated, as an infinite sequence $(c_1, c_2, ...)$, without repetition.

If A is countable (countably infinite) and $j : A \to B$ is a bijection, then B is countable (countably infinite). Not every set is countable. It was Cantor who first showed, in 1873, that there are different sizes of infinite set, showing that the set of real numbers is uncountable.

Proposition: Let A be a non-empty set. Then the following are equivalent.

(a) A is countable.

(b) There exists a surjection $f : \mathbb{N} \to A$.

(c) There exists an injection $g : A \to \mathbb{N}$.

Theorem: The set $\mathbb{N} \times \mathbb{N}$ is countable.

Proof: Define the mapping $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$f(k, l) = 2^k 2^l.$$

Use above proposition to prove that $\mathbb{N} \times \mathbb{N}$ is countable.

1.7 The Set of Rational Numbers

Construction of set of rational numbers: Let $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$ be the set of nonzero integers. Define a relation on $\mathbb{Z} \times \mathbb{Z}^*$ by setting $(p, q) \sim (r, s)$ if $ps = qr$.

Proposition: The relation $(p, q) \sim (r, s)$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^*$.

Proof:

Transitive: Suppose $(p, q) \sim (r, s)$ and $(r, s) \sim (t, u)$, we need to show that $(p, q) \sim (t, u)$. Consider

$$pusr = (ps)(ru) = (qr)(ts) = qtsr$$

$\mathbb{Q}$ is abelian group under addition: We define addition on $\mathbb{Q}$ by

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}.$$

We define the multiplication as $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right) = \frac{pr}{qs}$.

Proposition: Let $\mathbb{Q}^* = \mathbb{Q}/\{0/1\}$. Then, $(\mathbb{Q}^*, .)$ and $(\mathbb{Q}^*, +)$ are abelian group

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1.8.1 Partial Order

A partial order is a relation R on a set S such that for all $a, b, c \in S$, we have the following

- aRa (Reflexivity).
- If aRb and bRa implies $a = b$ (Anti Symmetric).
- If aRb, bRc then aRc (Transitve).

1.9 Lower Bound

Suppose the set S is an ordered set, and $E \subset S$. If there exists a b such that $x \geq b$ whenever $x \in E$. We say that E is bounded below. In this case, b is a lower bound of E.

If b is a lower bound of E, then so is any smaller number.

1.9.1 Infimum

If α is a lower bound of E, but no number greater than α is a lower bound of E, then α is an infimum of E, and we write

$$\alpha = \inf E.$$

Exampie: Recall the set $A = \{p \in \mathbb{Q} : p^2 > 2\}$.

1.10 Upper Bound

Suppose the set S is an ordered set, and $E \subset S$. If there exists a b such that $x \leq b$ whenever $x \in E$. We say that E is bounded above In this case, b is an upper bound of E.

If b is an upper bound of E, then so is any larger number

1.10.1 Supremum

If β is an upper bound of E, but no number less than β is an upper bound of E, then β is a supremum of E, and we write

$$\beta = \sup E.$$

Example: Recall the set $A = \{p \in \mathbb{Q} : p^2 < 2\}$.

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The set of rational numbers is defined as

$$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}.$$

Example: Find solution of the equation $p = \sqrt{2}$ in the set of rational numbers if possible.

Solution: See Lecture.

Example: Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$. We will show that for every $p \in A$, we can find a rational number q such that $p < q$.

To do this, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{1.2}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{1.3}$$

- If p is in A then $p^2 - 2 < 0$, (1.2) shows that $q > p$ and (1.3) shows that $q^2 < 2$. Thus q is in A.

Let $B = \{p \in \mathbb{Q} : p^2 > 2\}$. We will show that for every $p \in B$, we can find a rational number q such that $q < p$.

- If p is in B then $p^2 - 2 > 0$, (1.2) shows that $0 < q < p$ and (1.3) shows that $q^2 > 2$. Thus q is in B.

1.8 Ordered Set

Let S be a set. An order on S is a relation , denoted by $<$, with the following properties

- For each pair $a, b \in S$, exactly one of the following is true:

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

- If $a, b, c \in S$ such that If $a < b$ and $b < c$, then $a < c$. (The relation $<$ is *transitive*.)
- If $a < b$, then $a + c < b + c$ for any c, and if $0 < c$, then $ac < bc$.

Example: The set of rational numbers $\mathbb{Q}$ is an order set, if $r < s$ is defined to means that $s - r$ is a positive rational number.

An ordered set is a set S in which an order is defined.

The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which one of these two is hold.

What is the negation of $x > y$?

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all follow from the axioms of field.

Theorem: For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.

Proof: The uniqueness is clear, since if there are two positive numbers y_1 and y_2 then

$$0 < y_1 < y_2 \quad \Rightarrow \quad y_1^n < y_2^n.$$

Let E be the set containing of all positive real numbers t such that $t^n < x$.
The set E is not empty, as if $t = x/(x+1)$ then $0 \leq t < 1$ and $t^n < t < x$.
If $t > 1 + x$ then $t^n \geq t > x$, so that $t \notin E$. Thus $t \in E$ is an upper bound of E.
Then by least upper bound property there exists $y \in \mathbb{R}$ such that

$$y = \sup E.$$

We need to prove that $y^n = x$ we will show that each of the inequalities $y^n < x$ and $y^n > x$ leads to a contradiction.

Recall the identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \ldots + a^{n-1})$ yields the identity

$$b^n - a^n = (b-a)nb^{n-1},$$

when $0 < a < b$.

Assume $y^n < x$, choose h so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$

Put $a = y$, $b = y + h$. Then

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus $(y+h)^n < x$, and $y + h \in E$, which is contradiction to the fact that y is an upper bound of E.

Assume $y^n > x$, put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y$. If $t \geq y - k$, we conclude that

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x.$$

Thus $t^n > x$, and $t \notin E$. It follows that $y - k$ is an upper bound of E. But $y - k < y$, which contradicts the fact that y is the least upper bound of E.

Theorem: If a and b are any two real numbers, then

$$|a + b| \leq |a| + |b|. \tag{1.6}$$

Proof: There are four possibilities:

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2.1.1 Convergent Sequence

A sequence $\{s_n\}$ *converges to a limit* L if for every $\varepsilon > 0$ there is an integer N such that

$$|s_n - L| < \varepsilon \quad \text{if} \quad n \geq N.$$

In this case we say that $\{s_n\}$ is *convergent* and write

$$\lim_{n\to\infty} s_n = L.$$

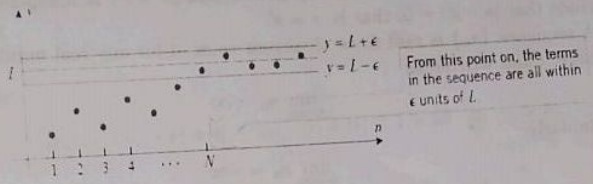


Figure 2.1: Convergence of the sequence

A sequence that does not converge *diverges*, or is *divergent*

Example: If $s_n = c$ for $n \geq k$, then $|s_n - c| = 0$ for $n \geq k$, and $\lim_{n\to\infty} s_n = c$.

Example: If

$$s_n = \left\{\frac{2n+1}{n+1}\right\},$$

then $\lim_{n\to\infty} s_n = 2$, since

$$\begin{aligned} |s_n - 2| &= \left|\frac{2n+1}{n+1} - \frac{2n+2}{n+1}\right| \\ &= \frac{1}{n+1}; \end{aligned}$$

Theorem: The limit of a convergent sequence is unique.
Proof: Suppose that

$$\lim_{n\to\infty} s_n = s \quad \text{and} \quad \lim_{n\to\infty} s_n = s'.$$

We must show that $s = s'$. Let $\varepsilon > 0$. From definition, there are integers N_1 and N_2 such that

$$|s_n - s| < \varepsilon \quad \text{if} \quad n \geq N_1$$

(because $\lim_{n\to\infty} s_n = s$), and

$$|s_n - s'| < \varepsilon \quad \text{if} \quad n \geq N_2$$

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Example: If $s_n = [1+(-1)^n]n$, then $\{s_n\}$ is bounded below ($s_n \geq 0$) but unbounded above.

$\{\ s_n\}$ is bounded above ($-s_n \leq 0$) but unbounded below.

If $s_n = (-1)^n$, then $\{s_n\}$ is bounded.

If $s_n = (-1)^n n$, then $\{s_n\}$ is not bounded above or below.

Theorem: A convergent sequence is bounded.
Proof: By taking $\varepsilon = 1$, we see that if $\lim_{n\to\infty} s_n = s$, then there is an integer N such that

$$|s_n - s| < 1 \quad \text{if} \quad n \geq N.$$

Therefore,

$$|s_n| = |(s_n - s) + s| \leq |s_n - s| + |s| < 1 + |s| \quad \text{if} \quad n \geq N,$$

and

$$|s_n| \leq \max\{|s_0|, |s_1|, \ldots, |s_{N-1}|, 1 + |s|\}$$

for all n, so $\{s_n\}$ is bounded.

Theorem: A sequence $\{s_n\}$ converges to s if and only if every neighborhood of s contains s_n for all but finitely many n.

2.3 Monotonic Sequences

A sequence $\{s_n\}$ is *nondecreasing* if $s_n \geq s_{n-1}$ for all n, or *nonincreasing* if $s_n \leq s_{n-1}$ for all n.

A *monotonic sequence* is a sequence that is either nonincreasing or nondecreasing. If $s_n > s_{n-1}$ for all n, then $\{s_n\}$ is *increasing*, while if $s_n < s_{n-1}$ for all n, $\{s_n\}$ is *decreasing*.

Theorem:

- If $\{s_n\}$ is nondecreasing, then $\lim_{n\to\infty} s_n = \sup\{s_n\}$.
- If $\{s_n\}$ is nonincreasing, then $\lim_{n\to\infty} s_n = \inf\{s_n\}$.

Recall: The definition of supremum and infimum of a set.
Proof: Let $\beta = \sup\{s_n\}$. If $\beta < \infty$, then by definition if $\varepsilon > 0$ then

$$\beta - \varepsilon < s_N \leq \beta$$

for some integer N.

Since $s_N \leq s_n \leq \beta$ if $n \geq N$, it follows that

$$\beta - \varepsilon < s_n \leq \beta \quad \text{if} \quad n \geq N.$$

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Although each y_j occurs as a term of $\{x_n\}$, $\{y_j\}$ is not necessarily a subsequence of $\{x_n\}$, because if we write

$$y_j = x_{n_j}.$$

There is no reason to expect that $\{n_j\}$ is an increasing sequence as required in definition of subsequence.

However, it is always possible to pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ that is increasing, and then the sequence $\{y_{j_k}\} = \{s_{n_{j_k}}\}$ is a subsequence of both $\{y_j\}$ and $\{x_n\}$.

2.6 Limit Superior and Limit Inferior

Theorem:

- If $\{s_n\}$ is bounded above and does not diverge to $-\infty$, then there is a unique real number $\overline{s}$ such that, if $\varepsilon > 0$,

$$s_n < \overline{s} + \varepsilon \quad \text{for large } n \tag{2.4}$$

and

$$s_n > \overline{s} - \varepsilon \quad \text{for infinitely many } n. \tag{2.5}$$

- If $\{s_n\}$ is bounded below and does not diverge to ∞, then there is a unique real number $\underline{s}$ such that, if $\varepsilon > 0$,

$$s_n > \underline{s} - \varepsilon \quad \text{for large } n \tag{2.6}$$

and

$$s_n < \underline{s} + \varepsilon \quad \text{for infinitely many } n. \tag{2.7}$$

Proof: Proof of the first part. Since $\{s_n\}$ is bounded above, there is a number β such that $s_n < \beta$ for all n. Since $\{s_n\}$ does not diverge to $-\infty$, there is a number α such that $s_n > \alpha$ for infinitely many n. If we define

$$M_k = \sup\{s_k, s_{k+1}, \ldots, s_{k+r}, \ldots\},$$

then $\alpha \leq M_k \leq \beta$, so $\{M_k\}$ is bounded. Since $\{M_k\}$ is nonincreasing (why?), it converges. Let

$$\overline{s} = \lim_{k\to\infty} M_k. \tag{2.8}$$

If $\varepsilon > 0$, then $M_k < \overline{s} + \varepsilon$ for large k, and since $s_n \leq M_k$ for $n > k$, $\overline{s}$ satisfies (2.4).

If (2.5) were false for some positive ε, there would be an integer K such that

$$s_n \leq \overline{s} - \varepsilon \quad \text{if} \quad n \geq K.$$

However, this implies that

$$M_k \leq \overline{s} - \varepsilon \quad \text{if} \quad k \geq K,$$

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Example:

$$\limsup_{n\to\infty} n^2 = \liminf_{n\to\infty} n^2 = \infty.$$

$$\limsup_{n\to\infty} (-1)^n \left(1 - \frac{1}{n}\right) = 1.$$

$$\liminf_{n\to\infty} (-1)^n \left(n - \frac{1}{n}\right) = 1.$$

and

$$\limsup_{n\to\infty} [1 + (-1)^n] n^2 = \infty.$$

$$\liminf_{n\to\infty} [1 + (-1)^n] n^2 = 0.$$

Theorem: If $\{s_n\}$ is a sequence of real numbers, then

$$\lim_{n\to\infty} s_n = s \tag{2.9}$$

if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s. \tag{2.10}$$

Proof: If $s = \pm\infty$, the equivalence of (2.9) and (2.10) follows immediately from their definitions. If $\lim_{n\to\infty} s_n = s$ (finite), then definition of a convergent sequence implies that (2.4)–(2.7) hold with $\overline{s}$ and $\underline{s}$ replaced by s. Hence, (2.10) follows from the uniqueness of $\overline{s}$ and $\underline{s}$. For the converse, suppose that $\overline{s} = \underline{s}$ and let s denote their common value. Then (2.4) and (2.6) imply that

$$s - \varepsilon < s_n < s + \varepsilon$$

for large n, and (2.9) follows from Definition and the uniqueness of $\lim_{n\to\infty} s_n$.

2.7 Cauchy Sequence

A sequence $\{s_n\}$ of real numbers is called a Cauchy sequence if for every $\varepsilon > 0$, there is an integer N such that

$$|s_n - s_m| < \varepsilon \quad \text{if} \quad m, n \geq N.$$

Theorem: If $\{s_n\}$ is a Cauchy sequence of real numbers, then $\{s_n\}$ is bounded.

Proof: See Lecture

Theorem (Cauchy's convergence criterion): A sequence $\{s_n\}$ of real numbers converges **if and only if**, for every $\varepsilon > 0$, there is an integer N such that

$$|s_n - s_m| < \varepsilon \quad \text{if} \quad m, n \geq N.$$

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where

$$A_n = a_k + a_{k+1} + \cdots + a_n, \quad n \geq k.$$

2.11 The Comparison Test

Theorem: Suppose that

$$0 < a_n < b_n, \quad n \geq k.$$

Then

- $\sum a_n < \infty$ if $\sum b_n < \infty$.
- $\sum b_n = \infty$ if $\sum a_n = \infty$.

Proof: If

$$A_n = a_k + a_{k+1} + \cdots + a_n \quad \text{and} \quad B_n = b_k + b_{k+1} + \cdots + b_n, \quad n \geq k,$$

then, we have

$$A_n \leq B_n.$$

If $\sum b_n < \infty$, then $\{B_n\}$ is bounded above implies that $\{A_n\}$ is also; therefore, $\sum a_n < \infty$.

On the other hand, if $\sum a_n = \infty$, then $\{A_n\}$ is unbounded above implies that $\{B_n\}$ is also; therefore, $\sum b_n = \infty$.

Example: Discuss the convergence of the series $\sum \frac{r^n}{n}$.

Solution: Since

$$\frac{r^n}{n} < r^n, \quad n \geq 1,$$

and $\sum r^n < \infty$ if $0 < r < 1$, the series $\sum r^n/n$ converges if $0 < r < 1$, by the comparison test.

Comparing these two series is **inconclusive** if $r > 1$, since it does not help to know that the terms of $\sum r^n/n$ are smaller than those of the divergent series $\sum r^n$.

If $r < 0$, the comparison test does not apply, since the series then have infinitely many negative terms. Since

$$r^n < nr^n$$

and $\sum r^n = \infty$ if $r \geq 1$, the comparison test implies that $\sum nr^n = \infty$ if $r \geq 1$.

Comparing these two series is inconclusive if $0 < r < 1$, since it does not help to know that the terms of $\sum nr^n$ are larger than those of the convergent series $\sum r^n$.

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However,

$$\lim_{n\to\infty} n\left(\frac{a_{n+1}}{a_n} - 1\right) = \lim_{n\to\infty} n\left(\frac{n+1}{\alpha+n} - 1\right)$$
$$= \lim_{n\to\infty} \frac{n(1-\alpha)}{\alpha+n} = 1 - \alpha,$$

so Raabe's test implies that $\sum a_n < \infty$ if $\alpha > 2$ and $\sum a_n = \infty$ if $0 < \alpha < 2$.

2.14 Cauchy's Root Test

Theorem: If $a_n \geq 0$ for $n \geq k$, then

- $\sum a_n < \infty$ if $\limsup_{n\to\infty} a_n^{1/n} < 1$.
- $\sum a_n = \infty$ if $\limsup_{n\to\infty} a_n^{1/n} > 1$.

The test is inconclusive if $\limsup_{n\to\infty} a_n^{1/n} = 1$.

Proof: If $\limsup_{n\to\infty} a_n^{1/n} < 1$, there is an r such that $0 < r < 1$ and $a_n^{1/n} < r$ for large n. Therefore, $a_n < r^n$ for large n. Since $\sum r^n < \infty$, the comparison test implies that $\sum a_n < \infty$.

If $\limsup_{n\to\infty} a_n^{1/n} > 1$, then $a_n^{1/n} > 1$ for infinitely many values of n, so $\sum a_n = \infty$.

Example: Cauchy's root test is inconclusive if

$$\sum a_n = \sum \frac{1}{n^p},$$

because then

$$\limsup_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^p}\right)^{1/n} = \lim_{n\to\infty} \exp\left(-\frac{p}{n}\log n\right) = 1$$

for all p.

However, we know that $\sum 1/n^p < \infty$ if $p > 1$ and $\sum 1/n^p = \infty$ if $p \leq 1$.

Example: If

$$\sum a_n = \sum \left(2 + \sin\frac{n\pi}{4}\right)^n r^n,$$

then

$$\limsup_{n\to\infty} a_n^{1/n} = \limsup_{n\to\infty} \left(2 + \sin\frac{n\pi}{4}\right) r = 3r,$$

and so $\sum a_n < \infty$ if $r < 1/3$ and $\sum a_n = \infty$ if $r > 1/3$. The test is inconclusive if $r = 1/3$, but then $|a_{8m+2}| = 1$ for $m \geq 0$, so $\sum a_n = \infty$.

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2.15 Absolute Convergence

2.16 Absolute Convergence

A series $\sum a_n$ *converges absolutely*, or is *absolutely convergent*, if $\sum |a_n| < \infty$.

Example: A convergent series $\sum a_n$ of nonnegative terms is absolutely convergent, since $\sum a_n$ and $\sum |a_n|$ are the same.

More generally, any convergent series whose terms are of the same sign for sufficiently large n converges absolutely.

Example: Consider the series

$$\sum \frac{\sin n\theta}{n^p},$$

where θ is arbitrary and $p > 1$.

Solution: Since

$$\left|\frac{\sin n\theta}{n^p}\right| \le \frac{1}{n^p}$$

and $\sum 1/n^p < \infty$ if $p > 1$, the comparison test implies that

$$\sum \left|\frac{\sin n\theta}{n^p}\right| < \infty, \quad p > 1.$$

Therefore, the given series converges absolutely if $p > 1$.

Example: If $0 < p < 1$, then the series

$$\sum \frac{(-1)^n}{n^p}$$

does not converge absolutely, since

$$\sum \left|\frac{(-1)^n}{n^p}\right| = \sum \frac{1}{n^p} = \infty.$$

However, the series converges, by the alternating series test, which we prove below.

Remark: Any test for convergence of a series with nonnegative terms can be used to test an arbitrary series $\sum a_n$ for absolute convergence by applying it to $\sum |a_n|$.

Example: Determine the series

$$\sum a_n = \sum (-1)^n \frac{n!}{\alpha(\alpha+1)\cdots(\alpha+n-1)}, \quad \alpha > 0,$$

for absolutely convergent or not?

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2.18 Alternating Series

Series whose terms alternate between positive and negative, called alternating series.

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}.$$

$$\sum_{k=1}^{\infty}(-1)^{k}\frac{1}{k}.$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k, \qquad \sum_{k=1}^{\infty}(-1)^{k+1}a_k.$$

Corollary (Alternating series test): The series $\sum(-1)^n a_n$ converges if $0 \leq a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$.

Proof: Let $b_n = (-1)^n$: recall that

$$B_n = b_k + b_{k+1} + \cdots + b_n, \quad n \geq k.$$

Then $\{|B_n|\}$ is a sequence of zeros and ones and therefore bounded. The conclusion now follows from Abel's test.

Example: Show that the series is convergent

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}.$$

Example: Show that the series is convergent

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}.$$

2.19 Grouping Terms in a Series

- The terms of a finite sum can be grouped by putting parenthesis arbitrarily.
- For example.

$$(1+7)+(6+5)+4 =$$
$$(1+7+6)+(5+4).$$

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CHAPTER 3

Continuity

Function: A rule f that assigns to each member of a nonempty set D a unique member of a set Y is a *function from D to Y*. The relationship between a member x of D and the member y of Y that f assigns to x as

$$y = f(x).$$

The set D is the *domain* of f, denoted by D_f. The members of Y are the possible *values* of f.

If $y_0 \in Y$ and there is an x_0 in D such that $f(x_0) = y_0$ then we say that f *attains* or *assumes* the value y_0. The set of values attained by f is the *range* of f.

A *real-valued function of a real variable* is a function whose domain and range are both subsets of the real numbers.

Examples: Consider the functions

$$f(x) = x^2, \quad g(x) = \sin x, \quad \text{and} \quad h(x) = e^x.$$

The functions f, g, and h defined on the extended real number system $(-\infty, \infty)$.

- The range of f is $[0, \infty)$.
- The range of g is $[-1, 1]$.
- The range of h is $(0, \infty)$.

The equation

$$[f(x)]^2 = x$$

does not define a function except on the singleton set $\{0\}$. If $x < 0$, no real number satisfies the $[f(x)]^2 = x$, while if $x > 0$, two real numbers satisfy $[f(x)]^2 = x$. However, the conditions

$$[f(x)]^2 = x \quad \text{and} \quad f(x) \geq 0$$

define a function f on $D_f = [0, \infty)$ with values $f(x) = \sqrt{x}$.

Similarly, the conditions

$$[g(x)]^2 = x \quad \text{and} \quad g(x) \leq 0$$

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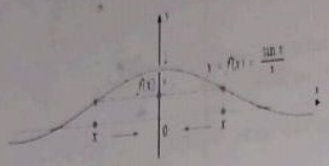


Figure 3.5: Limit $x \to 0$ when $f(x)$ is not defined at $x = 0$

Example: Investigate the limit

$$\lim_{x \to 0} \frac{|x|}{x}.$$

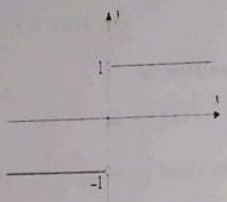


Figure 3.6: Limit $x \to 0$

3.1.2 Formal Definition of Limit

We say that $f(x)$ *approaches the limit* L *as* x *approaches* x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

if f is defined on some deleted neighborhood of x_0 and, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon, \qquad \text{whenever} \qquad 0 < |x - x_0| < \delta. \tag{3.1}$$

Example: For the function defined by $f(x) = cx$, where $c \in \mathbb{R}$. Show that

$$\lim_{x \to x_0} f(x) = cx_0.$$

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Solution: We write

$$|f(x) - cx_0| = |cx - cx_0| = |c||x - x_0|.$$

If $c \neq 0$, this yields

$$|f(x) - cx_0| < \varepsilon$$

if

$$|x - x_0| < \delta,$$

where δ is any number such that $0 < \delta \leq \varepsilon/|c|$.

If $c = 0$, then $f(x) - cx_0 = 0$ for all x, so $|f(x) - cx_0| < \varepsilon$ holds for all x.

Example: Prove that $\lim_{x\to 2}(3x - 5) = 1$.

Solution: See Lecture.

Example: For the function

$$f(x) = x \sin \frac{1}{x}, \quad x \neq 0,$$

show that

$$\lim_{x\to 0} f(x) = 0.$$

Solution: Even though f is not defined at $x_0 = 0$, because if

$$0 < |x| < \delta = \varepsilon,$$

then

$$|f(x) - 0| = \left| x \sin \frac{1}{x} \right| \leq |x| < \varepsilon.$$

On the other hand, the function

$$g(x) = \sin \frac{1}{x}, \quad x \neq 0,$$

has no limit as x approaches 0, since it assumes all values between -1 and 1 in every neighborhood of the origin.

Theorem: If $\lim_{x\to x_0} f(x)$ exists, then it is unique.

Proof: Suppose that limit of the function exists and it is L_1 and L_2. Let $\varepsilon > 0$. From definition of the limit, there are positive numbers δ_1 and δ_2 such that

$$|f(x) - L_i| < \varepsilon \quad \text{if} \quad 0 < |x - x_0| < \delta_i, \quad i = 1, 2.$$

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First observe that if $L_2 \neq 0$, there is a $\delta_3 > 0$ such that

$$|g(x) - L_2| < \frac{|L_2|}{2}, \qquad |g(x)| > \frac{|L_2|}{2} \quad \text{if} \quad 0 < |x - x_0| < \delta_3.$$

Now suppose that $0 < |x - x_0| < \min(\delta_1, \delta_2, \delta_3)$. Then

$$\begin{aligned}
\left|\left(\frac{f}{g}\right)(x) - \frac{L_1}{L_2}\right| &= \left|\frac{f(x)}{g(x)} - \frac{L_1}{L_2}\right| = \frac{|L_2 f(x) - L_1 g(x)|}{|g(x) L_2|} \\
&\leq \frac{2}{|L_2|^2}|L_2 f(x) - L_1 g(x)| \\
&= \frac{2}{|L_2|^2}|L_2[f(x) - L_1] + L_1[L_2 - g(x)]| \\
&\leq \frac{2}{|L_2|^2}[|L_2||f(x) - L_1| + |L_1||L_2 - g(x)|] \\
&< \frac{2}{|L_2|^2}(|L_2| + |L_1|)\varepsilon.
\end{aligned}$$

Example: Find

$$\lim_{x\to 2} \frac{9 - x^2}{x + 1} \quad \text{and} \quad \lim_{x\to 2}(9 - x^2)(x + 1).$$

Solution: If c is a constant, then $\lim_{x\to x_0} c = c$, and $\lim_{x\to x_0} x = x_0$. Therefore

$$\begin{aligned}
\lim_{x\to 2}(9 - x^2) &= \lim_{x\to 2} 9 - \lim_{x\to 2} x^2 \\
&= \lim_{x\to 2} 9 - (\lim_{x\to 2} x)^2 \\
&= 9 - 2^2 = 5,
\end{aligned}$$

and

$$\lim_{x\to 2}(x + 1) = \lim_{x\to 2} x + \lim_{x\to 2} 1 = 2 + 1 = 3.$$

Therefore,

$$\lim_{x\to 2} \frac{9 - x^2}{x + 1} = \frac{\lim_{x\to 2}(9 - x^2)}{\lim_{x\to 2}(x + 1)} = \frac{5}{3}$$

and

$$\lim_{x\to 2}(9 - x^2)(x + 1) = \lim_{x\to 2}(9 - x^2) \lim_{x\to 2}(x + 1) = 5 \cdot 3 = 15.$$

Example: The function

$$f(x) = 2x \sin\sqrt{x}$$

satisfies the inequality

$$|f(x)| < \varepsilon, \qquad \text{whenever} \quad 0 < x < \delta = \varepsilon/2.$$

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if f is defined on some open interval (x_0, b) and, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad x_0 < x < x_0 + \delta.$$

Example: Let

$$g(x) = \frac{x + |x|(1+x)}{x} \sin\frac{1}{x}, \quad x \neq 0.$$

Solution: If $x < 0$, then

$$g(x) = -x \sin\frac{1}{x}, \qquad \lim_{x \to 0-} g(x) = 0,$$

since

$$|g(x) - 0| = \left| x \sin\frac{1}{x} \right| \leq |x| < \varepsilon$$

if $-\varepsilon < x < 0$; that is, we have $\delta = \varepsilon$.

If $x > 0$, then

$$g(x) = (2 + x) \sin\frac{1}{x},$$

which takes on every value between -2 and 2 in every interval $(0, \delta)$.

Hence, $g(x)$ does not approach a right-hand limit at x approaches 0 from the right. This shows that a function may have a limit from one side at a point but fail to have a limit from the other side.

Example: Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution: See Lecture.

Example: Show that

$$\lim_{x \to 0+} \left(\frac{|x|}{x} + x \right) = 1, \qquad \lim_{x \to 0-} \left(\frac{|x|}{x} + x \right) = -1,$$

Solution: See Lecture

Theorem: A function f has a limit at x_0 if and only if it has left- and right-hand limits at x_0, and they are equal. More specifically,

$$\lim_{x \to x_0} f(x) = L$$

if and only if

$$\lim_{x \to x_0 +} f(x) = \lim_{x \to x_0 -} f(x) = L.$$

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3.3 Limits at $\pm\infty$

If f is defined on an interval (a, ∞), then $f(x)$ *approaches the limit* L *as* x *approaches* ∞, and write

$$\lim_{x\to\infty} f(x) = L.$$

For each $\varepsilon > 0$, there is a number β such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad x > \beta.$$

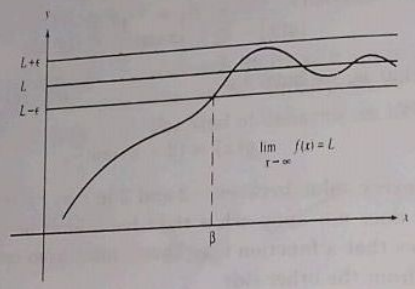


Figure 3.7: Limits at $\pm\infty$

Examples: Let

$$f(x) = 1 - \frac{1}{x^2}, \quad g(x) = \frac{2|x|}{1+x}, \quad \text{and} \quad h(x) = \sin x.$$

Find limits when $x \to \pm\infty$.

Solution: We have

$$\lim_{x\to\infty} f(x) = 1,$$

since

$$|f(x) - 1| = \frac{1}{x^2} < \varepsilon \quad \text{if} \quad x > \frac{1}{\sqrt{\varepsilon}}.$$

We have

$$\lim_{x\to\infty} g(x) = 2,$$

since

$$|g(x) - 2| = \left|\frac{2x}{1+x} - 2\right| = \frac{2}{1+x} < \frac{2}{x} < \varepsilon \quad \text{if} \quad x > \frac{2}{\varepsilon}.$$

However, $\lim_{x\to\infty} h(x)$ does not exist, since h assumes all values between -1 and 1 in any semi-infinite interval (τ, ∞). Discuss the limits at $x \to -\infty$.

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x^2}.$$

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Example Find
$$\lim_{x\to\infty}\frac{2x^2-x+1}{3x^2+2x-1}.$$

Example Find
$$\lim_{x\to 1}\frac{x^8-1}{x^4-1}.$$

$$f(x)=\begin{cases}\frac{1}{x+2}, & x<-2,\\ x^2-5, & -2<x<3,\\ \sqrt{x+13}, & x>3.\end{cases}$$

3.5 Continuity

Continuity (Informal):

- We say that f is *continuous at* x_0 if f is defined on an open interval (a,b) containing x_0 and
$$\lim_{x\to x_0} f(x)=f(x_0).$$

- We say that f is *continuous from the left at* x_0 if f is defined on an open interval (a,x_0) and
$$\lim_{x\to x_0^-} f(x)=f(x_0).$$

- We say that f is *continuous from the right at* x_0 if f is defined on an open interval (x_0,b) and
$$\lim_{x\to x_0^+} f(x)=f(x_0).$$

Continuity (Formal):

- A function f is continuous at x_0 if and only if f is defined on an open interval (a,b) containing x_0 and for each $\varepsilon>0$ there is a $\delta>0$ such that
$$|f(x)-f(x_0)|<\varepsilon \quad \text{whenever} \quad |x-x_0|<\delta.$$

- A function f is continuous from the right at x_0 if and only if f is defined on an interval $[x_0,b)$ and for each $\varepsilon>0$ there is a $\delta>0$ such that
$$|f(x)-f(x_0)|<\varepsilon \quad \text{whenever} \quad x_0\le x<x_0+\delta.$$

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Example: Discuss the continuity of the following function

$$r(x) = \frac{9 - x^2}{x + 1}.$$

Solution: See Lecture.

3.7 Removable Discontinuity

It can be shown that if f_1, f_2, $\cdots$, f_n are continuous on a set S, then so are $f_1 + f_2 + \cdots + f_n$ and $f_1 f_2 \cdots f_n$.

Therefore, any rational function

$$r(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m}$$

is continuous for all values of x except those for which its denominator vanishes.

Removable discontinuity: Let f be defined on a deleted neighborhood of x_0 and discontinuous (perhaps even undefined) at x_0. We say that f has a removable discontinuity at x_0 if $\lim_{x \to x_0} f(x)$ exists.

In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0, \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0, \end{cases}$$

is continuous at x_0.

Example: Consider the function

$$f(x) = x \sin \frac{1}{x}.$$

Solution: The function is not defined at $x_0 = 0$, and therefore certainly not continuous there, but $\lim_{x \to 0} f(x) = 0$.

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0, \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0, \end{cases}$$

The function

$$f_1(x) = \sin \frac{1}{x}$$

is undefined at 0 and its discontinuity there is not removable, since $\lim_{x \to 0} f_1(x)$ does not exist.

Composition of functions: Suppose that f and g are functions with domains D_f and D_g.

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If D_g has a nonempty subset T such that $g(x) \in D_f$ whenever $x \in T$, then the *composite function* $f \circ g$ is defined on T by

$$(f \circ g)(x) = f(g(x)).$$

Example: If

$$f(x) = \log x \quad \text{and} \quad g(x) = \frac{1}{1-x^2},$$

Can we define $f \circ g$?

Solution: The domain of the given functions are

$$D_f = (0, \infty) \quad \text{and} \quad D_g = x \in \mathbb{R} / \, x \neq \pm 1.$$

Since $g(x) > 0$ if $x \in T = (-1, 1)$, the composite function $f \circ g$ is defined on $(-1, 1)$ by

$$(f \circ g)(x) = \log \frac{1}{1-x^2}.$$

Can we define $g \circ f$?

The function $g \circ f$ is defined on $(0, 1/e) \cup (1/e, e) \cup (e, \infty)$ by

$$(g \circ f)(x) = \frac{1}{1 - (\log x)^2}.$$

Theorem: Suppose that g is continuous at x_0, $g(x_0)$ is an interior point of D_f, and f is continuous at $g(x_0)$. Then $f \circ g$ is continuous at x_0.

Proof: Since $g(x_0)$ is an interior point of D_f and f is continuous at $g(x_0)$, for every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that $f(t)$ is defined and

$$|f(t) - f(g(x_0))| < \varepsilon \quad \text{if} \quad |t - g(x_0)| < \delta_1.$$

Since g is continuous at x_0, there is a $\delta > 0$ such that $g(x)$ is defined and

$$|g(x) - g(x_0)| < \delta_1 \quad \text{if} \quad |x - x_0| < \delta.$$

We can imply that

$$|f(g(x)) - f(g(x_0))| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta.$$

Therefore, $f \circ g$ is continuous at x_0.

Example: For the functions

$$f(x) = \sqrt{x}, \qquad g(x) = \frac{9 - x^2}{x + 1}.$$

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is bounded on $[0,1]$, and

$$\sup_{0\le x\le 1} g(x) = 1, \quad \inf_{0\le x\le 1} g(x) = 0.$$

Therefore, g has no maximum or minimum on $[0,1]$, since it does not assume either of the values 0 and 1.

Example: The function

$$h(x) = 1 - x, \quad 0 \le x \le 1,$$

which differs from g only at 0 and 1, has the same supremum and infimum as g, but it attains these values at $x = 0$ and $x = 1$, respectively.

Therefore,

$$\max_{0\le x\le 1} h(x) = 1 \quad \text{and} \quad \min_{0\le x\le 1} h(x) = 0.$$

Example: The function

$$f(x) = e^{x(x-1)} \sin \frac{1}{x(x-1)}, \quad 0 < x < 1,$$

oscillates between $\pm e^{x(x-1)}$ infinitely often in every interval of the form $(0,\rho)$ or $(1-\rho,1)$, where $0 < \rho < 1$.

Furthermore, we have

$$\sup_{0<x<1} f(x) = 1, \quad \inf_{0<x<1} f(x) = -1.$$

However, f does not assume these values, so f has no maximum or minimum on $(0,1)$.

Theorem: If f is continuous on a finite closed interval $[a,b]$, then f is bounded on $[a,b]$.

Proof: Suppose that $t \in [a,b]$. Since f is continuous at t, there is an open interval I_t containing t such that

$$|f(x) - f(t)| < 1 \quad \text{if} \quad x \in I_t \cap [a,b].$$

The collection $\mathcal{H} = \{I_t : a \le t \le b\}$ is an open covering of $[a,b]$.

Since $[a,b]$ is compact, the Heine-Borel theorem implies that there are finitely many points $t_1, t_2, \ldots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b]$.

According to continuity of the function and with $t = t_i$,

$$|f(x) - f(t_i)| < 1 \quad \text{if} \quad x \in I_{t_i} \cap [a,b].$$

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The function $g(x)$ is continuous and has supremum 2 on the noncompact interval $(0, 1]$.

Since

$$\begin{aligned} g(x) &\leq 1 + (1 - x)\left|\sin\frac{1}{x}\right| \\ &< 1 + (1 \quad x) < 2 \quad \text{if} \quad 0 < x \leq 1. \end{aligned}$$

But does not assume its supremum on $(0, 1]$.
Consider the function

$$f(x) = e^{-x}.$$

The function $f(x)$ is continuous and has infimum 0, which it does not attain, on the noncompact interval $(0, \infty)$.

3.9 The Intermediate Value Theorem

Theorem: Suppose that f is continuous on $[a, b]$, $f(a) \neq f(b)$, and μ is between $f(a)$ and $f(b)$. Then $f(c) = \mu$ for some c in (a, b).

Proof: Suppose that $f(a) < \mu < f(b)$. The set

$$S = x : a \leq x \leq b \quad \text{and} \quad f(x) \leq \mu$$

is bounded and nonempty.

Let $c = \sup S$. We will show that $f(c) = \mu$.

If $f(c) > \mu$, then $c > a$ and, since f is continuous at c, there is an $\varepsilon > 0$ such that $f(x) > \mu$ if $c - \varepsilon < x \leq c$.

Therefore, $c - \varepsilon$ is an upper bound for S, which contradicts the definition of c as the supremum of S.

If $f(c) < \mu$, then $c < b$ and there is an $\varepsilon > 0$ such that $f(x) < \mu$ for $c \leq x < c + \varepsilon$, so c is not an upper bound for S.

This is also a contradiction. Therefore, $f(c) = \mu$.

3.10 Uniform Continuity

A function f is continuous on a subset S of its domain if for each $\varepsilon > 0$ and each x_0 in S, there is a $\delta > 0$, *which may depend upon* x_0 *as well as* ε, such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta \quad \text{and} \quad x \in D_f.$$

A function f is *uniformly continuous* on a subset S of its domain if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon \text{ whenever } |x - x'| < \delta \text{ and } x, x' \in S.$$

The function f is a uni for continuous at every a There exis $|f(x) - f(y)| < \epsilon$ $|x - y| < \delta$

Example: Check the uniform continuity of the function

$$f(x) = 2x.$$

Solution: For the function $f(x)$, we have

$$|f(x) - f(x')| = 2|x - x'| < \varepsilon \quad \text{if} \quad |x - x'| < \varepsilon/2.$$

The function $f(x)$ is uniformly continuous on $(-\infty, \infty)$.

Remark: A function f is *not* uniformly continuous on S if there is an $\varepsilon_0 > 0$ such that if δ is any positive number, there are points x and x' in S such that

$$|x - x'| < \delta$$

but

$$|f(x) - f(x')| \geq \varepsilon_0.$$

Example: Show that the function $g(x) = x^2$ is uniformly continuous on $[-r, r]$ for any finite r.

Solution: To see this, note that

$$|g(x) - g(x')| = |x^2 - (x')^2| = |x - x'|\,|x + x'| \leq 2r|x - x'|,$$

so

$$|g(x) - g(x')| < \varepsilon \quad \text{if} \quad |x - x'| < \delta = \frac{\varepsilon}{2r} \quad \text{and} \quad -r \leq x, x' \leq r.$$

Example: The function $g(x) = x^2$ is not uniformly continuous on $(-\infty, \infty)$.

Solution: To see this, we will show that if $\delta > 0$ there are real numbers x and x' such that

$$|x - x'| = \delta/2 \quad \text{and} \quad |g(x) - g(x')| \geq 1.$$

To this end, we write

$$|g(x) - g(x')| = |x^2 - (x')^2| = |x - x'|\,|x + x'|.$$

If $|x - x'| = \delta/2$ and $x, x' > 1/\delta$, then

$$|x - x'|\,|x + x'| > \frac{\delta}{2}\left(\frac{1}{\delta} + \frac{1}{\delta}\right) = 1.$$

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From (3.8), (3.10), and the triangle inequality,

$$|x' - t_r| = |(x' - x) + (x - t_r)| \le |x' - x| + |x - t_r| < \delta + \delta_{t_r} \le 2\delta_{t_r}.$$

Therefore, from continuity condition with $t = t_r$ and x replaced by x' implies that

$$|f(x') - f(t_r)| < \frac{\varepsilon}{2}.$$

This, (3.9), and (3.11) imply that $|f(x) - f(x')| < \varepsilon$.

3.11 Monotonic Functions

Monotonic functions: A function f is *nondecreasing* on an interval I if

$$f(x_1) \le f(x_2) \quad \text{whenever} \quad x_1, x_2 \in I, \quad x_1 < x_2. \tag{3.12}$$

or *nonincreasing* on I if

$$f(x_1) \ge f(x_2) \quad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2. \tag{3.13}$$

In either case, f is on I.

If $\le$ can be replaced by $<$ in (3.12), f is *increasing* on I. If $\ge$ can be replaced by $>$ in (3.13), f is *decreasing* on I. In either of these two cases, f is *strictly monotonic* on I.

Example: The function

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ 2, & 1 \le x \le 2, \end{cases}$$

is nondecreasing on $I = [0, 2]$.

Example: The function $g(x) = x^2$ is increasing on $[0, \infty)$. The function $h(x) = -x^3$ is decreasing on $(-\infty, \infty)$.

Theorem: Suppose that f is monotonic on (a, b) and define

$$\alpha = \inf_{a<x<b} f(x) \quad \text{and} \quad \beta = \sup_{a<x<b} f(x).$$

1. If f is nondecreasing, then $\lim_{x \to a+} f(x) = \alpha$ and $\lim_{x \to b-} f(x) = \beta$.
2. If f is nonincreasing, then $\lim_{x \to a+} f(x) = \beta$ and $\lim_{x \to b-} f(x) = \alpha$. (Here $a+ = -\infty$ if $a = -\infty$ and $b- = \infty$ if $b = \infty$.)

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(3.17)

Define $g(y_0) = x_0$,

we have $f(g(y_0)) = y_0, \quad g(f(x_0)) = x_0.$

Since this is true for all x_0 and y_0, so

$$g(f(x)) = x, \quad a \le x \le b, \quad f(g(y)) = y, \quad c \le y \le d.$$

The uniqueness of g follows from our assumption that f is increasing, and therefore only one value of x_0 can satisfy $f(x_0) = y_0$, for each y_0.

To see that g is increasing, suppose that $y_1 < y_2$ and let x_1 and x_2 be the points in $[a, b]$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Since f is increasing, $x_1 < x_2$. Therefore,

$$g(y_1) = x_1 < x_2 = g(y_2),$$

so g is increasing.

Since $R_g = \{g(y) : y \in [c, d]\}$ is the interval $[g(c), g(d)] = [a, b]$, therefore from theorem, we have proved with f and $[a, b]$ replaced by g and $[c, d]$ implies that g is continuous on $[c, d]$.

Example: If

$$f(x) = x^2, \quad 0 \le x \le R.$$

Solution: The inverse of the given function is

$$f^{-1}(y) = g(y) = \sqrt{y}, \quad 0 \le y \le R^2.$$

Example: If

$$f(x) = 2x + 4, \quad 0 \le x \le 2,$$

Solution: The inverse of the given function is

$$f^{-1}(y) = g(y) = \frac{y - 4}{2}, \quad 4 \le y \le 8.$$

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CHAPTER 4

Differentiability

4.1 Derivative

Derivative: A function f is *differentiable* at an interior point x_0 of its domain if the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0}, \quad x \neq x_0,$$

approaches a limit as x approaches x_0, in which case the limit is called the *derivative of f at x_0*, and is denoted by $f'(x_0)$.

Thus,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If we take $x = x_0 + h$ then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- If f is defined on an open set S, we say that f is *differentiable on* S if f is differentiable at every point of S.
- If f is differentiable on S, then f' is a function on S. We say that f is *continuously differentiable* on S if f' is continuous on S.
- If f is differentiable on a neighborhood of x_0, it is reasonable to ask if f' is differentiable at x_0. If so, we denote the derivative of f' at x_0 by $f''(x_0)$.
- This is the *second derivative of f at x_0*, and it is also denoted by $f^{(2)}(x_0)$. Continuing inductively, if $f^{(n-1)}$ is defined on a neighborhood of x_0, then the nth *derivative of f at x_0*, denoted by $f^{(n)}(x_0)$, is the derivative of $f^{(n-1)}$ at x_0.

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- For convenience we define the *zeroth derivative* of f to be f itself; thus

$$f^{(0)} = f.$$

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where $E(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0), & x \in D_f \text{ and } x \neq x_0, \\ 0, & x = x_0. \end{cases}$

Examples:

- $f(x) = x^2$ has derivative $f'(x) = 4x$.
- $f(x) = \frac{1}{x}$ has derivative $f'(x) = \frac{-1}{x^2}$.
- $f(x) = \sin x$ has derivative $f'(x) = \cos x$.

Is continuity implies differentiability?

Counter example: Consider the function

$$f(x) = |x|.$$

The functions can be written as

$$f(x) = \begin{cases} x, & x > 0, \\ -x, & x \leq 0, \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} 1, & x > 0, \\ -1, & x \leq 0. \end{cases}$$

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x}{x} \tag{4.1}$$

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{-x}{x} = -1 \tag{4.2}$$

are different,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist (Theorem ??); thus, f is not differentiable at 0, even though it is continuous at 0.

Theorem: If f and g are differentiable at x_0, then so are $f + g$, $f - g$, and fg, with

1. $(f + g)'(x_0) = f'(x_0) + g'(x_0)$;
2. $(f - g)'(x_0) = f'(x_0) - g(x_0)$;
3. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

The quotient f/g is differentiable at x_0 if $g(x_0) \neq 0$, with

- $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$.

Proof The trick is to add and subtract the right quantity in the numerator of the difference quotient for $(fg)'(x_0)$; thus.

$$\begin{aligned}
&\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0}.
\end{aligned}$$

The difference quotients on the right approach $f'(x_0)$ and $g'(x_0)$ as x approaches x_0, and $\lim_{x \to x_0} g(x) = g(x_0)$.

Example: Find $\frac{ds}{dt}$ where

$s(t) = (5 - t^2)t^{3/2}$.

Example: Find $\frac{dy}{dx}$ where

$y(x) = \frac{x^4 - 7x^2}{x - 5}$.

Lemma: If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

Theorem (The chain rule): Suppose that g is differentiable at x_0 and f is differentiable at $g(x_0)$.

Then the composite function $h = f \circ g$, defined by

$$h(x) = f(g(x)),$$

is differentiable at x_0, with

$$h'(x_0) = f'(g(x_0))g'(x_0).$$

Proof: Since f is differentiable at $g(x_0)$, we can write

$$f(t) - f(g(x_0)) = [f'(g(x_0)) + E(t)][t - g(x_0)], \tag{4.3}$$

where $\lim_{t \to g(x_0)} E(t) = E(g(x_0)) = 0.$

Letting $t = g(x)$ yields

$$f(g(x)) - f(g(x_0)) = [f'(g(x_0)) + E(g(x))][g(x) - g(x_0)].$$

Since $h(x) = f(g(x))$, this implies that

$$\frac{h(x) - h(x_0)}{x - x_0} = [f'(g(x_0)) + E(g(x))]\frac{g(x) - g(x_0)}{x - x_0}. \tag{4.4}$$

Since g is continuous at x_0 we have

$$\lim_{x \to x_0} E(g(x)) = E(g(x_0)) = 0.$$

Therefore, (4.4) implies that

$$h'(x_0) = \lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = f'(g(x_0))g'(x_0),$$

as stated.

Example: Calculate the derivative of

$$h(x) = \sin\frac{1}{x}, \quad x \neq 0.$$

If

$$f(x) = \sin x \quad \text{and} \quad g(x) = \frac{1}{x}, \quad x \neq 0,$$

then

$$h(x) = f(g(x)) = \sin\frac{1}{x}, \quad x \neq 0,$$

and

$$h'(x) = f'(g(x))g(x) = \left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \quad x \neq 0.$$

Example: What is wrong in the following justification?

$$\begin{aligned}\frac{h(x) - h(x_0)}{x - x_0} &= \frac{f(g(x)) - f(g(x_0))}{x - x_0} \\ &= \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)}\,\frac{g(x) - g(x_0)}{x - x_0}\end{aligned}$$

and arguing that

$$\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} = f'(g(x_0))$$

(because $\lim_{x \to x_0} g(x) = g(x_0)$) and

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0).$$

Is it a valid proof?

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4.2 One Sided Derivative

One sided derivatives: If f is defined on $[x_0, b)$, the *right-hand derivative of f at* x_0 is defined to be

$$f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0},$$

if the limit exists.

If f is defined on $(a, x_0]$, the *left-hand derivative of f at* x_0 is defined to be

$$f'_-(x_0) = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0}$$

if the limit exists.

The definition of limit implies that f is differentiable at x_0 if and only if $f'_+(x_0)$ and $f'_-(x_0)$ exist and are equal, in which case

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

The functions defined by $f(x) = |x|$ and $f(x) = \frac{|x|}{x}$ have one sided derivatives.

Example: For the piecewise defined function

$$f(x) = \begin{cases} x^3, & x \le 0, \\ x^2 \sin \frac{1}{x}, & x > 0, \end{cases}$$

Investigate the one sided derivatives.

Solution: For the given function, we have

$$f'(x) = \begin{cases} 3x^2, & x < 0, \\ 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x > 0. \end{cases}$$

Since neither formula in $f(x)$ holds for all x in any neighborhood of 0, we cannot simply differentiate either to obtain $f'(0)$.

We will calculate the one sided derivatives

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0+} x \sin \frac{1}{x} = 0,$$

$$f'(0) = \lim_{x \to 0-} \frac{x^3 - 0}{x - 0} = \lim_{x \to 0-} x^2 = 0;$$

hence, $f'(0) = f'_+(0) = f'_-(0) = 0$.

Remark: There is a difference between one sided derivatives and one sided limit of derivative

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4.5 Rolle's Theorem

Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$. Then $f'(c) = 0$ for some c in the open interval (a, b).

Proof: Since f is continuous on $[a, b]$, f attains a maximum and a minimum value on $[a, b]$. If these two extreme values are the same, then f is constant on (a, b), so $f'(x) = 0$ for all x in (a, b).

If the extreme values differ, then at least one must be attained at some point c in the open interval (a, b), and $f'(c) = 0$.

4.6 The Mean Value Theorem

Theorem: Suppose that f is differentiable on $[a, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$. Then $f'(c) = \mu$ for some c in (a, b).

Proof: Suppose first that

$$f'(a) < \mu < f'(b)$$

and define

$$g(x) = f(x) - \mu x.$$

Then

$$g'(x) = f'(x) - \mu, \quad a \leq x \leq b,$$

and by our supposition $f'(a) < \mu < f'(b)$, we have

$$g'(a) < 0 \quad \text{and} \quad g'(b) > 0.$$

Since g is continuous on $[a, b]$, g attains a minimum at some point c in $[a, b]$.

Due to Lemma and $g'(a) < 0$ and $g'(b) > 0$, imply that there is a $\delta > 0$ such that

$$g(x) < g(a), \quad a < x < a + \delta, \quad \text{and} \quad g(x) < g(b), \quad b - \delta < x < b.$$

Therefore $c \neq a$ and $c \neq b$. Hence, $a < c < b$, and therefore $g'(c) = 0$, by the fact that if f is differentiable at a local extreme point x_0 then $f'(x_0) = 0$. From , $f'(c) = \mu$. The proof for the case where $f'(b) < \mu < f'(a)$ can be obtained by applying this result to $-f$.

Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some c in (a, b).

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Proof: The function

$$h(x) = [b-a]f(x) - [f(b)-f(a)]x$$

is continuous on $[a,b]$ and differentiable on (a,b).
Furthermore

$$h(a) = h(b) = bf(a) - f(b)a.$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a,b).

Since $h'(c) = [b-a]f'(c) - [f(b)-f(a)]$.

4.7 Generalized Mean Value Theorem

Theorem: If f and g are continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then

$$[g(b)-g(a)]f'(c) = [f(b)-f(a)]g'(c)$$

for some c in (a,b).

Proof: The function

$$h(x) = [g(b)-g(a)]f(x) - [f(b)-f(a)]g(x)$$

is continuous on $[a,b]$ and differentiable on (a,b).
Furthermore

$$h(a) = h(b) = g(b)f(a) - f(b)g(a).$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a,b).

Since $h'(c) = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c)$.

Theorem: If $f'(x) = 0$ for all x in (a,b), then f is constant on (a,b).

Proof: See Lecture.

Theorem: If f' exists and does not change sign on (a,b), then f is monotonic on (a,b): increasing, nondecreasing, decreasing, or nonincreasing as

$$f'(x) > 0, \quad f'(x) \geq 0, \quad f'(x) < 0, \quad \text{or} \quad f'(x) \leq 0,$$

respectively, for all x in (a,b).

Proof: See Lecture.

Theorem: If f and g are differentiable on an interval, and if $f'(x) = g'(x)$ for all x in that interval, then $f - g$ is a constant function on the interval.

Proof: See Lecture.

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4.8 Lipschitz Continuity

A function that satisfies inequality

$$|f(x) - f(x')| \leq M|x - x'|, \quad x, x' \in (a, b),$$

for all x and x' in an interval, where $M > 0$ is some real number is said to satisfy a *Lipschitz condition* on the interval.

Remark: A differentiable real valued function is Lipschitz continuous, indeed we have

differentiability at $x \Rightarrow$ Lipschits continuity at $x \Rightarrow$ Continuity at x

Lipschitz continuity: Continuity at $x \not\Rightarrow$ Lipschits continuity at $x \not\Rightarrow$ differentiability at x

Example: The function $f(x) = \sqrt{|x|}$ is continuous at $x = 0$, but not Lipschits continuous at $x = 0$.

The function $f(x) = |x|$ is Lipschits continuous at $x = 0$ but not differentiable at $x = 0$.

Theorem: If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$[g(b) - g(a)]f'(c)$$
$$= [f(b) - f(a)]g'(c)$$

for some c in (a, b).

Theorem: Suppose that f and g are differentiable and g' has no zeros on (a, b). Let

$$\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = 0$$

or

$$\lim_{x \to b-} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to b-} g(x) = \pm\infty,$$

and suppose that

$$\lim_{x \to b-} \frac{f'(x)}{g'(x)} = L \quad (\text{finite or } \pm\infty).$$

Then

$$\lim_{x \to b-} \frac{f(x)}{g(x)} = L.$$

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4.9 Indeterminate Forms

The quotient function f/g *is of the form* $0/0$ *as* $x \to b-$ if

$$\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = 0,$$

or *of the form* ∞/∞ *as* $x \to b-$ if

$$\lim_{x \to b-} f(x) = \pm\infty$$

and

$$\lim_{x \to b-} g(x) = \pm\infty.$$

The corresponding definitions for $x \to b+$ and $x \to \pm\infty$ are similar.

If f/g is of one of these forms as $x \to b-$ and as $x \to b+$, then we say that it is of that form as $x \to b$.

Example: Evaluate the following limit

$$\lim_{x \to 0} \sin x/x.$$

Solution: The given limit limit $\lim_{x \to 0} \sin x/x$. is of the form $0/0$ as $x \to 0$. and L'Hospital's rule yields

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Example: Evaluate the limit

$$\lim_{x \to -\infty} \frac{e^{-x}}{x}.$$

Solution: See Lecture.

Example: Evaluate the limit

$$\lim_{x \to +\infty} \frac{x^{-4/3}}{\sin(1/x)}.$$

Solution: See Lecture.

Example: Using L'Hospital's rule may lead to another indeterminate form: thus.

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x}$$

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if the limit on the right exists in the extended reals. Applying L'Hospital's rule again yields

$$\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty.$$

Therefore,

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\infty.$$

More generally,

$$\lim_{x\to\infty}\frac{e^x}{x^\alpha}=\infty$$

for any real number α

Example: Evaluate $\lim_{x\to 0}\frac{4-4\cos x-2\sin^2 x}{x^4}$.

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{4-4\cos x-2\sin^2 x}{x^4} &= \lim_{x\to 0}\frac{4\sin x-4\sin x\cos x}{4x^3}\\ &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1-\cos x}{x^2}\right)\\ &= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{2x}\right)\\ &= \frac{1}{2}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\\ &= \frac{1}{2}(1)^2=\frac{1}{2}.\end{aligned}$$

Example: If

$$f(x)=x-x^2\sin\frac{1}{x}\quad\text{and}\quad g(x)=\sin x,$$

then

$$f'(x)=1-2x\sin\frac{1}{x}+\cos\frac{1}{x}\quad\text{and}\quad g'(x)=\cos x.$$

Therefore, $\lim_{x\to 0} f'(x)/g'(x)$ does not exist. However,

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1-x\sin(1/x)}{(\sin x)/x}=\frac{1}{1}=1.$$

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4.10 Indeterminate forms $(0)(\infty)$

A product of functions *fg is said to have the indeterminate form $0 \cdot \infty$ as $x \to b$* if one of the factors approaches 0 and the other approaches $\pm\infty$ as $x \to b$.

In this case, it may be useful to apply L'Hospital's rule after writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)g(x) = \frac{g(x)}{1/f(x)},$$

since one of these ratios is of the form $0/0$ and the other is of the form ∞/∞ as $x \to b-$. Similar statements apply to limits as $x \to b+$, $x \to b$, and $x \to \pm\infty$.

Example: Evaluate the limit

$$\lim_{x \to 0+} x \log x.$$

Solution: The product $x \log x$ is of the form $0 \cdot \infty$ as $x \to 0+$. Converting it to an ∞/∞ form yields

$$\begin{aligned} \lim_{x \to 0+} x \log x &= \lim_{x \to 0+} \frac{\log x}{1/x} \\ &= \lim_{x \to 0+} \frac{1/x}{-1/x^2} \\ &= -\lim_{x \to 0+} x = 0. \end{aligned}$$

Example: Evaluate the limit

$$\lim_{x \to \infty} x \log 1 + 1/x.$$

Solution: Converting it to a $0/0$ form yields

$$\begin{aligned} \lim_{x \to \infty} x \log(1 + 1/x) &= \lim_{x \to \infty} \frac{\log(1 + 1/x)}{1/x} \\ &= \lim_{x \to \infty} \frac{[1/(1 + 1/x)](-1/x^2)}{-1/x^2} \\ &= \lim_{x \to \infty} \frac{1}{1 + 1/x} = 1. \end{aligned}$$

In this case, converting to an ∞/∞ form complicates the problem:

$$\begin{aligned} \lim_{x \to \infty} x \log(1 + 1/x) &= \lim_{x \to \infty} \frac{x}{1/\log(1 + 1/x)} \\ &= \lim_{x \to \infty} \frac{1}{\left(\frac{-1}{[\log(1+1/x)]^2}\right)\left(\frac{-1/x^2}{1+1/x}\right)} \\ &= \lim_{x \to \infty} x(x + 1)[\log(1 + 1/x)]^2 = ? \end{aligned}$$

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4.10.1 Indeterminate form $\infty - \infty$

A difference $f - g$ is of the form $\infty - \infty$ as $x \to b-$ if

$$\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = \pm\infty.$$

In this case, it may be possible to manipulate $f - g$ into an expression that is no longer indeterminate, or is of the form $0/0$ or ∞/∞ as $x \to b-$.

Similar remarks apply to limits as $x \to b+$, $x \to b$, or $x \to \pm\infty$.

Example: Evaluate the limit

$$\lim_{x \to 0} \frac{\sin x}{x^2} - \frac{1}{x}.$$

Solution: See Lecture.

Example: Evaluate the limit

$$\lim_{x \to \infty} x^2 - x.$$

Solution: See Lecture.

4.10.2 Indeterminate forms $0^0, 1^\infty, \infty^0$

The function f^g is defined by

$$f(x)^{g(x)} = e^{g(x) \log f(x)} = \exp(g(x) \log f(x))$$

for all x such that $f(x) > 0$.

Therefore, if f and g are defined and $f(x) > 0$ on an interval (a, b), implies that

$$\lim_{x \to b-} [f(x)]^{g(x)} = \exp\left(\lim_{x \to b-} g(x) \log f(x)\right) \tag{4.6}$$

if $\lim_{x \to b-} g(x) \log f(x)$ exists in the extended real number system.

If this limit is $\pm\infty$ then (4.6) is valid if we define $e^{-\infty} = 0$ and $e^{\infty} = \infty$.

Indeterminate forms $0^0, 1^\infty, \infty^0$: The product $g \log f$ can be of the form $0 \cdot \infty$ in three ways as $x \to b-$:

1. If $\lim_{x \to b-} g(x) = 0$ and $\lim_{x \to b-} f(x) = 0$.
2. If $\lim_{x \to b-} g(x) = \pm\infty$ and $\lim_{x \to b-} f(x) = 1$.
3. If $\lim_{x \to b} g(x) = 0$ and $\lim_{x \to b} f(x) = \infty$.

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4.11 Taylor's Theorem

Recall the following lemma: If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0),$$

where E is defined on a neighborhood of x_0 and

$$\lim_{x \to x_0} E(x) = E(x_0) = 0.$$

To generalize this result, we first restate it: the polynomial

$$T_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

which is of degree ≤ 1 and satisfies

$$T_1(x_0) = f(x_0), \quad T_1'(x_0) = f'(x_0).$$

$T_1(x)$ approximates f so well near x_0 that

$$\lim_{x \to x_0} \frac{f(x) - T_1(x)}{x - x_0} = 0.$$

Taylor polynomial: Suppose that f has n derivatives at x_0 and T_n is the polynomial of degree $\leq n$ such that

$$T_n^{(r)}(x_0) = f^{(r)}(x_0), \quad 0 \leq r \leq n.$$

How well does T_n approximate f near x_0?

Since T_n is a polynomial of degree $\leq n$, it can be written as

$$T_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n,$$

where $a_0, \ldots, a_n$ are constants.

Differentiating yields

$$T_n^{(r)}(x_0) = r!a_r, \quad 0 \leq r \leq n.$$

We obtained a_r uniquely as

$$a_r = \frac{f^{(r)}(x_0)}{r!}, \quad 0 \leq r \leq n.$$

Therefore,

$$\begin{aligned} T_n(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}(x - x_0)^r. \end{aligned}$$

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CHAPTER 5

Riemann Integration

5.1 Riemann Sums

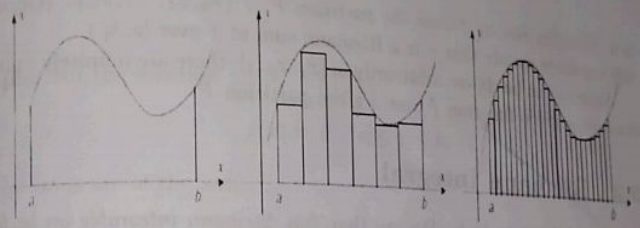


Figure 5.1: Area under the curve

Partition of an interval: Let f be a real valued function defined on a finite interval $[a, b]$.

A *partition of* $[a, b]$ is a set of subintervals

$$[x_0, x_1],\ [x_1, x_2], \ldots, [x_{n-1}, x_n],$$

where

$$a = x_0 < x_1 \cdots < x_n = b.$$

Thus, any set of $n + 1$ points satisfying $a = x_0 < x_1 \cdots < x_n = b$, defines a partition P of $[a, b]$, which we denote by

$$P = \{x_0, x_1, \ldots, x_n\}.$$

The points $x_0, x_1, \ldots, x_n$ are the *partition points* of P.

The largest of the lengths of the subintervals $[x_0, x_1],\ [x_1, x_2], \ldots, [x_{n-1}, x_n]$, is the *norm* of P, written as $\|P\|$: thus,

$$\|P\| = \max_{1 \le i \le n} (x_i - x_{i-1}).$$

Refinement of a partition: If P and P' are partitions of $[a, b]$, then P' is a *refinement of* P if every partition point of P is also a partition point of P'; that is, if P' is obtained by inserting additional points between those of P.

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Riemann sums: If f is defined on $[a, b]$, then a sum

$$\sigma = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}),$$

where

$$x_{j-1} \le c_j \le x_j, \quad 1 \le j \le n,$$

is a *Riemann sum of f over the partition* $P = \{x_0, x_1, \ldots, x_n\}$. (Occasionally we will say more simply that σ is a Riemann sum of f over $[a, b]$.)

Since c_j can be chosen arbitrarily in $[x_j, x_{j-1}]$, there are infinitely many Riemann sums for a given function f over a given partition P.

5.2 Riemann Integral

Let f be defined on $[a, b]$. We say that f is *Riemann integrable on* $[a, b]$ if there is a number L with the following property:

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|\sigma - L| < \varepsilon.$$

If σ is any Riemann sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta$. In this case, we say that L is *the Riemann integral of f over* $[a, b]$, and write

$$\int_a^b f(x)\,dx = L.$$

Theorem: Prove that the Riemann integral $\int_a^b f(x)dx$, if it exists, is unique.

Proof: See Lecture.

- Prove: If $\int_a^b f(x)\,dx$ exists, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|\sigma_1 - \sigma_2| < \varepsilon$ if σ_1 and σ_2 are Riemann sums of f over partitions P_1 and P_2 of $[a, b]$ with norms less than δ.

Example: Determine f is Riemann integrable over the given interval or not? Find $\int_a^b f(x)dx$, where

$$f(x) = 1, \quad a \le x \le b.$$

Solution: For the given function, we have

$$\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} (x_j - x_{j-1}).$$

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contradicting the assumption that f is unbounded on $[a,b]$.

Consider the Riemann sum

$$\sigma' = \sum_{j=1}^{n} f(c_j')(x_j - x_{j-1})$$

over the same partition P, where

$$c_j' = \begin{cases} c_j, & j \neq i, \\ c, & j = i. \end{cases}$$

Since

$$|\sigma - \sigma'| = |f(c) - f(c_i)|(x_i - x_{i-1}),$$

hence due to

$$|f(c) - f(c_i)| \geq \frac{M}{x_i - x_{i-1}}$$

implies (5.5).

5.3 Upper and Lower Integrals

Upper and lower integrals: If f is bounded on $[a,b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, let

$$M_j = \sup_{x_{j-1} < x \leq x_j} f(x) \quad \text{and} \quad m_j = \inf_{x_{j-1} \leq x < x_j} f(x).$$

The *upper sum of f over P* is

$$S(P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}),$$

and the *upper integral of f over* $[a,b]$, denoted by

$$\overline{\int_a^b} f(x)\,dx,$$

is the infimum of all upper sums.

Lower integrals: The *lower sum of f over P* is

$$s(P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}),$$

and the *lower integral of f over* $[a,b]$, denoted by

$$\underline{\int_a^b} f(x)\,dx,$$

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The Riemann sum produced in this way is

$$\sigma \geq \sum_{j=1}^{n} f(\bar{c}_j)(x_j - x_{j-1}) > \sum_{j=1}^{n} \left[M_j - \frac{\varepsilon}{n(x_j - x_{j-1})} \right] (x_j - x_{j-1})$$
$$= S(P) - \varepsilon.$$

Now from definition $S(P)$ is the supremum of the set of Riemann sums of f over P.

Example: Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since every interval contains both rational and irrational numbers

$$m_j = 0 \quad \text{and} \quad M_j = 1, \quad 1 \leq j \leq n.$$

Hence,

$$S(P) = \sum_{j=1}^{n} 1 \cdot (x_j - x_{j-1}) = b - a$$
$$s(P) = \sum_{j=1}^{n} 0 \cdot (x_j - x_{j-1}) = 0.$$

Since all upper sums equal $b - a$ and all lower sums equal 0, from definition we have

$$\overline{\int_a^b} f(x)\,dx = b - a \quad \text{and} \quad \underline{\int_a^b} f(x)\,dx = 0.$$

Example: Let f be defined on $[1, 2]$ by $f(x) = 0$ if x is irrational and $f(p/q) = 1/q$ if p and q are positive integers with no common factors.

If $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[1, 2]$, then $m_j = 0$, $1 \leq j \leq n$, so $s(P) = 0$; hence,

$$\underline{\int_1^2} f(x)\,dx = 0. \tag{5.6}$$

We now show that

$$\overline{\int_1^2} f(x)\,dx = 0$$

also.

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Theorem: If f is bounded on $[a,b]$, then f is integrable on $[a,b]$ if and only if for each $\varepsilon > 0$ there is a partition P of $[a,b]$ for which

$$S(P) - s(P) < \varepsilon.$$

Theorem: If f is monotonic on $[a,b]$, then f is integrable on $[a,b]$.

Proof: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Since f is nondecreasing,

$$\begin{aligned} f(x_j) &= M_j = \sup_{x_{j-1} \le x \le x_j} f(x) \\ f(x_{j-1}) &= m_j = \inf_{x_{j-1} \le x \le x_j} f(x). \end{aligned}$$

Hence,

$$S(P) - s(P) = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))(x_j - x_{j-1}).$$

Since $0 < x_j - x_{j-1} \le \|P\|$ and $f(x_j) - f(x_{j-1}) \ge 0$,

$$\begin{aligned} S(P) - s(P) &\le \|P\| \sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) \\ &= \|P\|(f(b) - f(a)). \end{aligned}$$

Therefore,

$$S(P) - s(P) < \varepsilon \quad \text{if} \quad \|P\|(f(b) - f(a)) < \varepsilon,$$

so f is integrable on $[a,b]$.

Theorem: If f and g are integrable on $[a,b]$, then so is $f+g$, and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof: Any Riemann sum of $f+g$ over a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ can be written as

$$\begin{aligned} \sigma_{f+g} &= \textstyle\sum_{j=1}^{n} [f(c_j) + g(c_j)](x_j - x_{j-1}) \\ &= \textstyle\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) + \sum_{j=1}^{n} g(c_j)(x_j - x_{j-1}) \\ &= \sigma_f + \sigma_g, \end{aligned}$$

where σ_f and σ_g are Riemann sums for f and g.

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Proof: See Lecture.

Theorem: If f and g are integrable on $[a,b]$ and $f(x) \le g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof: Since $g(x) - f(x) \ge 0$, every lower sum of $g - f$ over any partition of $[a,b]$ is nonnegative.

Therefore,

$$\underline{\int_a^b} (g(x) - f(x))\,dx \ge 0.$$

Hence,

$$\begin{aligned} \int_a^b g(x)\,dx - \int_a^b f(x)\,dx &= \int_a^b (g(x) - f(x))\,dx \\ &= \underline{\int_a^b} (g(x) - f(x))\,dx \ge 0. \end{aligned} \tag{5.24}$$

which yields . (The first equality in (5.24) follows from Theorems ?? and ??; the second, from Theorem ??.)

Theorem: If f is integrable on $[a,b]$, then so is $|f|$, and

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

Proof: Let P be a partition of $[a,b]$ and define

$$\begin{aligned} M_j &= \sup\{f(x) : x_{j-1} \le x \le x_j\}, \\ m_j &= \inf\{f(x) : x_{j-1} \le x \le x_j\}, \\ \overline{M}_j &= \sup\{|f(x)| : x_{j-1} \le x \le x_j\}, \\ \overline{m}_j &= \inf\{|f(x)| : x_{j-1} \le x \le x_j\}. \end{aligned}$$

Then

$$\begin{aligned} \overline{M}_j - \overline{m}_j &= \sup\{|f(x)| - |f(x')| : x_{j-1} \le x, x' \le x_j\} \\ &\le \sup\{|f(x) - f(x')| : x_{j-1} \le x, x' \le x_j\} \\ &= M_j - m_j. \end{aligned} \tag{5.25}$$

Therefore,

$$\overline{S}(P) - \overline{s}(P) \le S(P) - s(P),$$

where the upper and lower sums on the left are associated with $|f|$ and those on the right are associated with f.

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Proof: Thus $P = \{x_r, x_{r+1}, \ldots, x_s\}$ is a partition of $[a_1, b_1]$ over which the upper and lower sums of f satisfy

$$S(P) - s(P) < \epsilon.$$

Therefore, f is integrable on $[a_1, b_1]$.

If f is integrable on $[a, b]$ and $a \le a_1 < b_1 \le b$, then f is integrable on $[a_1, b_1]$.

Theorem: If f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{5.36}$$

So far we have defined $\int_\alpha^\beta f(x)\,dx$ only for the case where $\alpha < \beta$.

If $\alpha < \beta$, we define

$$\int_\beta^\alpha f(x)\,dx = -\int_\alpha^\beta f(x)\,dx.$$

$$\int_\alpha^\alpha f(x)\,dx = 0.$$

Theorem: If f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{5.37}$$

Proof: See Lecture.

5.4 Fundamental Theorem of Calculus

Theorem: If f is integrable on $[a, b]$ and $a \le c \le b$, then the function F defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition on $[a, b]$, and is therefore continuous on $[a, b]$.

Proof: If x and x' are in $[a, b]$, then

$$F(x) - F(x') = \int_c^x f(t)\,dt - \int_c^{x'} f(t)\,dt = \int_{x'}^x f(t)\,dt.$$

Since $f(t) < K$ $(a < t < b)$ for some constant K,

$$\left| \int_{x'}^x f(t)\,dt \right| \le K|x - x'|, \quad a \le x, x' \le b.$$

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Proof: Thus $\overline{P} = \{x_r, x_{r+1}, \ldots, x_s\}$ is a partition of $[a_1, b_1]$ over which the upper and lower sums of f satisfy

$$S(\overline{P}) - s(\overline{P}) < \varepsilon.$$

Therefore, f is integrable on $[a_1, b_1]$.

If f is integrable on $[a, b]$ and $a \le a_1 < b_1 \le b$, then f is integrable on $[a_1, b_1]$.

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Theorem: If f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

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Proof: See Lecture.

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Proof: If x and x' are in $[a, b]$, then

$$F(x) - F(x') = \int_c^x f(t)\,dt - \int_c^{x'} f(t)\,dt = \int_{x'}^x f(t)\,dt.$$

Since $|f(t)| < K$ $(a \le t \le b)$ for some constant K,

$$\left|\int_{x'}^x f(t)\,dt\right| \le K|x - x'|, \quad a \le x, x' \le b.$$

Proof: Thus $\overline{P} = \{x_r, x_{r+1}, \ldots, x_s\}$ is a partition of $[a_1, b_1]$ over which the upper and lower sums of f satisfy

$$S(\overline{P}) - s(\overline{P}) < \varepsilon.$$

Therefore, f is integrable on $[a_1, b_1]$.

If f is integrable on $[a, b]$ and $a \le a_1 < b_1 \le b$, then f is integrable on $[a_1, b_1]$.

Theorem: If f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{5.36}$$

So far we have defined $\int_\alpha^\beta f(x)\,dx$ only for the case where $\alpha < \beta$.

If $\alpha < \beta$, we define

$$\int_\beta^\alpha f(x)\,dx = -\int_\alpha^\beta f(x)\,dx.$$

$$\int_\alpha^\alpha f(x)\,dx = 0.$$

Theorem: If f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx. \tag{5.37}$$

Proof: See Lecture.

5.4 Fundamental Theorem of Calculus

Theorem: If f is integrable on $[a, b]$ and $a \le c \le b$, then the function F defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition on $[a, b]$, and is therefore continuous on $[a, b]$.

Proof: If x and x' are in $[a, b]$, then

$$F(x) - F(x') = \int_c^x f(t)\,dt - \int_c^{x'} f(t)\,dt = \int_{x'}^x f(t)\,dt.$$

Since $|f(t)| \le K$ $(a < t < b)$ for some constant K,

$$\left| \int_{x'}^x f(t)\,dt \right| \le K|x - x'|, \quad a \le x, x' \le b.$$

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Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then the function F defined by

$$F(x) = \int_c^x f(t)\,dt$$

satisfies a Lipschitz condition on $[a,b]$, and is therefore continuous on $[a,b]$.

Proof: Consequently,

$$|F(x) - F(x')| \le K|x - x'|, \quad a \le x, x' \le b.$$

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then $F(x) = \int_c^x f(t)\,dt$ is differentiable at any point x_0 in (a,b) where f is continuous, with $F'(x_0) = f(x_0)$.

If f is continuous from the right at a, then $F'_+(a) = f(a)$. If f is continuous from the left at b, then $F'_-(b) = f(b)$.

Proof: Since

$$\frac{1}{x - x_0}\int_{x_0}^x f(x_0)\,dt = f(x_0).$$

We can write

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x |f(t) - f(x_0)|\,dt.$$

From this

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \le \frac{1}{|x - x_0|}\left|\int_{x_0}^x |f(t) - f(x_0)|\,dt\right|.$$

Since f is continuous at x_0, there is for each $\varepsilon > 0$ a $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{if} \quad |x - x_0| < \delta$$

and t is between x and x_0.

Therefore, we have

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \le \varepsilon\frac{|x - x_0|}{|x - x_0|} = \varepsilon \quad \text{if} \quad 0 < |x - x_0| < \delta.$$

Hence, $F'(x_0) = f(x_0)$.

Theorem: If f is integrable on $[a,b]$ and $a \le c \le b$, then $F(x) = \int_c^x f(t)\,dt$ is differentiable at any point x_0 in (a,b) where f is continuous, with $F'(x_0) = f(x_0)$.

If f is continuous from the right at a, then $F'_+(a) = f(a)$. If f is continuous from the left at b, then $F'_-(b) = f(b)$.

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Example: If

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ x+1, & 1 < x \le 2. \end{cases}$$

Then the function

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \frac{x^2}{2}, & 0 < x \le 1, \\ \frac{x^2}{2} + x - 1, & 1 < x \le 2, \end{cases}$$

is continuous on $[0,2]$.

We can conclude

$$F'(x) = \begin{cases} x = f(x), & 0 < x < 1, \\ x+1 = f(x), & 1 < x < 2. \end{cases}$$

$$\begin{aligned} F'_+(0) &= \lim_{x\to 0+} \frac{F(x) - F(0)}{x} = \lim_{x\to 0+} \frac{(x^2/2) - 0}{x} = 0 = f(0), \\ F'_-(2) &= \lim_{x\to 2-} \frac{F(x) - F(2)}{x-2} = \lim_{x\to 2-} \frac{(x^2/2) + x - 1 - 3}{x-2} \\ &= \lim_{x\to 2-} \frac{x+4}{2} = 3 = f(2). \end{aligned}$$

F does not have a derivative at $x = 1$, where f is discontinuous, since

$$F'_-(1) = 1 \quad \text{and} \quad F'_+(1) = 2.$$

Theorem: Suppose that F is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), and f is integrable on $[a,b]$.

Suppose also that

$$F'(x) = f(x), \quad a < x < b.$$

Then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5.3}$$

Proof: If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, then

$$F(b) - F(a) = \sum_{j=1}^{n} (F(x_j) - F(x_{j-1})). \tag{5.3}$$

From mean value theorem, there is in each open interval (x_{j-1}, x_j) a point c_j such that

$$F(x_j) - F(x_{j-1}) = f(c_j)(x_j - x_{j-1}).$$

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