

Let H be a subgroup of G and let N be a normal subgroup of G Then $(HN) \cap N =$ _____

page 1009

Answer (Please select your correct option)

☐ $H \cap (H \cup N)$

☒ $H \cap (H \cap N)$

☐ $H \cup (H \cap N)$

☐ $H \cap (H \cap N)$



A non-abelian group G may have _____ of some order d dividing $|G|$.

page 1035

Answer (Please select your correct option)

☐ abelian subgroup

☐ subset

☒ no subgroup

☐ none of these



Question No : 8 of 11

A _____ contained in no larger p -subgroup.

page 1064

Answer (Please select your correct option)

☒ p - subgroup



☐ cyclic subgroups

☐ proper subgroups

☐ trivial subgroups

Each orbit in G under conjugation by G is a _____ in G .

page 1082

Answer (Please select your correct option)

☒ conjugate class



☐ subgroup

☐ trivial group

☐ none of these

Question No : 6 of 41

For a prime number p , every group G of order p^2 is _____

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Answer (Please select your correct option)

☒ abelian



☐ non-abelian

☐ infinite

☐ none of these

Question No : 5 of 11

If H and N are subgroups of a group G , then the their join $H \vee N$ is the smallest subgroup of G containing both H and N .

page 1004

Answer (Please select your correct option)

True



False

Question No : 4 of 11

Let G be a group acting on a set X , and let $x \in X$. Then the set $Gx = \{ax \mid a \in G\}$

is called the orbit of x in G .

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Answer (Please select your correct option)

True

☒

False

☐



Question No : 3 of 41

Sometimes a set X is used to study G via a group action of G on X

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Answer (Please select your correct option)

Yes

☐



No

Generally, for any sets A , B , and C , we can view a map $\cdot : A \times B \rightarrow C$ as defining a "multiplication," where any element a of A times any element b of B has as value some element c of C .

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Answer (Please select your correct option)

True



MTH633 Group Theory

Question No : 1 of 41

A_3 is a normal subgroup of S_3

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Answer (Please select your correct option)

Yes



No

Let the complex numbers of magnitude 1 form a subgroup U of C^* , then the cosets of U are _____

page 718

Answer (Please select your correct option)

☐ Parallel lines

☒ circles with center at the origin.

☐ Non-parallel lines

☐ None of these



$\text{Ker}(A)$ is called the null space of A if it consists of all $v \in \mathbb{R}^n$ such that _____

page 700

Answer (Please select your correct option)

☐ $Av = a$

☐ $Av = b$

☐ $Av = c$

☒ $Av = 0$



Let ϕ be a homomorphism function from group $(G, *)$ to group $(G', \cdot)$. Then, $(\text{Ker } \phi, *)$ is _____ subgroup of $(G, *)$

page 675

Answer (Please select your correct option)

☐ commutative

☐ factor

☐ simple

☒ normal



If N is _____ subgroup of a group G , the left cosets of N in G are the same as the right cosets of N in G .

page 652

Answer (Please select your correct option)

☒ normal



☐ proper

☐ improper

☐ trivial

Let H be a subgroup of G and a is in G , then _____

page 564

Answer (Please select your correct option)

☒ $|aH| = |bH|$



☐ $|aH| \neq |bH|$

☐ $|aH| < |bH|$

☐ $|aH| > |bH|$

Question No : 15 of 41

$\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic group with generator -----

page 496

Answer (Please select your correct option)

☒ (1, 1)



☐ (0, 0)

☐ (1, 0)

☐ (0, 1)

$\prod_{i=1}^n G_i$ is the _____ notation for all the group operations for $G_1, \dots, G_n$.

page 484

Answer (Please select your correct option)

☐ additive

☒ multiplicative

☐ subtractive

☐ division



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Question No : 13 of 41

S_3 is NOT the group D_3 of symmetries of an equilateral triangle.

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Answer (Please select your correct option)

Yes

No



Question No : 12 of 41

Every subgroup of a cyclic group is ----- group.

page 385

Answer (Please select your correct option)

☒ Cyclic



☐ Non- cyclic

☐ Non- abelian

☐ Finite

$\langle \mathbb{Z}, + \rangle$ is of $\langle \mathbb{R}, \cdot \rangle$

page 305

Answer (Please select your correct option)

☐ subgroup

☒ not subgroup

☐ trivial subgroup

☐ non-trivial subgroup



QUESTION 1

Let $\phi: G \rightarrow G'$ be a group morphism. Then, ϕ is injective if and only if $\text{Ker } \phi = \{e\}$.

Proof :

If $\text{Ker}(\phi) = \{e\}$, then for every $a \in G$, the elements mapped into $\phi(a)$ are precisely the elements of the left coset $a \{e\} = \{a\}$, which shows that ϕ is one to one.

Conversely, suppose ϕ is one to one. Now, we know that $\phi(e) = e'$, the identity element of G' . Since ϕ is one to one, we see that e is the only element mapped into e' by ϕ , so $\text{Ker}(\phi) = \{e\}$.

QUESTION 2

A homomorphism $h: G \rightarrow G'$ is injective if and only if $\text{Ker } h = \{e\}$

Proof:

Suppose h is injective, and let $x \in \text{Ker } h$.

Then $h(x) = e' = h(e)$. Hence $x = e$.

Conversely, suppose $\text{Ker } h = \{e\}$.

Then $h(x) = h(y)$

$$\Rightarrow h(xy^{-1}) = h(x)h(y^{-1})$$

$$= h(x)h(y)^{-1} = e'$$

$$\Rightarrow xy^{-1} \in \text{Ker } h$$

$$\Rightarrow xy^{-1} = e$$

$$\Rightarrow x = y.$$

Hence, h is injective.

QUESTION 3

Classify the group $(\mathbb{Z}_4 \times \mathbb{Z}_2) / (\{0\} \times \mathbb{Z}_2)$ according to the fundamental theorem of finitely generated abelian groups.

Solution :

The projection map

$\pi_1: \mathbb{Z}_4 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ given by $\pi_1(x, y) = x$ is a homomorphism of $\mathbb{Z}_4 \times \mathbb{Z}_2$ onto $\mathbb{Z}_4$ with kernel $\{0\} \times \mathbb{Z}_2$. By fundamental theorem of homomorphism, we know that the given factor group is isomorphic to $\mathbb{Z}_4$.

The projection map

$\pi_1: \mathbb{Z}_4 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ given by

$$\pi_1(x, y) = x.$$

$$K = \text{Ker } \pi_1 = \{0\} \times \mathbb{Z}_2$$

$$= \{(0, 0), (0, 1)\}.$$

$$(1, 0) + K = \{(1, 0), (1, 1)\}$$

$$(2, 0) + K = \{(2, 0), (2, 1)\}$$

QUESTION 4

The set of all inner automorphisms of G is a subgroup of $\text{Aut}(G)$.

Proof:

(1) Let $i_a, i_b \in \text{Inn}(G)$.

$$\begin{aligned}\text{Then } i_a(i_b(x)) &= a(i_b(x))a^{-1} = abxb^{-1}a^{-1} \\ &= abx(ab)^{-1} = i_{ab} \in \text{Inn}(G).\end{aligned}$$

Hence the conjugation by b composed by conjugation by a is conjugation by ab .

(2) The inverse of i_a is conjugation by $a' = a^{-1}$.

$$i_a((i_{a'}) (x)) = i_a(a'x(a')^{-1}) = aa'xa'^{-1}a^{-1} = aa'x(aa')^{-1} = x.$$

Thus $\text{Inn}(G)$ is a subgroup.

QUESTION 5

A factor group of a cyclic group is cyclic.

Proof:

Let G be cyclic with generator a , and let N be a normal subgroup of G . We claim the coset aN generates G / N . We must compute all powers of aN . But this amounts to computing, in G , all powers of the representative a and all these powers give all elements in G . Hence the powers of aN certainly give all cosets of N and G / N is cyclic.

QUESTION 6

Here the first factor $\mathbb{Z}_4$ of $\mathbb{Z}_4 \times \mathbb{Z}_6$ is left alone. The $\mathbb{Z}_6$ factor, on the other hand, is essentially collapsed by a subgroup of order 3, giving a factor group in the second factor of order 2 that must be isomorphic to $\mathbb{Z}_2$. Thus $(\mathbb{Z}_4 \times \mathbb{Z}_6)/((0, 2))$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. The trivial subgroup $N = \{0\}$ of $\mathbb{Z}$ is, of course, a normal subgroup. Compute $\mathbb{Z}/\{0\}$.

Solution:

Since $N = \{0\}$ has only one element, every coset of N has only one element. That is, the cosets are of the form $\{m\}$ for $m \in \mathbb{Z}$. There is no collapsing at all, and consequently, $\mathbb{Z}/\{0\} \cong \mathbb{Z}$. Each $m \in \mathbb{Z}$ is simply renamed $\{m\}$ in $\mathbb{Z}/\{0\}$.

QUESTION 7

Let $G = H \times K$ be the direct product of groups H and K . Then $\bar{H} = \{(h, e) \mid h \in H\}$ is a normal subgroup of G . Also $G/\bar{H}$ is isomorphic to K in a natural way. Similarly, $G/\bar{K} \simeq H$ in a natural way.

Proof

Consider the map $\pi_2: H \times K \rightarrow K$ given by

$\pi_2(h, k) = k$. The map π_2 is homomorphism since

$$\pi_2(h_1 h_2, k_1 k_2) = k_1 k_2 = \pi_2(h_1, k_1) \pi_2(h_2, k_2).$$

Because $\text{Ker}(\pi_2) = \bar{H}$, we see that $\bar{H}$ is a normal subgroup of $H \times K$. Because π_2 is onto K , Fundamental Theorem of Homomorphism tells us that $(H \times K)/\bar{H} \simeq K$.

QUESTION 8

The cyclic group $G = \mathbb{Z}/5\mathbb{Z}$ of congruence classes modulo 5 is simple.

Proof

If H is a subgroup of this group, its order must be a divisor of the order of G which is 5.

Since 5 is prime, its only divisors are 1 and 5, so either H is G , or H is the trivial group.

QUESTION 9

M is a maximal normal subgroup of G if and only if G/M is simple.

Proof :

Let M be a maximal normal subgroup of G . Consider the canonical homomorphism $\gamma: G \rightarrow G/M$. Now γ^{-1} of any nontrivial proper normal subgroup of G/M is a proper normal subgroup of G properly containing M . But M is maximal, so this can not happen. Thus G/M is simple.

Conversely, if N is a normal subgroup of G properly containing M , then $\gamma[N]$ is normal in G/M . If also $N \neq G$, then $\gamma[N] \neq G/M$ and $\gamma[N] \neq \{M\}$.

Thus, if G/M is simple so that no such $\gamma[N]$ can exist, no such N can exist, and M is maximal.

QUESTION 10

Show that $Z(G)$ is a normal and an abelian subgroup of G .

Solution :

For each $g \in G$ and $z \in Z(G)$ we have $gzg^{-1} = zgg^{-1} = ze = z$, we see at once that $Z(G)$ is a normal subgroup of G . It implies that $gz = zg$ for $g \in G$ and $z \in Z(G)$. If G is abelian, then $Z(G) = G$; in this case, the center is not useful.

QUESTION 11

Let G be a group. The set of all commutators $aba^{-1}b^{-1}$ for $a, b \in G$ generates a subgroup C of G .

Proof:

Let $a, b \in G$. Then,

$$(aba^{-1}b^{-1})(aba^{-1}b^{-1})$$

$$= aba^{-1}b^{-1}bab^{-1}a^{-1}$$

$$= e \in C$$

since $e = eee^{-1}e^{-1}$ is a commutator.

QUESTION 12

The Klein 4-group $V = \{e, a, b, c\}$ is generated.

Proof:

The Klein 4-group $V = \{e, a, b, c\}$ is generated by $\{a, b\}$ since $ab=c$.

It is also generated by $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$.

If a group G is generated by a subset S , then every subset of G containing S generates G .

QUESTION 13

The group $\mathbb{Z}_6$ is generated.

Proof:

The group $\mathbb{Z}_6$ is generated by $\{1\}$ and $\{5\}$. It is also generated by $\{2,3\}$ since $2+3=5$, so that any subgroup containing 2 and 3 must contain 5 and must therefore be $\mathbb{Z}_6$. It is also generated by $\{3,4\}$, $\{2,3,4\}$, $\{1,3\}$, and $\{3,5\}$. But it is not generated by $\{2, 4\}$ since $\langle 2 \rangle = \{0, 2, 4\}$ contains 2 and 4.

QUESTION 14

If N is a normal subgroup of G , then G/N is abelian if and only if $C \leq N$.

Proof :

If N is a normal subgroup of G and G/N is abelian, then

$$(a^{-1}N)(b^{-1}N) = (b^{-1}N)(a^{-1}N); \text{ that is, } aba^{-1}b^{-1}N = N,$$

$$\text{so } aba^{-1}b^{-1} \in N, \text{ and } C \leq N.$$

Finally, if $C \leq N$, then

$$(aN)(bN) = abN$$

$$= ab(b^{-1}a^{-1}ba)N$$

$$= (abb^{-1}a^{-1})baN$$

$$= baN$$

$$= (bN)(aN).$$

QUESTION 15

The set $\text{Aut}(G)$ of all automorphisms of a group G is a group under composition of mappings, and $\text{Inn}(G) \triangleleft \text{Aut}(G)$. Moreover, $G/Z(G) \cong \text{Inn}(G)$.

Proof:

Clearly, $\text{Aut}(G)$ is nonempty. Let $\sigma, \tau \in \text{Aut}(G)$. Then for all $x, y \in G$, $\sigma\tau(xy) = \sigma((\tau(x) \tau(y))) = (\sigma\tau(x))(\sigma\tau(y))$.

Hence, $\sigma\tau \in \text{Aut}(G)$. Again,

$$\begin{aligned}\sigma(\sigma^{-1}(x)\sigma^{-1}(y)) &= \\ \sigma\sigma^{-1}(x)\sigma\sigma^{-1}(y) &= \\ = xy.\end{aligned}$$

Hence $\sigma^{-1}(x)\sigma^{-1}(y) = \sigma^{-1}(xy)$. Therefore, $\sigma^{-1} \in \text{Aut}(G)$. This proves that $\text{Aut}(G)$ is a subgroup of the symmetric group S_G and, hence, is itself a group.

QUESTION 16

Let G be a finite abelian group of order n , and let m be a positive integer relative prime to n . Then the mapping $\sigma: x \rightarrow x^m$ is an automorphism of G .

Solution:

$(m, n) = 1 \Rightarrow$ there exist integers u and v such that $mu + nv = 1 \Rightarrow$ for all $x \in G$, $x^{mu+nv} = x^{mu} x^{nv} = x^{um}$ since $o(G)=n$.

Now for all $x \in G$, $x = (x^u)^m$ implies that σ is onto. Further,

$x^m = e \Rightarrow x^{mu} = e \Rightarrow x = e$, showing that σ is 1-1.

That σ is a homomorphism follows from the fact that G is abelian. Hence, σ is an automorphism of G .

QUESTION 17

Let $G = \langle a \mid a^n = e \rangle$ be a finite cyclic group of order n . Then the mapping $\sigma : a \rightarrow a^m$ is an automorphism of G iff $(m, n) = 1$. Further, if $(m, n) = d$, then $(a^m)^{n/d} = (a^n)^{m/d} = e$. Thus, the order of a^m divides n/d ; that is, $n \mid n/d$. Hence, $d = 1$, and the solution is complete. Let X be a G -set. Then G_x is a subgroup of G for each $x \in X$.

Proof:

Let $x \in X$ and let $g_1, g_2 \in G_x$. Then $g_1 x = x$ and $g_2 x = x$. Consequently, $(g_1 g_2) x = g_1 (g_2 x) = g_1 x = x$, so $g_1 g_2 \in G_x$, and G_x is closed under the induced operation of G . Of course $e x = x$, so $e \in G_x$. If $g \in G_x$, then $g x = x$, so $x = e x = (g^{-1} g) x = g^{-1} (g x) = g^{-1} x$, and consequently $g^{-1} \in G_x$. Thus G_x is a subgroup of G .

QUESTION 18

Let X be a G -set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Then $\sim$ is an equivalence relation on X .

Proof:

For each $x \in X$, we have $ex = x$, so $x \sim x$ and $\sim$ is reflexive.

Suppose $x_1 \sim x_2$, so $gx_1 = x_2$ for some $g \in G$. Then $g^{-1}x_2 = g^{-1}(gx_1) = (g^{-1}g)x_1 = ex_1 = x_1$, so $x_2 \sim x_1$, and $\sim$ is symmetric. Finally, if $x_1 \sim x_2$ and $x_2 \sim x_3$, then $g_1x_1 = x_2$ and $g_2x_2 = x_3$ for some $g_1, g_2 \in G$. Then $(g_2g_1)x_1 = g_2(g_1x_1) = g_2x_2 = x_3$, so $x_1 \sim x_3$ and $\sim$ is transitive.

QUESTION 19

If N is a normal subgroup of G , and if H is any subgroup of G , then

$H \vee N = HN = NH$. Furthermore, if H is also normal in G , then HN is normal in G .

Proof:

We show that HN is a subgroup of G , from which

$H \vee N = HN$ follows at once. Let $h_1, h_2 \in H$ and $n_1, n_2 \in N$.

Since N is a normal subgroup, we have $n_1 h_2 = h_2 n_3$ for

some $n_3 \in N$. Then $(h_1 n_1)(h_2 n_2) = h_1 (n_1 h_2) n_2 = h_1 (h_2 n_3) n_2 =$

$(h_1 h_2)(n_3 n_2) \in HN$, so HN is closed under the induced

operation in G . Clearly $e = ee$ is in HN . For $h \in H$ and $n \in N$,

we have $(hn)^{-1} = n^{-1} h^{-1} = h^{-1} n_4$ for some $n_4 \in N$, since N is a

normal subgroup. Thus $(hn)^{-1} \in HN$, so $HN \leq G$.

A similar argument shows that NH is a subgroup, so $NH = H \vee N = HN$.

Now suppose that H is also normal in G , and let $h \in H$, $n \in N$, and $g \in G$. Then

$ghng^{-1} = (ghg^{-1})(gng^{-1}) \in HN$, so HN is indeed normal in G .

QUESTION 20

Let G be a group such that for some fixed integer $n > 1$, $(ab)^n = a^n b^n$ for all $a, b \in G$. Let $G_n = \{a \in G \mid a^n = e\}$ and $G^n = \{a^n \mid a \in G\}$. Then $G_n \triangleleft G$, $G^n \triangleleft G$, and $G/G_n \cong G^n$.

Solution:

Let $a, b \in G_n$ and $x \in G$. Then $(ab^{-1})^n = a^n (b^{-1})^n = e$, so $ab^{-1} \in G_n$. Also, $(xax^{-1})^n = (xax^{-1}) \dots (xax^{-1}) = xa^n x^{-1} = e$ implies $xax^{-1} \in G_n$. Hence, $G_n \triangleleft G$.

Let $a, b, x \in G$. Then $a^n (b^{-1})^n = (ab^{-1})^n \in G^n$.

Also, $xa^n x^{-1} = (xax^{-1}) \dots (xax^{-1}) = (xax^{-1})^n \in G^n$. Therefore, $G^n \triangleleft G$.

QUESTION 21

Third Isomorphism Theorem

Let H and K be normal subgroups of a group G with $K \leq H$. Then $G/H \cong (G/K)/(H/K)$.

Proof:

Let $\phi: G \rightarrow (G/K)/(H/K)$ be given by $\phi(a) = (aK)(H/K)$ for $a \in G$. Clearly ϕ is onto $(G/K)/(H/K)$, and for $a, b \in G$, $\phi(ab) = [(ab)K](H/K)$

$$= [(aK)(bK)](H/K)$$

$= [(aK)(H/K)][(bK)(H/K)] = \phi(a) \phi(b)$, so ϕ is a homomorphism.

The kernel consists of those $x \in G$ such that $\phi(x) = H/K$.

These x are just the elements of H .

Then first isomorphism theorem shows that $G/H \cong (G/K)/(H/K)$.

QUESTION 22

Let G be a group of order p^n and let X be a finite G -set. Then

$$|X| \equiv |X_G| \pmod{p}.$$

Proof:

Recall $|X| = |X_G| + \sum_{i=s+1}^r |Gx_i|$. In the notation of above Equation,

we know that

$$|Gx_i| \text{ divides } |G|.$$

Consequently p divides $|Gx_i|$ for $s+1 \leq i \leq r$. Above equation then shows that $|X| - |X_G|$ is divisible by p , so $|X| \equiv |X_G| \pmod{p}$.

QUESTION 23

Let H be a p -subgroup of a finite group G . Then

$$(N[H]:H) \equiv (G:H) \pmod{p}.$$

Proof:

Let $\mathcal{L}$ be the set of left cosets of H in G , and let H act on $\mathcal{L}$ by left translation, so that $h(xH) = (hx)H$. Then $\mathcal{L}$ becomes an H -set. Note that $|\mathcal{L}| = (G:H)$. Let us determine $\mathcal{L}_H$, that is, those left cosets that are fixed under action by all elements of H . Now $xH = h(xH)$ if and only if $H = x^{-1}hxH$, or if and only if $x^{-1}hx \in H$.

Thus $xH = h(xH)$ for all $h \in H$ if and only if $x^{-1}hx$
 $= x^{-1}h(x^{-1})^{-1} \in H$ for all $h \in H$, or if and only if $x^{-1} \in N[H]$, or if and only if $x \in N[H]$. Thus the left cosets in $\mathcal{L}_H$ are those contained in $N[H]$. The number of such cosets is $(N[H]:H)$, so $|\mathcal{L}_H| = (N[H]:H)$.

Since H is a p -group, it has order a power of p . Then $|\mathcal{L}| \equiv |\mathcal{L}_H| \pmod{p}$, that is, $(G:H) \equiv (N[H]:H) \pmod{p}$.

QUESTION 24

Second Sylow Theorem

Let P_1 and P_2 be Sylow p -subgroups of a finite group G . Then P_1 and P_2 are conjugate subgroups of G .

Proof:

Here we will let one of the subgroups act on left cosets of the other. Let $\mathcal{L}$ be the collection of left cosets of P_1 , and let P_2 act on $\mathcal{L}$ by $z(xP_1) = (zx)P_1$ for $z \in P_2$. Then $\mathcal{L}$ is a P_2 -set. We have $|\mathcal{L}_{P_2}| \equiv |\mathcal{L}| \pmod{p}$, and $|\mathcal{L}| = (G : P_1)$ is not divisible by p , so $|\mathcal{L}_{P_2}| \neq 0$. Let $xP_1 \in \mathcal{L}_{P_2}$.

Then $zxP_1 = xP_1$ for all $z \in P_2$, so $x^{-1}zxP_1 = P_1$ for all $z \in P_2$.

Thus $x^{-1}zx \in P_1$ for all $z \in P_2$, so $x^{-1}P_2x \leq P_1$. Since $|P_1| = |P_2|$, we must have $P_1 = x^{-1}P_2x$, so P_1 and P_2 are indeed conjugate subgroups.

QUESTION 25

The center of a finite nontrivial p -group G is nontrivial.

Proof:

We have $|G| = c + n_{c+1} + \dots + n_r$, where n_i is the number of elements in the i th orbit of G under conjugation by itself.

For G , each n_i divides $|G|$ for $c+1 \leq i \leq r$, so p divides each n_i , and p divides $|G|$. Therefore p divides c . Now $e \in Z(G)$, so $c \geq 1$. Therefore $c \geq p$, and there exists some $a \in Z(G)$ where $a \neq e$.

QUESTION 26

For a prime number p , every group G of order p^2 is abelian.

Proof:

If G is not cyclic, then every element except e must be of order p .

Let a be such an element. Then the cyclic subgroup $\langle a \rangle$ of order p does not exhaust G . Also let $b \in G$ with $b \notin \langle a \rangle$. Then $\langle a \rangle \cap \langle b \rangle = \{e\}$, since an element c in $\langle a \rangle \cap \langle b \rangle$ with $c \neq e$ would generate both $\langle a \rangle$ and $\langle b \rangle$, giving $\langle a \rangle = \langle b \rangle$, contrary to construction.

From first Sylow theorem, $\langle a \rangle$ is normal in some subgroup of order p^2 of G , that is, normal in all of G . Likewise $\langle b \rangle$ is normal in G .

Now $\langle a \rangle \vee \langle b \rangle$ is a subgroup of G properly containing $\langle a \rangle$ and of order dividing p^2 . Hence $\langle a \rangle \vee \langle b \rangle$ must be all of G .

Thus the hypotheses of last lemma are satisfied, and G is isomorphic to $\langle a \rangle \times \langle b \rangle$ and therefore abelian.