Mon☐ Tues☐ Wed☐ Thurs☐ Fri☐ Sat☐ Lec#1 Date: 30-Oct-18

Topic 1:-

"Introduction To Topology"

(i) What is Topology?

Topology is a combination of two words.

(i) Topos (Greek) means "Place".

(ii) Logy (Latin) means "study/Analysis".

The study of some thing with respect to its place.

(ii) What kind of Math it is?

Topology is a kind of Math in which we study shapes.

Q:- Does it mean, Topology is Geometry?

Ans:- No.

So what is a difference between Topology and Geometry?

In Geometry we care about the exact measurements such as lengths, angles and areas.

But in Topology we don't care about the measurements.

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Geometerically rectangle, triangle, circle and square are different. But from Topology point of view these shapes are same. There is no difference b/w the triangle, rectangle, circle and square.

Instead of measurements, in Topology we care more about

(i) Arrangements of shapes

(ii) Relative positions of shapes

(iii) Functions ~~and~~ deformations between the shapes.

So we may say that topology is "Geometry of Positions."

In Topology two shapes are considered the same if one shapes can be deformed to other shapes.

As an example a circle can be deformed into a square.

So topologically circle and square are of the same shapes.

But a line and circle are

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different shapes.

(iii) How useful this subject is if we can't tell the difference between shapes?

In fact it's a matter of nature of the study.

Topological study is more quatitative in nature, rather than in quantitative.

There are many problem/mathematical models where qualitative study is more effective than the quantitative study.

So topology is a very useful tool in such problems or one may say some time the only effective tool.

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Topic 2:-
Topology on a set:-

Power Set:-

Let X be a set. Power set of X, denoted by $P(X)$, is a set of all subsets of X. Usually we use the word class for a set of sets.

Example:-

Let $X = \{1,2,3\}$

Then $P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Remarks:-

1. ϕ and X are also members of $P(X)$.

If is X finite and has n elements then

$$|P(X)| = 2^n$$

Let X be a non-empty set and $\mathcal{T}$ be a subclass of $P(X)$, then $\mathcal{T}$ is called a Topology on X. If $\mathcal{T}$ satisfied the following

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axioms.

1. $\phi, X \in \tau$

2. The union of any number of sets in τ belongs to τ.

3. The intersect of finite numbers of sets in τ belongs to τ.

arbitrary use hota h kap kitnewords ly rhe hain jo word finite be hoskta h.

Open Set:

The elements of τ are called open sets or more specifically τ-open sets.

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Topic 3:-

Topology on a set (Contd.)

Topology on a set can also be defined by starting only 2nd and 3rd axiom.

1st axiom will be followed from 2nd and 3rd.

$$\cup \{U \in \mathcal{T} \mid U \in \phi\} = \phi$$

$$\cap \{U \in \mathcal{T} \mid U \in \phi\} = X$$

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Topic 4:- Examples Of Topology on a Set:-

Example 1:

Let $X = \{a, b, c, d\}$ Then

$P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}$

Let us consider the subcollection

$\mathcal{T} = \{\phi, \{a\}, \{c\}, \{a,c\}, X\}$

Let us check all three conditions one by one on $\mathcal{T}$.

(i) It is clear that

$\phi, X \in \mathcal{T}$ (means ϕ, X both are in $\mathcal{T}$)

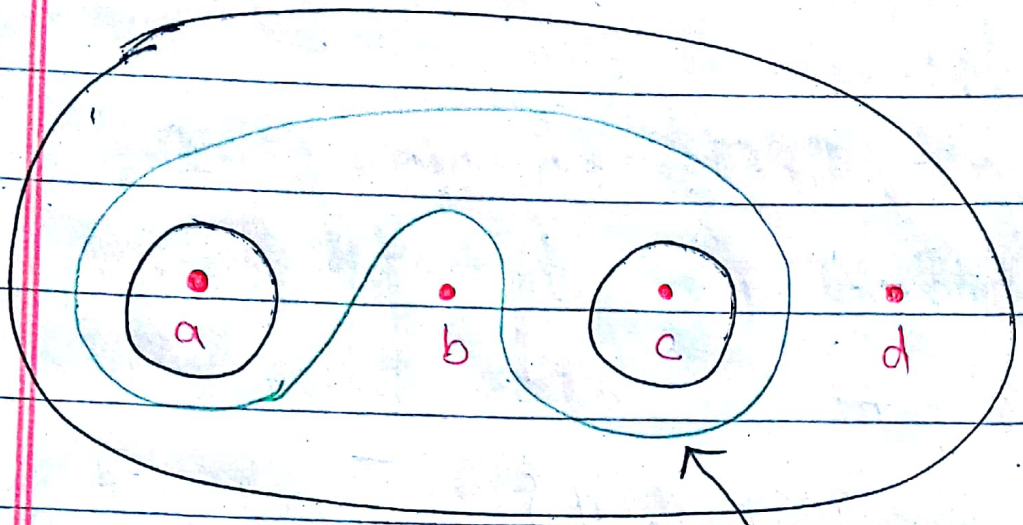
(ii) Now if we take union of any number of elements of $\mathcal{T}$. It belongs to $\mathcal{T}$. (means comp. $\mathcal{T}$ main sb elements ka 1 dosre sy union ho)

(iii) The same is true for intersection. (means sb elements sy koi sy. 2 ka milain to intersection ho).

So, $\mathcal{T}$ is topology on X.

Let us make a graph. for idea

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$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, c\}, X\}$

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Lec. 2:-

Topic 5:-

Examples:-

Topology On a Set:-

Example 2:

Let $X = \{a, b, c, d\}$

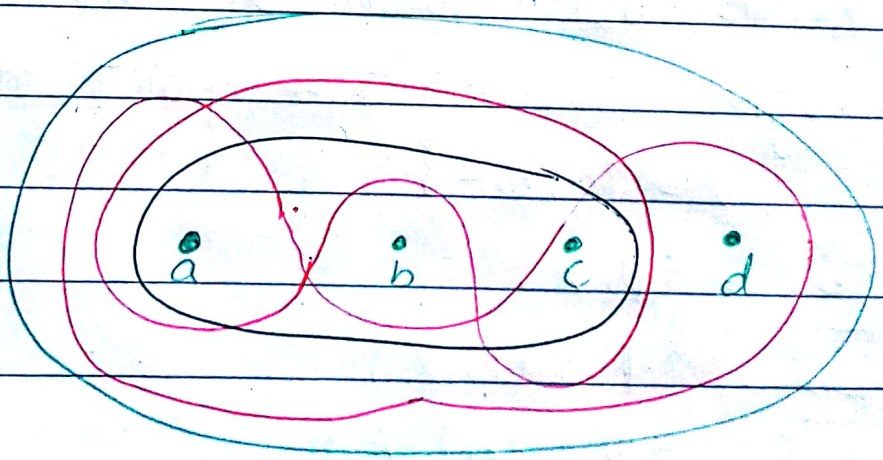
Let us consider

$\mathcal{T}' = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$

1. Again in this example it is clear that $\phi, X \in \mathcal{T}'$

Sol:-

$\mathcal{T}' = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$



کسی بھی دو سیٹ کے intersection کو دیکھیں تو اس میں آ جائے گا۔

Remark 1:-

More then 1 topologies can

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be defined on a set.

Let us define all the possible topologies on 1-point, 2-point and 3-point sets.

1- point set:-

$X = \{a\}$

$\tau_S = \{\phi, X\}$

2- point set:

$Y = \{a, b\}$

$\tau_1 = \{\phi, \{a\}, Y\}$

$\tau_2 = \{\phi, \{b\}, Y\}$

$\tau_3 = \{\phi, \{a, b\}, Y\}$

$\tau_4 = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$

3-point set:-

let us list down, all the topologies on a three point set.

$X = \{a, b, c\}$

$\tau_1 = \{\phi, X\}$

$\tau_2 = \{\phi, \{a\}, X\}$

$\tau_3 = \{\phi, \{b\}, X\}$

$\tau_4 = \{\phi, \{c\}, X\}$

$\tau_5 = \{\phi, \{a, b\}, X\}$

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$\tau_6 = \{\phi, \{a,c\}, X\}$

$\tau_7 = \{\phi, \{b,c\}, X\}$

$\tau_8 = \{\phi, \{a\}, \{a,b\}, X\}$

$\tau_9 = \{\phi, \{a\}, \{a,c\}, X\}$

$\tau_{10} = \{\phi, \{a\}, \{b,c\}, X\}$

$\tau_{11} = \{\phi, \{b\}, \{a,b\}, X\}$

$\tau_{12} = \{\phi, \{b\}, \{a,c\}, X\}$

$\tau_{13} = \{\phi, \{b\}, \{b,c\}, X\}$

$\tau_{14} = \{\phi, \{c\}, \{a,c\}, X\}$

$\tau_{15} = \{\phi, \{c\}, \{b,c\}, X\}$

$\tau_{16} = \{\phi, \{c\}, \{a,b\}, X\}$

$\tau_{17} = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$

$\tau_{18} = \{\phi, \{a\}, \{b\}, \{a,c\}, X\}$

$\tau_{19} = \{\phi, \{b\}, \{c\}, \{b,c\}, X\}$

$\tau_{20} = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

$\tau_{21} = \{\phi, \{b\}, \{a,b\}, \{b,c\}, X\}$

$\tau_{22} = \{\phi, \{c\}, \{a,c\}, \{b,c\}, X\}$

$\tau_{23} = \{\phi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\}$

$\tau_{24} = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$

$\tau_{25} = \{\phi, \{a\}, \{c\}, \{a,c\}, \{b,c\}, X\}$

$\tau_{26} = \{\phi, \{a\}, \{c\}, \{a,c\}, \{a,b\}, X\}$

$\tau_{27} = \{\phi, \{b\}, \{c\}, \{b,c\}, \{a,b\}, X\}$

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$\tau_{28} = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$

$\tau_{29} = P(X)$

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Topic 6:-
Examples 2:-
Topology on a set:-

Remark 2:-

Not every collection of subsets of a set is eligible to be a topology on the set unless that collection doesn't satisfy all the stated conditions in the defination.

Now let us look into some examples those are not topologies on a set X.

Example 3:-

Let $X = \{a, b, c, d\}$ and consides the following collection of subsets of X?

$\mathcal{T}_1 = \{\{b\}, \{c\}, \{b, c\}, X\}$

$\mathcal{T}_2 = \{\phi, \{a\}, \{c\}, \{a, c\}\}$

$\mathcal{T}_3 = \{\phi, \{a\}, \{b, c\}, X\}$

$\mathcal{T}_4 = \{\phi, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$

None of the above $\mathcal{T}_i$'s is a topology on X.

Reasons:-

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1. For $\tilde{\tau}_1$:

$\phi \notin \tilde{\tau}_1$

$\tilde{\tau}_1 = \{\{b\}, \{c\}, \{b,c\}, X\}$

2. For $\tilde{\tau}_2$:

$X \notin \tilde{\tau}_2$

$\tilde{\tau}_2 = \{\phi, \{a\}, \{c\}, \{a,c\}\}$

3. For $\tilde{\tau}_3$:

$\{a\} \cup \{b,c\} = \{a,b,c\} \notin \tilde{\tau}_3$?

$\tilde{\tau}_3 = \{\phi, \{a,\}, \{b,c\}, X\}$

4. For $\tilde{\tau}_4$

$\{a,B\} \cap \{b,c\} = \{b\} \notin \tilde{\tau}_4$

$\tilde{\tau}_4 = \{\phi, \{a,b\}, \{b,c\}, \{a,b,c\}, X\}$

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Topic 7:-
Some More Examples:
Topology on a set.

Example 4:

Let $X = \mathbb{R}$ and $\mathcal{T}$ be the class consisting ϕ, $\mathbb{R}$ and all infinite open intervals of the following the kind

$$I_a = (a, \infty)$$

Where a be any real numbers. Then $\mathcal{T}$ is a topology on $\mathbb{R}$.

Let us justify our claim.

It is given that

$$\phi, \mathbb{R} \in \mathcal{T}$$

For union:

Note that for any subcollection

$$A = \{I_i \mid i \in J\}$$

of $\mathcal{T}$, where $J \subset \mathbb{R}$.

Now

$$\bigcup_i I_i = \begin{cases} (j, \infty) & \text{if } j = \inf(J) \\ (-\infty, \infty) & \text{if No inf exists} \end{cases}$$

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In both cases union belongs to τ.

For intersection

Let $A = (a, \infty)$ and $B = (b, \infty)$

Consider $a < b$

$A \cap B = (b, \infty) \in \tau$

Example 5:-

Let $X = \mathbb{R}$ and consider the class T consisting of $\phi, \mathbb{R}$ along with all infinite open intervals

$A_q = (-\infty, q)$

where $\forall q \in Q$.

This ~~topology~~ collection is not a topology on real line

Reason:

The condition of union is not satisfied by the given collection T.

Let us consider the following subclass of T.

$S = \{(-\infty, q) \mid q \in Q, q < \sqrt{1}\}$

Since.

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$$\bigcup_{\substack{q \in \mathbb{Q} \\ q < x}} (-\infty, q) = (-\infty, x) \in T$$

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Topic 8:
Some Special Topologies

For any set there are always two topologies those are very special.

1. Indiscreate Topology
2. Discrete Topology

Indiscrete Topology:

A topology on a set X consists of empty set and the set itself is called "Indiscrete topology". It is also called trivial topology.

Remark:

Indiscrete topology is the smallest topology one can define on some set.

Task:

Show that such collection is a topology on any set.

X

$\mathcal{T}_{ind} = \{\phi, X\}$.

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→ $\phi, X \in \mathcal{T}_{ind}$

→ $\phi \cup X = X \in \mathcal{T}_{ind}$

→ $\phi \cap = \phi \in \mathcal{T}_{ind}$

Discrete topology:

A topology on a set X consists of all subsets of X is called Discrete Topology. ~~one can define on some set X.~~

~~Task~~ i.e.

$$\mathcal{T} = P(X)$$

Remarks:-

Discrete topology is the largest topology one can define on some set X.

Task:-

let us show that $\mathcal{T} = P(X)$ is a topology on set X

$P(X)$ = The set of all subset of

→ $X, \phi \in P(X) = \mathcal{T}$

→ $A_i \subset \mathcal{T}$

$$\bigcup_{i \in I} A_i \subset X$$

$$\Rightarrow \bigcup_{i=I} A_i \in P(X) = \mathcal{T}$$

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$\Rightarrow A_j \subset \mathcal{T}$

$\bigcap_{j=1}^{n} A_j \subset X$

$\Rightarrow \bigcap_{j=1}^{n} A_j \in P(X) = \mathcal{T}$

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Topic 9:-

Cofinite Topology:-

"Cofinite Topology" is also known as "Finite Complement Topology."

For a set X "cofinite Topology" τ_{cof} be the collect of all subsets U of X such that U^c is either finite or complete X.

ie.

$$\tau_{cof} = \{U \subset X \mid U^c \text{ is finite or } X\}$$

or

$$\tau_{cof} = \{U \subset X \mid U^c \text{ is finite or } U = \phi\}$$

Task:

Show that $\tau_{cof} = \{U \subset X \mid U^c$ is finite or $X\}$ defines a topology on X.

Proof:-

$\phi \in \tau_{cof}$ $\because \phi^c = X$

(i) $X \in \tau_{cof}$ $\because X^c = \phi$ finite

(2) Let $A_i \in \tau_{cof}$ $i \in I$

$\Rightarrow A_i^c =$ finite $\because X$

Now we have to show that

$$\bigcup_{i \in I} A_i \in \tau_{cof}$$

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De Morgen's Law

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} (A_i)^c = \text{finite on } X$$

(3) Let $A_i \in \tilde{\tau}_{cof}$

$\Rightarrow (A_i)^c$ = finite on X_i $\quad i = 1, 2, 3, \ldots n$

Now we have to show that

$$\bigcap_{i=1}^{n} A_i \in \tilde{\tau}_{cof}$$

De-Morgen's law

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} (A_i)^c = \text{finite on } X$$

Finite union of finite sets is finite

$$\Rightarrow \bigcap_{i=1}^{n} A_i \in \tilde{\tau}_{cof}$$

So, $\tilde{\tau}_{cof}$ is a topology on X.

Remark:-

If X is finite then the cofinite topology on X is discrete

i.e

$\tilde{\tau}_{cof} = P(X)$ if X is finite

Proof:- P(X) the set of all subsets of X

$\tilde{\tau}_{cof} \subset P(X)$ | Now $U \in P(X)$:

$\therefore$ X is finite so U^c = finite

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$\Rightarrow U \in \tilde{\mathcal{S}}_{cof}$

$\Rightarrow \tilde{\mathcal{S}}_{cof} = P(X)$

X is finite

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Topic 10:-
Topological Space:-

Let $\mathcal{T}$ be a topology on X Then the pair $(X, \mathcal{T})$ is a "Topological space" or some time called $\mathcal{T}$-topological space."

Example

Consider $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$

Then $(X, \mathcal{T})$ is a topological space.

Example:

Consider $X = \{a, b, c, d\}$ with $\mathcal{T}' = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$

Then $(X, \mathcal{T}')$ be another topological space.

Note that in both examples set X is the same but we define two different topological spaces.

Remark:

Correspond to single set one may define multiple topological

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Spaces and all can be different from each other.

Discrete space:

A set X with discrete topology is called "Discrete topological space" or simply "Discrete space".

Indiscrete Space:-

A set X with indiscrete topology is called "Indiscrete topological space". or simply "Indiscrete space."

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Topic 11:-
Arbitrary Intersection:-

Let us recall the 3rd axiom of defination of topology on a set.

"Finite Intersection"

Q:- What about arbitrary intersection? Specifically "Infinite Intersection".

Before giving the answer to this Question let us look into the next two examples.

Example 1:-

Consider $\mathbb{R}$ "Real line" and consider τ to be the set of all open intervals and their unions. This collection is a topology on $\mathbb{R}$.

Consider the following subcollection

$$A = \{A_n = \left(\frac{-1}{n}, \frac{1}{n}\right) \mid n \in \mathbb{N}^*\}$$

Now

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$$\bigcap_{n\in\mathbb{N}^*}\left[\frac{-1}{n},\frac{1}{n}\right]=\{0\}\notin\mathcal{T}$$

Example 2:-

Consider $\mathbb{R}^2$ "Reeal plane" and consider $\mathcal{T}$ the set of all open disks, their arbitrary unions and their finite intersection. This collection is a topology on $\mathbb{R}^2$.

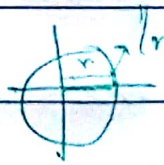
Consider the following subcollection of open disks.

$$A=\{A_n / n\in\mathbb{N}^*\}$$

where

$$A_n=\left\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<\left(\frac{1}{n}\right)^2\right\}$$

$$\bigcap_{n\in\mathbb{N}^*} A_n=(0,0)\in\mathcal{T}$$



Represent the radius of circle

Remark:

Arbitrary intersection of open sets may or may not be open.

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Topic 12
Open Sets, Close Sets:-

Open Sets:

Let $\mathcal{T}$ be at a topology on X. Then the elements of $\mathcal{T}$ are called the open sets, more precisely open subsets of X.

Let us look into an example for better understanding of the openness.

Example 1:-

Let $X = \{a, b, c, d\}$

consider

$\mathcal{T} = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$

Then $\phi, \{a\}, \{c\}, \{a, c\}, X$ are open subsets of X.

Example 2:-

Let $X = \{a, b, c, d\}$ consider

$\mathcal{T} = \{\phi, \{c\}, \{a, c\}, \{b, c, d\}, X\}$

Then $\phi, \{c\}, \{a, c\}, \{b, c, d\}, X$ are open subsets of X.

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Remark:

Openness is a relative term. ϕ and X are always open.

Closed set:

Let $(X, \mathcal{T})$ be a topological space. A subset A of X is closed iff the complement of A in X is open. i.e.

$$X - A \in \mathcal{T}$$

let us discuss ~~the~~ in example.

Example 3:-

$$X = \{a, b, c, d\}$$

$$\mathcal{T} = \{\phi, \{c\}, \{a, c\}, \{b, c, d\}, X\}$$

Now, $\{a\}$ is closed since

$$X - \{a\} = \{b, c, d\}$$

is open. i.e. belongs to $\mathcal{T}$.

Similarly $\{b, d\}$ is closed since

$$\{b, d\}^c = X - \{b, d\} = \{a, c\}$$

Example 4:-

$$X = \{a, b, c, d\}$$

$$\mathcal{T} = \{\phi, \{c\}, \{a, c\}, \{b, d\}, \{b, c, d\}, X\}$$

$\{a, c\}$ open

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closed ← $\{a, c\}^c = \{b, d\} \in \tau$

Remark:

closeness like openness is a relative term.

ϕ and X are always closed.

A set can be open and closed at the same time.

In discrete space all sets are open and close at the same time.

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Lec. 4

Topic 13:- Properties of collection of closed sets:-

Let (X, T) be a topological space.

Consider the collection

$$A = \{A \subset X \mid A^c \in T\}$$

A is set of all those subsets of X whose complement is open.

Simply "collection of all closed subsets."

Example:-

Let $X = \{a, b, c\}$

and $T = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$

then

$$A = \{X, \{b, c\}, \{a, b\}, \{c\}, \{b\}, \phi\}$$

is the collection of all closed subsets of X.

Example:

Let $X = \{a, b, c, d\}$

and

$$T = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$$

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Then

$A = \{X, \{b,c,d\}, \{b,e\}, \{b\}, \phi\}$

is a collection of all closed subsets of X.

Example:

Let $X = \mathbb{R}$

and

$T = \{(a, \infty) \; a \in \mathbb{R}\}$

Then

$A = \{(-\infty, a] \mid a \in \mathbb{R}\}$

is a collection of all closed subsets of $\mathbb{R}$ with respect to T.

Theorem:

Let (X, T) be a topological space. Then A collection of all closed subsets of X, satisfies the following properties.

1. $\phi, X \in A$
2. A is closed under finite set union. i.e. $A_i \in A \; (i \in \{1, 2, \dots n\})$

$\Rightarrow \bigcup_{i=1}^{n} A_i \in A$

3. A is closed under arbitrary set intersection.

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i.e. $A_i \in A \ (i \in I \Rightarrow \cap_{i \in I} A_i \in A)$

Proof:-

$$A = \{A \subset X \mid A^c \in \tau\}$$

1. $\phi \in A \quad \because \phi^c = X \in \tau$

$X \in A \quad \because X^c = \phi \in \tau$

2. Let $A_i \in A$ i.e. $A_i^c \in \tau, \ i = 1, 2, \ldots n$

$\bigcup_{i=1}^{n} A_i \in A \ = ?$

$$(\cup_{i=1}^{n} A_i)^c = \cap_{i=1}^{n} (A_i)^c \in \tau$$

$\therefore$ The finite intersection of open set is always open.

3. Let $A_i \in A \quad i \in I$

$\Rightarrow A_i^c \in \tau \quad \forall \ i \in I$

$\cap_{i=1}^{n} A_i \in A$

$$(\cap_{i=1}^{n} A_i)^c = \cup_{i \in I} (A_i)^c$$

A_i^c are open. so, the union of open sets is always open. i.e.

$$\cup_{i \in I} U_i \in \tau$$

$$\Rightarrow \bigcap_{i \in I} A_i \in A$$

—

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Topic 14:-

Intersection Of Topologies:-

Let T_1 and T_2 be two topologies on X. Now what can we say about the intersection of T_1 and T_2?

Does T_1 and T_2 define a topology on X?

Before answering to this question let us look in to an example.

Example:-

Let $X = \{a, b, c\}$

and

$T_1 = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$

$T_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$

then

$T_1 \cap T_2 = \{\phi, \{a\}, \{a, b\}, X\}$

It is clear that $T_1 \cap T_2$ is a topology on X.

Theorem:-

Let $\{T_i \mid i \in I\}$

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be any collection of topologies on set X. Then $\cap_{i\in I} T_i$ is also a topology on X.

Proof:-

(1) $\{T_i \mid i \in I\}$

Now we have to show That $\cap_{i\in I} T_i$ is also a topology on X.

(2) $\phi \in \cap_{i\in I} T_i$ $\because T_i$ is a top. on X $\forall i$

So $\phi \in T_i$ $\forall i \in I$

$X \in \cap_{i\in I} T_i$ $\because T_i$ is a top. on X $\forall i$

So $X \in T_i$ $\forall i \in I$

(2) Let $U_j \in \cap_{i\in I} T_i$

$\Rightarrow U_j \in T_i$ $\forall i$

But T_i is topology on X $\forall i$

So, $\cup_{j\in J} \in T_i$ $\forall i \in I$

$\Rightarrow \cup_{j\in J} U_j \in \cap_{i\in I} T_i$

(3) let $U_j \in \cap_{i\in I} T_i$ $j = 1, 2, \cdots n$

$\Rightarrow U_j \in T_i$ $\forall i \in I$, $\forall j$

But T_i is a topology on X $\forall i \in I$.

So $\cap_{j=1} U_j \in T_i$ $\forall i \in I$

$\Rightarrow \cap_{j=1}^{n} U_j \in \cap_{i\in I} T_i$

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Topic 15:-
Union of Topology:-

Let T_1 and T_2 be two topologies on X. Now what can we say about the union of T_1 and T_2?

As like intersection, we can say that $T_1 \cup T_2$ defines a topology on X?

Let us see some examples.

Example:

Let $X = \{a, b, c\}$

and $T_1 = \{\phi, \{a\}, \{a, b\}, X\}$

$T_2 = \{\phi, \{b\}, \{a, b\}, X\}$

then

$T_1 \cup T_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$.

In this example $T_1 \cup T_2$ is a topology on X.

Does it mean that it is true in general?

No, unless we prove it.

Example:

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Let $X = \{a, b, c\}$

and

$T_1 = \{\phi, \{a\}, \{c\}, \{a,b\}, \{a,c\}, X\}$

$T_2 = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$

Then

$T_1 \cup T_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, X\}$

$T = T_1 \cup T_2$ is not a topology on X. Since $\{b\} \cup \{c\} = \{b,c\} \notin T$.

Remark:

Union of topologies on a set X may or may not be a topology on X.

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Topic 16:- Usual Topology on $\mathbb{R}$:-

Real line:-

The real number set, $\mathbb{R}$ one of the most studied/dominated object in Mathematics. Most of the concepts in Mathematics are the generalization/abstraction of the properties of the real numbers. To study $\mathbb{R}$ some one needs a usual structure, so the rest of the terminologies can be defined according to that.

This usual structure is provided by usual topology. Here we define the usual topology on $\mathbb{R}$ by using order relation.

Open Interval:-

Let $a, b \in \mathbb{R}$. with usual order relation "$a < b$" then the open interval from a to b is

$$(a, b) = \{x \mid a < x < b\} \subset \mathbb{R}$$

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Open set in $\mathbb{R}$:

A subset A of $\mathbb{R}$ is open iff for every $a \in A$ there exists an open interval I_a, containing a which contained in A i.e.

$a \in I_a \subset A$.

Usual Topology on $\mathbb{R}$:-

The set of all open sets of $\mathbb{R}$ T_u, (as defined in above def.) is a topology on $\mathbb{R}$ called usual topology on $\mathbb{R}$.

Task:

Prove that T_u is a topology on $\mathbb{R}$.

Proof:-

criteria
$A \subset \mathbb{R}$
$A \in T_u$ iff
$\forall a \in A \ \exists I_a$
s.t $a \in I_a \subset A$

① $\phi \in T_u$ $\because \nexists x \in \phi$

$X = \mathbb{R} \in T_u$ $\forall x \in \mathbb{R}$

$I_x \subset \mathbb{R}$

② Let $U_i \in T_u$ $i \in I$

Let $x \in \bigcup_{i \in I} U_i$

$\Rightarrow x \in U_i$ for some $i \in I$

But U_i is open in $\mathbb{R}$. So, ~~$\exists I_x \subset U_i$~~

$\exists I_x \subset U_i \subset \bigcup_{i \in I} U_i$

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$\Rightarrow x \in I_x \subset \bigcup_{i \in I} U_i$

$\Rightarrow \bigcup_{i \in I} U_i \in \tau_u$.

(3) Let $U_1, U_2 \in \tau_u$

$x \in U_1 \cap U_2$

$\Rightarrow x \in U_1$ and $x \in U_2$

But U_1 and U_2 both are open

So, $\exists\ I_{1x}, I_{2x}$ s.t.

$I_{1x} \subset U_1$ and $I_{2x} \subset U_2$

Now $I_{1x} \cap I_{2x} = I_{3x} \subset U_1, U_2$

$x \in I_{3x} \subset U_1 \cap U_2$ which is open interval.

$\Rightarrow U_1 \cap U_2 \in \tau_u$

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Topic 17:-

Usual Topology on $\mathbb{R}^2$

Real plane $\mathbb{R}^2$:-

The real plane is the cartesian product of two real lines. i.e. $\mathbb{R} \times \mathbb{R}$. The real plane is usually written as $\mathbb{R}^2$.

Open Disk:-

An open disk $D_{(x,y)}$ in $\mathbb{R}^2$, centered at $(x,y) \in \mathbb{R}^2$, is the set of all those points (x', y') in $\mathbb{R}^2$ whose distance from (x,y) is less than r, where r is a positive real number.

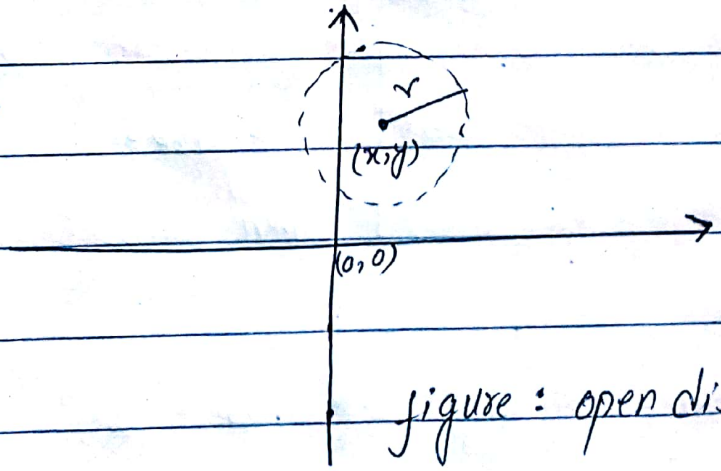


figure : open disk.

$$D(x,y) = \{(x',y') \mid \sqrt{(x-x')^2 + (y-y')^2} < r\} \subset \mathbb{R}^2$$

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Open set in $\mathbb{R}^2$:-

A subset U of $\mathbb{R}^2$ is open in $\mathbb{R}^2$ iff for every $a \in (x,y) \in U$ there exists an open ~~interval~~ disk D_a; which contained in U. i.e.

$$a \in D_a \subset U.$$

Usual Topology on $\mathbb{R}^2$:-

The set of all open sets of $\mathbb{R}^2$ τ_u (as defined in above def.) is a topology on $\mathbb{R}^2$, called usuall topology on $\mathbb{R}^2$

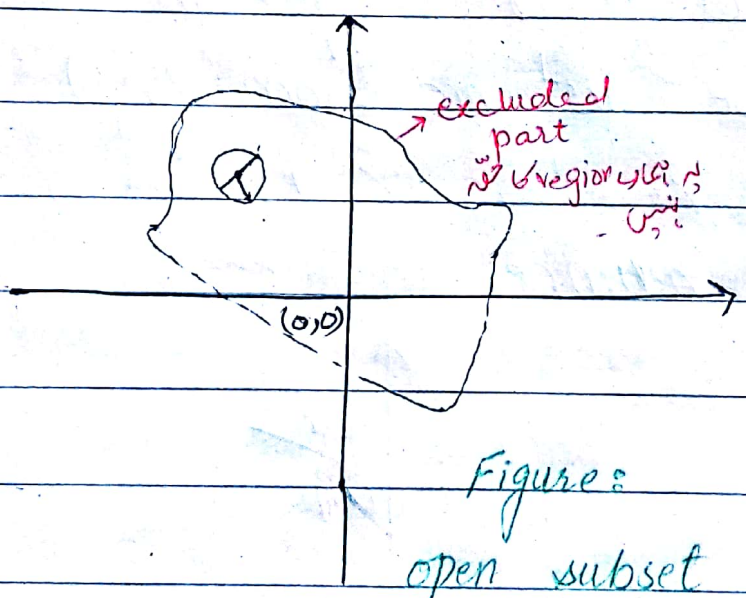


Figure:
open subset of $\mathbb{R}^2$

Task:

Prove that τ_u is a topology on $\mathbb{R}^2$.

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Topic 18
Usual Topology on $\mathbb{R}^n$

Open n-Ball:-

An open ball B_x in $\mathbb{R}^n$, centered at $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$, is the set of all those points $y = (y_1, y_2, \dots y_n)$ in $\mathbb{R}^n$, whose distance from x is less than r, where r is a positive real number.

$$B_x = \left\{ y ; \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < r \right\} \subset \mathbb{R}^n$$

Open set in $\mathbb{R}^n$:

A subset U of $\mathbb{R}^n$ is open in $\mathbb{R}^n$ iff for every $a \in U$ there exists an open n-Ball B_a contained in U, i.e. for every $a \in U$ $\exists, B_a$ such that

$$a \in B_a \subset U.$$

Usual Toplogy on $\mathbb{R}^n$:-

The set of all open sets of $\mathbb{R}^n$, T_u, (as defined in above def.) is a topology on $\mathbb{R}^n$, called

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usual topology on $\mathbb{R}^n$.

Task:-

Prove that $\mathcal{T}_u$ is a topology on $\mathbb{R}^n$.

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Topic 19:-
Comparison of Topology

As we have already seen that more than one topologies can be defined on a set X with more than one element. So, it is quite natural to ask the following Question.

Question :-

Is there any relation between two different topologies?

As the topologies consist of collection of subsets of X. So one may put a question on the size of two topologies. But this question may not be answered property unless will not consider the relation between the elements of two topologies and the elements are the subsets of a set. Naturally we have the relation that is, the set

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inclusion. And we have the positive answer of our question that is "Yes". There can be class inclusion relation between two topologies.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a non-empty set X. Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ i.e.

$$\forall U \in \mathcal{T}_2 \Rightarrow U \in \mathcal{T}_1$$

If $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ then we also say that $\mathcal{T}_2$ is weaker than $\mathcal{T}_1$

Example:-

(i) Let $X = \{a, b, c\}$

and $\mathcal{T}_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$

$\mathcal{T}_2 = \{\phi, \{a\}, \{a, b\}, X\}$

It is clear that

$$\mathcal{T}_2 \subset \mathcal{T}_1$$

(ii) Let X be any non-empty set. Consider $\mathcal{T}$ be any topology

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on X and T_{IND} is indiscrete topology on X. Then it is not vary hard to show that

$$T_{IND} \subseteq T$$

∵ every T contains ϕ and X and $T_{IND} = \{\phi, X\}$.

Remark:-

In discrete topology is the weakest topology on a set X.

Example:-

Let X be any non-empty set. Considers T be any topology on X and T_{dis} is discrete topology on X. Then it is not very hard to show that $T \subseteq T_{dis}$.

∵ every $T \subseteq P(X) = T_{dis}$.

Remark:-

Discrete topology is the finest topology on a set X.

Comparable Topologies (Defination):-

Two topdogies on a set X are comparable iff one is finer

(weaker) than the other.

Remark:-

Not all topologies on a set are comparable.

Example:-

Let $X = \{a, b, c, d\}$ and consider.

$T_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$

$T_2 = \{\phi, \{c\}, \{c, d\}, X\}$

T_1 and T_2 are not comparable since $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$ i.e. no one is finer, no one is weaker in comparison of each other.

Theorem:-

The collection $T = \{T_i\}$ of all topologies on X is partially ordered by class inclusion.

Hint:-

Show that the following properties hold for all elements of T

(i) Reflexivity: $T \subseteq T$.

(ii) Antisymmetry: If $T \subseteq T'$ and $T' \subseteq T$, then $T = T'$.

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(iii) Transitivity:- If $T \subseteq T'$ and $T' \subseteq T''$, then $T \subseteq T''$.

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Topic 20:- Limit point of a set:-

Limit point:-

Let (X, T) be a topological space and A be a subset of X. A point $x \in X$ is a limit of A iff every open set U_x containing x contains a point of A different from x i.e.

$$\forall U_x \in T,\ x \in U_x$$
$$A \cap (U_x \setminus \{x\}) \neq \phi$$

Limit point is also called accumulation point or drived point.

Example:-

Let $X = \{a, b, c, d, e\}$ and

$T = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$

be a topology of X. Consider $A = \{a, b, c\} \subset X$. Let us check all the elements of X one by one for limit point of A.

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For a:-

a is not a limit point of A. Since there exists an open set $\{a\}$ containing a such that

$$A \cap (\{a\} \setminus \{a\}) = \phi$$

$T = \{\phi, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}, X\}$

For b:-

b is a limit point of $A = \{a,b,c\}$. Since all the open sets containing b intersect A other than b.

$$A \cap (\{b,c,d,e\} \setminus \{b\}) = \{c\} \neq \phi$$

$$A \cap (X \setminus \{b\}) = \{a,c\} \neq \phi$$

For c:-

c is not a limit point of $A = \{a,b,c\}$. Since there exist an open set $\{c,d\}$ containing c such that

$$A \cap (\{c,d\} \setminus \{c\}) = \phi$$

For d:-

d is a limit point of $A = \{a,b,c\}$. Since all the open sets containing d intersect A other than d.

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$A \cap (\{c,d\} \setminus \{d\}) = \{c\} \neq \phi$

$A \cap (\{a,c,d\} \setminus \{d\}) = \{a,c\} \neq \phi$

$A \cap (\{b,c,d,e\} \setminus \{d\}) = \{b,c\} \neq \phi$

$A \cap (X \setminus \{d\}) = A \neq \phi$

For e:-

e is a limit point of $A = \{a,b,c\}$. Since all the ~~poe~~ open sets containing e intersect~~ion~~ A other then e.

$A \cap (\{b,c,d,e\} \setminus \{e\}) = \{b,c\} \neq \phi$

$A \cap (X \setminus \{e\}) = A \neq \phi$

Examples:-

Let $X = \mathbb{R}$ with usual top.

$A = (0,3)$

7 is not a limit point $7 \in (6,8)$ of $A = (0,3)$

$A \cap [(6,8) \setminus \{7\}] = \phi$

1.5 is a limit point of $A = (0,3)$

$A \cap [(1.5-\varepsilon, 1.5+\varepsilon) \setminus \{1.5\}] \neq \phi \quad \because \varepsilon \in \mathbb{R}^+$

3 is a limit point of $A = (0,3)$

$A \cap [(3-\varepsilon, 3+\varepsilon) \setminus \{3\}] \neq \phi$

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Example:-

$X = \mathbb{R}$ with usual topology.

$B = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \{\frac{1}{x} \mid n \in N^+\}$

0 is a limit point of B

$0 \in (-\varepsilon, \varepsilon)$ $\quad \varepsilon = \mathbb{R}^+$

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Topic 21

Drived Set

Defination:-

Let $(X, \mathcal{T})$ is a topological space and A be a subset of X. Then the drived set of A, denoted as A' is the set of all limit points of A.

i.e., $A' = \{x \in X \mid \forall\ U_x \in \mathcal{T},\ A \cap (U_x \setminus \{x\}) \neq \phi\}$

Remarks:-

* Drived set of A may or may not be subset of A.
* Drived set of empty set is empty.

Example:-

Let $X = \{a, b, c, d, e\}$ and $\mathcal{T} = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$ be a topology of X.

Consider $A = \{a, b, c\} \subset X$.

Then

$A' = \{b, d, e\}$

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Consider $B = \{b, e\} \subset X$.

Then

$$B' = \{b, e\}$$

Example:-

Let $X = \mathbb{R}$ with usual topology and $A = \{1, 1/2, 1/3, 1/4, 1/5, \dots\}$

Then

$$A' = \{0\}$$

Example:-

Let $X = \mathbb{R}$ with usual topology and $A = (-1, 2)$

Then

$$A' = [-1, 2]$$

Example:-

Let $X = \mathbb{R}$ with usual topology and $A = \mathbb{Q}$.

Then

$$A' = \mathbb{R}.$$

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Topic 22:-

Drived set: Special Topologies:-

A we ase studied two special topologies on a set X named as "9ndiscrete" Topology" and "discrete topology".

Let us figuse out the drived set of a subset of X with respect to these special topologies.

Consider an indiscrete space X i.e. X with indiscrete topology

$$T_{IND} = \{\phi, X\}$$

Note that for any point ~~in~~ x in X there is one and only one open set containing x and that is X itself.

So a point $x \in X$ will be limit point of some subset A of X iff

$$A \cap (X \setminus \{x\}) \neq \phi$$

Now, if $A = \{y\}$ then any point

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$x \neq y$ of X satisfies the above said condition. i.e.

$$\{y\} \cap (X \setminus \{x\}) = \{y\} \neq \phi$$

So ~~case~~ in this case

$A' = A^c$.

Now, If A contains two are more points then any point x of X satisfies

$$A \cap (X \setminus \{x\}) = A \setminus \{x\} \neq \phi$$

So, in this case $A' = X$

Let us summarize it as our concluding remark.

Remarks-

In an indiscrete space X, the derived set A' of any subset A of X is as following

$$A' = \begin{cases} \phi & \text{if } A = \phi \\ A^c = X \setminus \{y\} & \text{if } A = \{y\} \\ X & \text{if otherwise} \end{cases}$$

Now let us discuss for discrete topology.

Consider a discrete ~~topology~~

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space X. i.e., X with discrete topology

$$\mathcal{T}_{dis} = P(X).$$

Note that for any point x in X every subset of X containing x is open.

So a point $x \in X$ will be a limit point of some subset A of X iff every subset U of X containing x must intersect with A other than x.

But for every x in a discrete space there is a single to set containing x. i.e., $\{x\}$ which does not intersect with A other than x.

So no point of x can be limit point of any subset A of X.

Following is our concluding remark.

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Remarks:-

In a discrete space X, the derived set A' of any subset A of X is empty.

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Topic 23:-

Derived Set: Some Results:-

In this topic we prove some results related to drived set. Before going towards the theorems let us see when does a point $x \in X$ not belong to derived set of some set A i.e. not be a limit point of A.

Recall that $x \in A'$ iff every open set U_x containing x such that $(U_x \setminus \{x\}) \cap A \neq \phi$.

So it means that $x \notin A'$ iff there exist an open set U_x containing x such that $(U_x \setminus \{x\}) \cap A = \phi$ or equivalently iff there exist an open set U_x containing x such that either $U_x \cap A = \phi$ or $U_x \cap A = \{x\}$ or equivalently iff there exist an open set U_x containing x such that $U_x \cap A \subset \{x\}$.

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Theorem :-

Let (X, T) be a topological space. let $A, B \subset X$. If $A \subset B$ then $A' \subset B'$.

Proof :-

Let $x \in A' \Rightarrow \forall U_x$ (open set containing x)

$(U_x \setminus \{x\}) \cap A \neq \phi$

Now given $A \subset B$ so,

$\phi \neq (U_x \setminus \{x\}) \cap A \supset (U_x \setminus \{x\}) \cap B$

$\Rightarrow \quad x \in B'$

Proved.

Theorem :-

Let (X, T) be a topological space, Let $A, B \subset X$.

Then $(A \cup B)' = A' \cup B'$.

Proof :-

Let us first show that $A' \cup B' \subset (A \cup B)'$

$A \subset A \cup B \Rightarrow A' \subset (A \cup B)'$

$B \subset A \cup B \Rightarrow B' \subset (A \cup B)'$

So,

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$A' \cup B' \subset (A \cup B)'$

Now we show that $(A \cup B)' \subset A' \cup B'$.
For this we take an element from $(A' \cup B')^c$ and show that it belongs to $((A \cup B)')^c$.

Let $x \notin (A' \cup B')$ this implies that $x \notin A'$ and $x \notin B'$

$\Rightarrow \exists U_x$ and V_x (open set containing x) such that

$U_x \cap A \subset \{x\}$ and $V_x \cap B \subset \{x\}$

Now $U_x \cap V_x$ is an open set containing x and

$(U_x \cap V_x) \cap (A \cup B) = ((U_x \cap V_x) \cap A) \cup ((U_x \cap V_x) \cap B)$

$\subset (U_x \cap A) \cup (V_x \cap B) \subset \{x\} \cup \{x\} = \{x\}$

$\Rightarrow x \notin (A \cup B)'$

$\Rightarrow (A \cup B)' \subset A' \cup B'$

So

$(A \cup B)' = A' \cup B'$ is proved.

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Topic 24:-
Closed set and Derived set:-

We have already studied that a subset A of a topological space X is closed iff A^c is open.

In this topic we characterize the closed set it terms of its limit points. It provides us a criterion to check the closeness of a set.

Theorem:-

A subset A of a topological space X is closed iff $A' \subset A$.

Example:-

Consider $X = \{a, b, c, d\}$

with

$$\mathcal{T} = \{\phi, \{a\}, \{a, c\}, \{a, b, d\}, X\}$$

Consider

$A = \{b, d\}$

$B = \{d\}$

$C = \{b, c, d\}$

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Now A and C are closed since $A' = \{b, d\} = A$ and $C' = \{b, d\} \subset C$ but B is not closed because $B' = \{b\} \not\subset B$.

Example:-

Consides $\mathbb{R}$ with usual topology.

Consides following subsets of $\mathbb{R}$.

$A = [0, 1)$ Not closed $A' = [0, 1] \not\subset A$

$B = \{0, 1\}$ closed $B' = \phi \subset B$

$C = \{1, 1/2, 1/3, \cdots\}$ Not closed $C' = \{0\} \not\subset C$

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Topic 25:-
Closed set and Derived set

~~Proof~~ Theorem:-

A subset of A of a topological space X is closed iff $A' \subset A$.

Proof:-

Let A is closed. Now we show that $A' \subset A$.

Let $x \notin A \Rightarrow x \in A^c$

∵ A is closed. So A^c is open and we have an open set A^c containing x does not intersect A at all i.e.,

$A^c \cap A = \phi \Rightarrow (A^c \setminus \{x\}) \cap A = \phi$

so $x \notin A' \Rightarrow A' \subset A$

Let us prove the converse.

Let $A' \subset A$. Now we show that A is closed or equivalently A^c is open.

Let

$x \in A^c \Rightarrow x \notin A \Rightarrow x \notin A'$ as

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given $A' \subset A$.

$\Rightarrow \exists\ U_x$ (open set containing x) s.t. $(U_x \setminus \{x\}) \cap A = \phi$

But $x \notin A$ so, $U_x \cap A = \phi$

$\Rightarrow U_x \subset A^c$.

So, for every x in A^c we have U_x (open set containing x) s.t. $U_x \subset A^c$

And we can write A^c as union of all those open subset. i.e.

$$A^c = \bigcup_{x \in A^c} U_x$$

This shows that A^c is open and this implies that A is closed.

proved.

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Topic 26:- Criterion for openness

Let $(X, \mathcal{T})$ be a topological space and we have already studied that the elements of $\mathcal{T}$ are called the open sets. Now the question is "Does there exist any other criterion to check the openness?" The answer is "Yes".

Let us state the following theorem.

Theorem:-

A subset A of a topological space X is open iff $\forall x \in A \; \exists U_x$ (open set containing x) s.t. $U_x \subset A$.

Example:-

Consider $X = \{a, b, c, d\}$

with $\mathcal{T} = \{\phi, \{a\}, \{a, c\}, \{a, d\}, \{a, b, d\}, X\}$

Consider

$A = \{a, b, d\}$

$B = \{a, b\}$

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$C = \{d\}$

Now A is open and B and C are not open.

Example:-

Consider $\mathbb{R}$ with usual Topology.

$A = (0,1]$ $\quad 1 \notin U_1$ s.t. $U_1 \not\subset A = (0,1]$

Not an open set.

$B = \{1,4\}$ $\quad 1 \notin U_1$ s.t. $U_1 \not\subset B$

not an open set

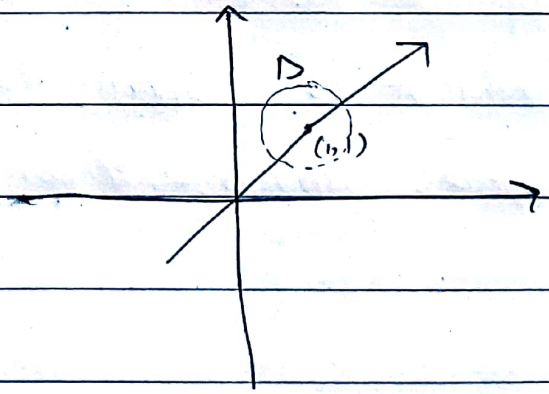
Example:-

Consider $\mathbb{R}^2$ with usual topology?

$A = \{(x,y) \in \mathbb{R}^2 \mid x = y\}$

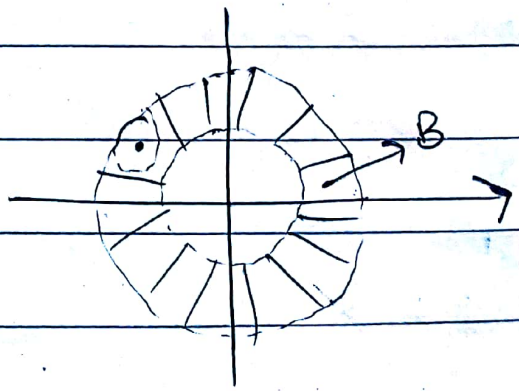
Not an open set

$\nexists\ U_{n(1,1)} \not\subset A$.



$B = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$

is open set of $\mathbb{R}^2$.



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Proof :-

One way is very obvious. i.e. Let A is open. It is obvious that $\forall\ x \in A\ \exists\ U_x$ (open set containing x) s.t. $U_x \subset A$.

∵ For every $x \in A$ there exists always the best candidate that satisfies the condition, is A itself.

So, $\forall\ x \in A\ \exists\ U_x = A$ (open set containing x) s.t. $U_x \subset A$. So, we write ~~union as~~ $\overset{A}{}$ union of all U_x i.e.,

$$A = \bigcup_{x \in A} U_x.$$

Which is shows that A is open. proved.

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Topic 27:-
Closure Of a set:-

Closure of a set:-

Let (X, T) be a topological space and A be a subset of X. The closure of A, denoted as $\bar{A}$, is the intersection of all closed subsets of X containing A.

i.e.,

$$\bar{A} = \bigcap_{C \in C_A} C$$

Where C_A is the class of all closed subsets of X containing A.

Example 1:-

Let $X = \{a, b, c, d, e\}$ and $T = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$ be a topology of X. So, here we have the closed collection

$C = \{X, \{c, d, e\}, \{a, b, e\}, \{e\}, \{a\}, \phi\}$

Consider,

$A = \{c\} \subset X$. Then $C_A = \{X, \{c, d, e\}\}$

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is the class of all closed subsets of X containing A. So,

$$\bar{A} = \bigcap_{C \in C_A} C = \{c,d,e\} \cap X = \{c,d,e\}$$

Example 2:-

Let $X = \{a,b,c,d,e\}$ and

$T = \{\phi, \{a,b\}, \{c,d\}, \{a,b,c,d\}, \{b,c,d,e\}, X\}$

So closed collection

$C = \{X, \{c,d,e\}, \{a,b,e\}, \{e\}, \{a\}, \phi\}$

Consider

$B = \{a,b\} \subset X$. Then $C_B = \{X, \{a,b,e\}\}$ is the class of all closed subsets of X ~~set~~ containing B. So,

$$\bar{B} = \bigcap_{C \in C_B} C = \{a,b,e\} \cap X = \{a,b,e\}$$

Remarks:-

$A \subset \bar{A}$

$\bar{A}$ is the smallest closed ~~subsets~~ superset of A.

$\bar{\phi} = \phi$

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$\bar{X} = X$

Example:-

Let $X = \{a, b, c, d, e\}$ and $T = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$ be a topology of X. So, here we have the closed ~~subsets of X containing A.~~ collection

$C = \{X, \{c, d, e\}, \{a, b, e\}, \{e\}, \{a\}, \phi\}$

Consider $A = \{c\} \subset X$. Then $C_A = \{X, \{c, d, e\}\}$ is the class of all closed subsets of X containing A. So,

$$\bar{A} = \bigcap_{C \in C_A} C = \{c, d, e\} \cap X = \{c, d, e\}.$$

Example:-

let $X = \{a, b, c, d, e\}$ and $T = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$

So closed collection

$C = \{X, \{c, d, e\}, \{a, b, e\}, \{e\}, \{a\}, \phi\}$

Consider $B = \{a, b\} \subset X$. Then $C_B = \{X, \{a, b, e\}\}$ is the class

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of all closed subsets of X containing B.

$$\bar{B} = \bigcap_{C \in C_B} C = \{a,b,e\} \cap X = \{a,b,e\}.$$

Remarks:-

$A \subset \bar{A}$ $\bar{A} \cap_{C \in C_A} C$ $C \subset X$ is closed s.t. $A \subset C$. $\Rightarrow A \subset \bar{A}$

$\bar{A}$ is the smallest closed superset of A.

$\bar{\phi} = \phi$ $\phi \subset \phi \Rightarrow \bar{\phi} = \phi$

$\bar{X} = X$. $\bar{X} \cap_{C \in C_X} C$ $\exists$ only one subset of X that is closed and containing X. X".

$\bar{X} = X$

If $\exists F \subset_{closed} X$ $A \subset F$.

$A \subset F$ is closed. $\bigcap_{C \in C_A} C$ $\exists C = F$

Then $A \subset \bar{A} \subset F$

$\Rightarrow \bar{A} \subset F$

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Topic 28:-
Computation of Closure:-

For a given topological space X and its subset A, one way to compute the closure of A is to apply the defination in following three steps.

(i) List all the closed subset of X.

(ii) Write C_A, all the closed subset of X containing A.

(iii) Take intersection of all the elements of C_A.

The second way to compute $\bar{A}$ is to figure out the smallest closed superset of A.

Examples-

Let $X=\mathbb{R}$ with usual topology and $A=(0,2]$.

Then the smallest closed superset of A is $[0,2]$.

So $\bar{A}=[0,2]$

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Example:-

Let $X = \mathbb{R}^2$ with usual topology and $A = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 < 1\}$ Then the smallest closed superset of A is

$\{(x,y) \in \mathbb{R}^2 / x^2+y^2 \leq 1\}$.

So,

$\bar{A} = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 \leq 1\}$.

The third way to compute $\bar{A}$ is to use the relation b/w the $\bar{A}$, A and A' given in the following theorem.

Theorem 1:-

Let A be a subset of a topological space X. Then

$\bar{A} = A \cup A'$

Theorem 2:-

Let A be a subset of a topological space X. Then $x \in \bar{A}$ iff for every open set U_x containing x intersect A.

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Theorem 1:-

Let A be a subset of a topological space X. Then

$\bar{A} = A \cup A'$

Proof:-

Let us first prove that

$A \cup A' \subset \bar{A}$

Now $A \subset \bar{A}$

Now let $x \in A'$

$\Rightarrow \forall U_x$ (open set containing x)

$(U_x \setminus \{x\}) \cap A \neq \phi$

$\Rightarrow U_x \cap A \neq \phi \Rightarrow x \in \bar{A}$ (By Th. 2)

$\Rightarrow A' \subset A$

$\Rightarrow A \cup A' \subset \bar{A}$

Now let us prove the converse

$\bar{A} \subset A \cup A'$

Let $x \in \bar{A}$ if $x \in A$ ~~also~~ we have done.

$\Rightarrow \bar{A} \subset A \subset A \cup A'$

If $x \notin A$ But $x \in \bar{A}$ (By Thr. 2)

$\forall U_x, U_x \cap A \neq \phi$

$(U_x \cap A \setminus \{x\}) \neq \phi$

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$\Rightarrow x \in A'$

$\Rightarrow x \in \underline{A} \cup A'$

$\Rightarrow \bar{A} \subset A \cup A'$

$\Rightarrow \bar{A} = A \cup A'$

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Topic 29:-

Properties of Closure of a set:-

Let us prove some results related to closure of a set.

Proposition:-

Let A be a subset of topological space X. Then

(i) $\bar{A}$ is closed subset of X.

(ii) A is closed iff $\bar{A} = A$.

(iii) $\overline{(\bar{A})} = \bar{A}$

Proof:-

(i) $\bar{A}$ is a closed subset of X.

By def. $\bar{A} = \bigcap_{C \in C_A} C$ C is closed superset of A.

We know that the intersection of closed set is closed.

$\Rightarrow$ $\bar{A}$ is closed.

(ii) A is closed iff $\bar{A} = A$

Let A is closed $\because A \subset \bar{A}$ and $\bar{A}$ is smallest closed superset of A.

$\Rightarrow \bar{A} \subset A$.

$\bar{A} = A$

If $\bar{A} = A$ is proved

$\downarrow$

closed

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(iii) $\overline{(\bar{A})} = \bar{A}$ $\because \bar{A}$ is closed (By 2)

$\overline{(\bar{A})} = \bar{A}$

Theorem :-

Let X be a topological space and $A, B \subset X$. If $A \subset B$ then $\bar{A} \subset \bar{B}$.

Proof :-

$A \subset B$

$\Rightarrow A' \subset B'$

$\Rightarrow (A \cup A') \subset (B \cup B')$

$\bar{A} \subset \bar{B}$.

Theorem :-

Let X be a topological space and $A, B \subset X$. Then

$\overline{A \cup B} = \bar{A} \cup \bar{B}$

Proof :-

First we prove that $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$

By privious theorem

$A \subset A \cup B$

$\Rightarrow \bar{A} \subset \overline{A \cup B}$

Similarly $\bar{B} \subset \overline{A \cup B}$

$\Rightarrow \bar{A} \cup \bar{B} \subset \overline{A \cup B}$

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Now let us prove that

$\overline{A \cup B} \subset \bar{A} \cup \bar{B}$

We know that

$A \subset \bar{A}$, $B \subset \bar{B}$

$\Rightarrow A \cup B \subset \bar{A} \cup \bar{B}$

$\Rightarrow \overline{A \cup B} \subset \overline{\bar{A} \cup \bar{B}} = \bar{A} \cup \bar{B}$

$\because$ But $\bar{A}$, $\bar{B}$ both are closed

$\Rightarrow \bar{A} \cup \bar{B}$ is closed

$\because$ But we know that a set is closed iff set = closed itself.

$\Rightarrow \overline{A \cup B} \subset \bar{A} \cup \bar{B}$

$\Rightarrow \overline{A \cup B} = \bar{A} \cup \bar{B}$.

proved. ——.

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Topic # 30:-
Dense Set:-

Defination:-

Let A be a subset of a topological space X. Then A is said to be dense in X iff every point $p \in X$ is either point of A or limit point of A, or in other words

A is dense in X iff $\bar{A} = X$.

Example 1:-

Let $X = \{a, b, c, d, e\}$ and

$\tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$

be a topology of X.

Consider $A = \{a, b, c\}$, $B = \{b, e\} \subset X$.

Then A is dense in X since

$\bar{A} = \{a, b, c, d, e\}$

But $B = \{b, e\}$ is not dense in X since

$\bar{B} = \{b, e\} \neq X$.

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Example 2:-

Let $X = \mathbb{R}$ with usual topologies and consider $\mathbb{Q}$. Then

$$\overline{\mathbb{Q}} = \mathbb{R}$$

So $\mathbb{Q}$ is dense in $\mathbb{R}$.

Now consider $\mathbb{N}$

Then $\overline{\mathbb{N}} = \mathbb{N} \neq \mathbb{R}$

So $\mathbb{N}$ is not dense in $\mathbb{R}$.

Remarks:-

→ X is dense in itself.

→ A proper closed subset of X can never dense in X.

Proof:-

$\Rightarrow \overline{X} = X$ $\quad$ X is dense in Y

$\Rightarrow A \underset{\text{closed}}{\subset} X$ $\quad$ $\bar{A} = A \neq X$

Proved.

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Topic 31:
Convergent Sequence :-

Let (X, T) be a topological space and $\{x_1, x_2, x_3 \cdots\} \subset X$ be a sequence of points in X. The sequence $\{x_n\}$ converges to a point $x \in X$ if for every U_x (open set containing x) in X there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the point $x_n \in U_x$

If $\{x_n\}$ converges to x then we say that x is the limit of the sequence $\{x_n\}$ and it is written as

$\lim_n x_n =$ or $x_n \to x$

Examples:-

$\mathbb{R}$ is usual topology:

$\{1, 1, \cdots\} \to 1$

$1 \in U$, open set

$(a, b) \in 1$

$\to \{1, 2, 3, \cdots 1\mathcal{B}, \cdots\}$, $x \in \mathbb{R}$.

$(a, b) \in 1\mathcal{B}$

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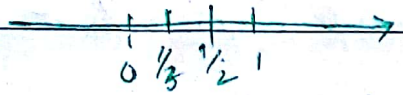
Topic 32:-

Example: Convergent Sequence:-

Example:-

Consider $\mathbb{R}$ with usual topology and consider the sequence

$$\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$$



$\forall\ U_0\ \exists\ N \in \mathbb{N}$

s.t. $x_n \in U_0\ \forall\ n \geq N$

Examples:-

Let X be an indiscrete space and $\{x_n\} \subset X$ be a sequence of points in X. Let y be a point in X. Now note that

There is only one open set containing y that is X itself.

X contains all the terms of $\{x_n\}$

So $x_n \to y$.

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Remark:-

In an indiscrete space X any sequence in X converges to any point in X.

Example:-

Let X be a discrete space and $\{x_1, x_2, x_3, \ldots\} \subset X$ be a sequence of points in X. Let y be a point in X. Now ~~Note~~ note that

→ There is a single ton open set containing y that is $\{y\}$. So x_n can not converge to y unless there exists an $N \in \mathbb{N}$ such that $x_n = y$ for all $n \geq N$.

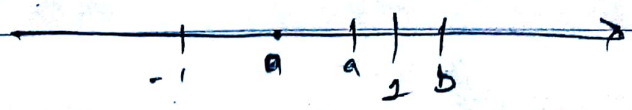
$\{x_1, x_2, x_3 \ldots x_N = y_1, y_2, \ldots\} \Rightarrow y$

Example:-

Consider $\mathbb{R}$ with usual topology and consider the sequence

$\{(-1)^n\}$

$= \{-1, 1, -1, \ldots\}$



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$n \in \mathbb{N}$ such that

$x_n \in (a, b)$

$x_n \in (1, -1)$

Example:-

Consider $\mathbb{R}$ with partition topology

$T_p = \{[4a-2, 4a+2] \mid a \in \mathbb{Z}\}$

and consides the sequence

$\{(-1)^n\}$.

$T_p = \{\ldots [-6, -2), [-2, 2), [2, 6), [6, 10), \ldots\}$

$(-1)^n \longrightarrow x$

$x \in [-2, 2)$.

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Topic 33:-
Interior Point:-

Defination :-

Let (X, T) be a topological space and A be a subset of X. A point $a \in A$ is an interior point of A iff there exists U_a (open set containing a), such that $U_a \subset A$.

Example 1:-

Let $X = \{a, b, c, d\}$ and

$T = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$

be a topology of X. Let

$$A = \{a, b, c\}$$

$a \in A$

a is an interior point of A.

$b \in A$

b is an interior point of A.

$c \in A$

c is an interior point of A.

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Example 2:-

Let $X = R$ with usual topology and $A = [0,1)$.

$\frac{1}{2} \in (0.3; 0.7) \subset A$

is open subset.

$\frac{1}{2}$ is an interior of A.

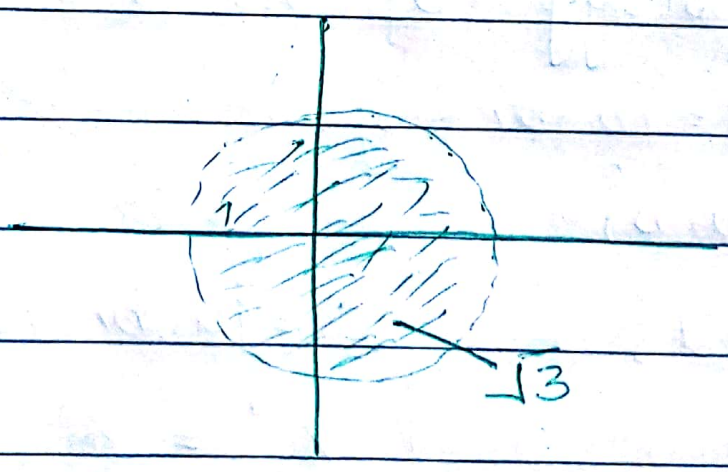
$1 \notin A$

$0 \in A$ $\nexists\ U_0 \subset A \in O$

Example 3:-

$X = \mathbb{R}^2$ with usual topology.

$A = \{(x,y) \mid x^2 + y^2 < 3\}$



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Topic 34:- Interior of a set

Interior of a set:-

Let (X, T) be a topological space and A be a subset of X. Interior of A is the set of all interior points of A.

i.e., $\{a \in A / \exists U_a \in T, a \in U_a \subset A\}$

Interior of A is denoted by $\text{int}(A)$ or A°.

Example 1:-

Let $X = \{a, b, c, d\}$ and $T = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X. Let $A = \{a, b, c\}$.

$A^\circ = \{a, b\}$

$B = \{c, d\}$ | $C = \{a, b, c, d\} = X$

$B^\circ = \phi$ | $X^\circ = X$

Example 2:-

Let $X = \mathbb{R}$ with usual topology?

$A = (0, 1]$

$A^\circ = (0, 1)$

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$B = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$B^\circ = \phi$

ϕ $\mathbb{Q}$, $\mathbb{Q}^\circ = \phi$

Example 4:-

Let $X = \mathbb{R}^2$ is a usual topology.

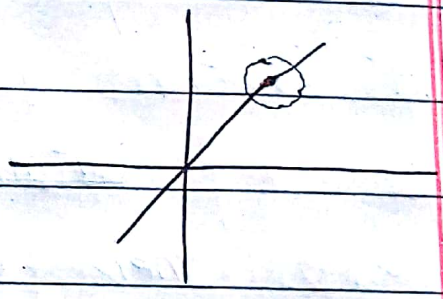
$A = \{(x,y) \mid x^2 + y^2 \leq 1\}$

$A^\circ = \{(x,y) \mid x^2 + y^2 < 1\}$



$B = \{(x,y) \mid x = y\}$

$B^\circ = \phi$



Remarkse-

$A^\circ \subset A$

$\phi^\circ = \phi$

$X^\circ = X$.

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Topic 35:- Properties of Interior

Proposition:-

Let A be a subset of topological space X. Then

(i) A° is the union of all open subsets of X contained in A.

(ii) A° is open in X.

(iii) A° is the largest open subset of X ~~cotain~~ contained in A.
i.e., If there exists an open subset B contained in A then $B \subset A^\circ \subset A$.

(iv) A is open iff $A = A^\circ$.

Proof:-

(i) $A^\circ \subseteq \bigcup U_i$

Let $a \in A^\circ$

$\exists\ U_a$ s.t.

$a \in U_a \subset A$

$\Rightarrow A^\circ \subset \bigcup U_i \rightarrow (1)$

Let U = The class of all open subsets of X contained in A $U_i \in U$.

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Now let $x \in \bigcup U_i \Rightarrow \exists\, j$ s.t. $x \in U_j$

But U_j is open subset of A.

$x \in A^\circ$

$\Rightarrow x \in \bigcup_i U_i A^\circ \longrightarrow (2)$

(1) and (2)

$\Rightarrow A^\circ = \bigcup_i U_i$

(2) A° is open

(1) $A^\circ = \bigcup_i U_i$ is open subset.

(3) A° is the largest open subset of X contained in A

i.e.,

If $\exists B$ is of open $\in \mathbb{R}$ s.t. $B \subset A$

then $B \subset A^\circ \subset A$

$A^\circ = \bigcup_i U_i \quad \exists\, j$ s.t. $U_j = B$

$B \subset A^\circ$

(4) A is open iff $A^\circ = A$

If A is open then by (3)

$A \subset A^\circ \subset A$

$\Rightarrow A^\circ = A$

If $A = A^\circ$ is open subset.

$\Rightarrow A$ is open subset.

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Topic 36
Exterior of a set

Exterior Point:-

Let (X, τ) be a topological space and A be a subset of X. A point $b \in A^c$ is an exterior point of A iff there exists U_b (open set containing b). such that $U_b \subset A^c$.

i.e., Exterior point of A is interior point of A^c.

Exterior of a set:-

Let (X, τ) be a topological space and A be a subset of X. The exterior of A, denoted as $Ext(A)$ is the interior of A^c i.e.,

$$Ext(A) = (A^c)^\circ$$

Example:-

Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X. Let

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$A = \{a, c\}$

$Ext(A) = (A^c)^\circ$

$A^c = \{b, d\}$

$(A^c)^\circ = \{b\}$

$B = \{d\}$

$B^c = \{a, b, c\}$

$(B^c)^\circ$ $Ext(B) = B^c$

Example :-

Let $X = \mathbb{R}$ with usual topology and $A = (-1, 2)$.

$Ext(A) = (A^c)^\circ$

$= ((-\infty, 1] \cup [2, \infty))^\circ$

$= (-\infty, 1) \cup (2, \infty)$

(2)

$Ext = (Q^c)^\circ = (Q')^\circ = \phi$

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Topic 37
Boundary of a set

Boundary Point :-

Let (X, T) be a topological space and A be a subset of X. A point $x \in X$ is called a boundary point of A iff for all U_x. (open set containing x).

$$U_x \cap A \neq \emptyset$$

and $$U_x \cap A^c \neq \emptyset$$

Boundary point is also called frontier point.

Boundary of a set :-

Let (X, T) be a topological space and A be a subset of X. Boundary of A is the set of all boundary points of A. Boundary of A is denoted as $fr(A)$ are $bd(A)$.

$$bd(A) = (A^\circ \cup Ext(A))^c$$

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Example 1:-

Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X. Let

$A = \{a, b, c\} \Rightarrow A^c = \{d\}$

c is a boundry point of A.

$\{a, c, d\} \cap A \neq \phi$ | $\{a, c, d\} \cap A' \neq \phi$

d is also a boundary point of A.

"a" $\subset \{a\}$ ✗

$\{a\} \cap A^c = \phi$

Example 2:-

Consider $X = \mathbb{R}$ with usual topology.

$A = [0, 2)$ "1" No

'0' Yes.

$(0.9, 1.5) \cap A^c = \phi$

$0 \in (c, d)$

$c < 0 < d$

$bd(A) = \{0, 2\}$

$\mathbb{B} = \mathbb{Q}$ $x \in \mathbb{R}$ $x \in U_x$ open interval

$bd(\mathbb{Q}) = \mathbb{R}$ $U_x \cap \mathbb{R} \neq \phi$ | $U_x \cap \mathbb{Q} \neq \phi$

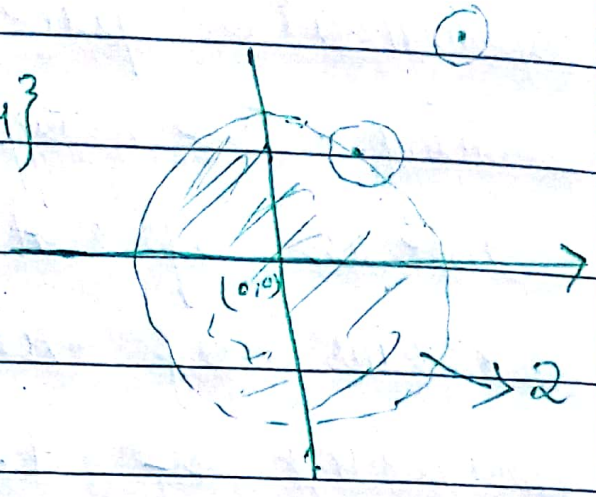
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Example 3:-

Consider $X = \mathbb{R}^2$ with usual topology.

Let $A = \{(x,y) \mid x^2+y^2 < 4\}$

$bd(A) = \{(x,y) \mid x^2+y^2 = 4\}$



Remarks:-

* bd(A) is closed set.
* Boundary of a set A is empty iff A is both open and closed.

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Topic 38

Partition of a set :-

For a subset A of a topological space X we discussed following three types of sets.

Interior of A.

Exterior of A.

Boundary of A.

The set X can be partitioned by these three sets i.e.,

$$X = A^\circ \cup Ext(A) \cup bd(A)$$

$X = \mathbb{R}$, $A = (0, 1]$

$$A^\circ = (0, 1]$$

$$Ext(A) = (A^c)^\circ$$

$$= (-\infty, 0) \cup (1, \infty)$$

$$bd(A) = \{0, 1\}$$

$$X = \mathbb{R} = \underbrace{(0, 1)}_{\text{int}} \cup \underbrace{(-\infty, 0) \cup (1, \infty)}_{\text{exterior}} \cup \underbrace{\{0, 1\}}_{\text{bd}}$$

$A \subset X$, $X = A^\circ \cup Ext(A) \cup bd(A)$ → (1)

$A^\circ \subset X$, $Ext(A) \subset X$, $bd(A) \subset X$

$\Rightarrow A^\circ \cup Ext(A) \cup bd(A) \subset X$

Let $x \in X$, $A \subset X$

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$\Rightarrow x \in A$ or $x \in A^c$

$x \in A^\circ$, ~~$x \notin A^\circ$~~

$x \subset A^\circ \cup Ext(A) \cup bd(A)$

$x \notin A^\circ$

$\nexists U_x$, $x \in U_x \subset A$

$\Rightarrow \forall U_x$, $U_x \cap A \neq \phi$

and $U_x \cap A^c \neq \phi$

$\Rightarrow x \in bd(A)$

Hence proved.

$x \in A^c$

$x \in Ext A = (A^c)^\circ$

$\Rightarrow x \subset Ext A \subset A^\circ \cup Ext(A) \cup bd(A)$

$x \notin (A^c)^\circ$

$\forall U_x$

$U_x \not\subset A^c$

$\Rightarrow \forall$ U_x superset containing y.

$U_x \cap A^c \neq \phi$

$U_x \cap A \neq \phi$

$x \subset A^\circ \cup Ext(A) \cup bd(A)$

$\Rightarrow x = A^\circ \cup Ext(A) \cup bd(A)$

$\Rightarrow x \in bd(A)$

$x \subset bd(A)$

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Topic 39:-

Neighborhood System:-

Neighborhood of a point:-

Let p be a point in a topological space (X, τ). A subset N of X is a neighborhood of p iff there exists an open set U_p such that

$$p \in U_p \subset N.$$

One may say that N is a neighborhood of p iff p is an interior point of N.

Open Neighbor hood:-

A neighborhood N of a point p is called an open neighborhood iff N is a neighborhood of all of its points. i.e., N is a open set.

~~Neib~~ Neighborhood ~~system~~:-

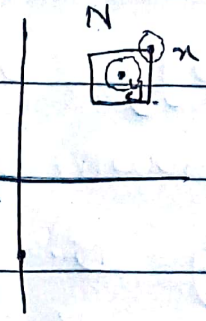
The class of all neighborhoods of $p \in X$ is ~~called~~ called the neighborhood system of p. It

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is denoted by N_p.

Examples:-

$X = \mathbb{R}^2$



Example:-

Let $X = \{a, b, c\}$ and $T = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ be a topology of X.

$\{a, b\}$ is a neighborhood of b.

Is $\{b, c\}$ is neighborhood of b.

No. $\nexists\ U_b$

$b \in U_b \subset \{b, c\}$

Examples:-

Let $X = \mathbb{R}$ with usual topology?

$A = [0, 1)$, $B = (-1/2, 1/2)$, $C = \{1, 2, 3\}$

A is not a neighborhood of '0'

"1" $1 \notin A$

B is a neighborhood of (0)

C is neighborhood of 1, 2, 3.

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Example-

Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X.

Let us list down all the neighborhoods of b.

$\{a, b\}, \{a, b, c\}, \{a, b, d\}, X$

So the neighborhood system of b is

$N_b = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$.

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Topic 40:-

Properties: Neighborhood System

Proposition:-

Let N_p is the neighborhood system of p. Then

(i) N_p is non-empty.

(ii) $p \in \bigcap_{N_p \in N_p} N_p$.

(iii) N_p is closed under finite set intersection.

(iv) Every superset of a member of N_p belongs to N_p.

Proof:-

(i) Note that X always is the neighborhood of p i.e., $X \in N_p$
N_p is non-empty.

(ii) $p \in N \quad \forall \quad N \in N_p$
By def. N is neighborhood of p iff $p \in N$ and $\exists$ up s.t. $p \in \cup p \subset N$
$\Rightarrow p \in \bigcap_{N_p \in N_p} N_p$.

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(iii) Let $N, M \in \mathcal{N}_p$

$\Rightarrow \exists\ U_p$ is open, such that $p \in U_p \subset N$ and $p \in U_p \subset M$

$\Rightarrow p \in U_p \cap V_p \subset N \cap M$

$N \cap M$ is a neighborhood of p.

$\Rightarrow N \cap M \in \mathcal{N}_p$.

(iv) $N \in \mathcal{N}_p \Rightarrow \exists\ U_p$ s.t. $p \in U_p \subset N$

Let $N \subset M$

$p \in U_p \subset M$

M is a neighborhood of p.

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Topic 41:- Subspace Topology

Subspace Topology

Let A be a nonempty subset of a topological space $(X, \mathcal{T})$. Then the collection

$\mathcal{T}_A = \{V = U \cap A \mid U \in \mathcal{T}\}$

of subsets of defines a topology on A.

$\mathcal{T}_A$ is called subspace topology or relative topology on A relatives to $(X, \mathcal{T})$.

Example :-

Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X. Let $A = \{a, b, c\}$.

Solution :-

$\mathcal{T}_A = \{\phi, \{a\}, \{a, b\}, \{a, c\}, A\}$

Example :-

$X = \mathbb{R}$ with usual topology for $A = \mathbb{Z}$.

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Solution:-

Let $(-1/2, 1/2) \cap \mathbb{Z} = \{0\}$ $\forall x \in \mathbb{Z}$

$\exists \cdot U \in \mathcal{T}_R$.

s.t. $\{x\} = U \cap \mathbb{Z}$

$\mathcal{T}_R$ is discrete topology on $\mathbb{Z}$ relative to $\mathcal{T}$.

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Topic 42 :- Proof: Subgroup Space Topology

Proof:

$T_A = \{V = U \cap A \mid U \in T\}$

(i) $\cdot \phi \in T_A$ $\because \phi \in T$, $\phi = \phi \cap A$

(ii) $\cdot A \in T_A$ $\because X \in T$, $A = X \cap A$

(iii) $\rightarrow V_i \in T_A \Rightarrow U_i V_i \in T_A$

$\Rightarrow \exists\ U_i \in T$

s.t. $V_i = U_i \cap A$

Now $U_i V_i = U_i (U_i \cap A)$

$= (\cup U_i) \cap A$

$\in T$

(iii) Let $V_1, V_2 \in T_A$

$\Rightarrow \exists\ U_1$ and $U_2 \in T$

s.t. $V_1 = U_1 \cap A$ and

$V_2 = U_2 \cap A$

$\Rightarrow V_1 \cap V_2 = (U_1 \cap A) \cap (U_2 \cap A)$

$= (U_1 \cap U_2) \cap A$

$\Rightarrow V_1 \cap V_2 \in T_A$

T_A is a topology on A.

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Topic 43
Relative open sets

Relative open sets:-

Let (A, T_A) be a subspace of a topological space (X, T). Then a subset H of A is open in A ~~and~~ or relative open to A iff there exist an open subset G of X such that

$$H = G \cap A$$

One may say that the elements of T_A are the open subset of A or called relative open subset of A

Example:-

Let $X = \{a, b, c, d\}$ and

$T = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$

be a topology of X. Let

$A = \{a, b, c\}$

$T_A = \{\phi, \{a\}, \{a, b\}, \{a, c\}, A\}$

Solution:-

$\{a\} = \{a\} \cap A$

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$\{b\} \subset A$ N.

$\{b\} = \square \cap A$

Example :-

Let $X = \{a, b, c, d\}$ and

$T = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$

be a topology of X. Let

$A = \{a, b, c\}$.

Solution :-

$\{c\} \subset A$, $\{c\} = \square \cap A$

$\{a, c\} \subset A$

$\{a, c\} = \{a, b, c\} \cap A$

Example:-

Let $X = \mathbb{R}$ with usual Toplogy.

Consider $A = (0, 1]$.

Solution :-

$(0, 1/2) \subset A$

$(-1/2, 1/2) \cap A = (0, 1/2)$

Is $(0.7, 1)$ open in A? Yes.

$(0.7, 1] = (0.7, 3) \cap A$.

Example:-

Let $X = \mathbb{R}$ with usual topology?

Consider $A = \mathbb{Z}$.

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Solution:-

J_x @ is discrete.

$\{n\} \subset \mathbb{Z}$.

which is open.

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Topic 44:-
Basis for a topology:-

Basis for a topology:-

Let (X, T) a topological space. A sub collection $B \subset T$ is called a basis for T iff $\forall\ U \in T$ and $\forall\ x \in U$. There exists $B \in B$ s.t.

$$x \in B \subset U.$$

Remark:-

If B is a basis for a topology T on X then every element of T can be written as union of the elements of B.

Proof:-

By def. of Basis $\forall\ U \in T$

$\forall\ x \in U \quad \exists\ B_x \in B$ s.t.

$$x \in B_x \subset U.$$

$$U_x = \bigcup_{x \in U} B_x.$$

Remark: Let B be the basis for T on X. Then every supper

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class B^x of open subsets of x of B is also a basis for T. i.e.,

$\forall \; B \subset B^x \subset T$

B^x is a basis.

Proof:-

B is a basis so, $\forall \; U \in T$

$\forall \; x \in U \;\; \exists \; B \in B \cap B^x$ s.t.

$x \in B \subset U$.

$\Rightarrow \forall \; U_x \in T \;\; , \;\; \forall x \in U$

$B \in B^x$ s.t.

$x \in B \subset U$.

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Topic 45:-

Examples: Basis for a Topology

Example 1:-

Let $X = \{a, b, c, d\}$ and $T = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X.

Consider $B = \{\{a\}, \{b\}, \{a, c, d\}\}$.

Solution:-

$\phi = \cup$ {empty subclass of B}

$\{a\} = \{a\}$

$\{b\} = \{b\}$

$\{a, b\} = \{a\} \cup \{b\}$

$\{a, c, d\} = \{a, c, d\}$

$X = \{b\} \cup \{a, c, d\}$

Example 2:-

Let $X = \{a, b, c, d\}$ and $T = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X. Consides $B = \{\{a\}, \{b\}, \{a, b, c\}\}$

Solution:-

$X \bigcup_{i \in I} B_i$

$U = x \in T$

$d \in X \nexists B \in B.$

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Example 3:-

Consider $\mathbb{R}$ is usual Topology.
consider $\mathcal{B} = \{(a,b) : a,b \in \mathbb{R}\}$

Solution:-

$(c,d) \in T_x$, $(c,d) \in \mathcal{B}$

$x \in (c,d) \cup (e,f) \in T_x$

$B \in \mathcal{B}$

$x \in B \subset (c,d) \cup (e,f)$

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Topic 46:-

~~Criticism~~ Criteria for a collection to be Basis

As we discuss for a given topology one can write a collection of open subsets of X that satisfies the condition of basis.

Now the question is "Can a topology be formed by a given collection B of subsets of X by using set union operation? If "yes" then what should be the conditions on that collection?

One condition is very much clear that is

$$X = \bigcup \{B : B \in \mathcal{B}\}$$

because X is always open in every topology on X.

Is this condition enough for any collection to be a basis?

To understand this let us

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look example:-

Example:-

Let $X=\{a,b,c\}$ and $B=\{\{a,b\},\{b,c\}\}$.

Let us generate the elements from this collection.

Solution:-

$T=\{\phi,\{a,b\},\{b,c\},\{a,b,c\}\}$

$\{a,b\}\cap\{b,c\}=\{b\}\notin T$

T is not a topology on X.

Theorem :-

Let B be a class of subsets of non-empty set X. Then B forms a basis for some topological on X if and if B satisfies the following conditions:

(i) $\forall x\in X \ \exists B$ s.t. $x\in B$

(ii) For $B_1, B_2\in B$ and $\forall x\in B_1\cap B_2$, $\exists B_3\in B$ such that $x\in B_3\subset B_1\cap B_2$.

~~Solution:~~ Example:-

Let $X=\{a,b,c\}$ and $B=\{\{c\},\{a,c\},\{b,c\}\}$. It is clear that B satisfies both conditions stated

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in the theorem. Let us see generate the elements from this collection.

Solution:-

$T = \{\phi, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X\}$.

Example:-

Let $X = \{a,b,c,d\}$ and $B = \{\{a,b\}, \{c,d\}, \{d\}\}$. Let us generate the elements from this collection.

Solution:-

$T = \{\phi, \{d\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{a,b,d\}, \{a,b,c,d\}, X\}$

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Topic 47:-

Basis for some Topologies:-

In this topic we will discuss the basis of indiscrete topology and discrete topology.

Basis for indiscrete Topology:-

Consider the indiscrete space. That is, a non-empty set with $T = \{\phi, X\}$.

It is clear that there is only one nonempty open set in this collection that is X so basis of indiscrete space is $\{x\}$.

Basis of Discrete Topology:-

Example:-

Let $X = \{a, b, c\}$, consider the set of all single tons i.e.,

$$B = \{\{a\}, \{b\}, \{c\}\}$$

It is clear that this collection satisfies both conditions of basis. Let us generate the topology

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from B. Following is the topology

$T = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, X\}$

It is discrete topology.

Remark:-

Consider a discrete space. That is, a nonempty set X with $T = P(X)$. Then the set of all single tons forms a basis for discrete topology.

i.e; $B = \{\{x\} | x \in X\}$

Proof:-

$\rightarrow \forall \; x \in X \;\; \exists \;\; B = \{x\} \in B$

$B_1, B_2 \in B, \; x \in B_1 \cap B_2 \; \exists \; B_3 \in B$

$x \in B_3 \subset B_1 \cap B_2$

$B_1 \cap B_2 = \phi$

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Topic 48
Basis for Usual Topology on $\mathbb{R}^n$:-

Let us discuss usual basis for the usual topologies $\mathbb{R}^n_u$ on $\mathbb{R}^n$.

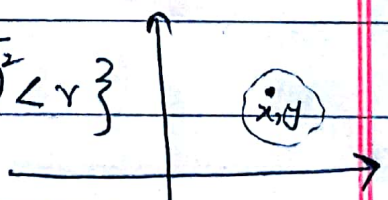
Basis for usual on $\mathbb{R}$:-

Consider $\mathbb{R}_u$ i.e., $\mathbb{R}$ with usual topology. The usual topology basis for $\mathbb{R}_u$ is the set of all open intervals.

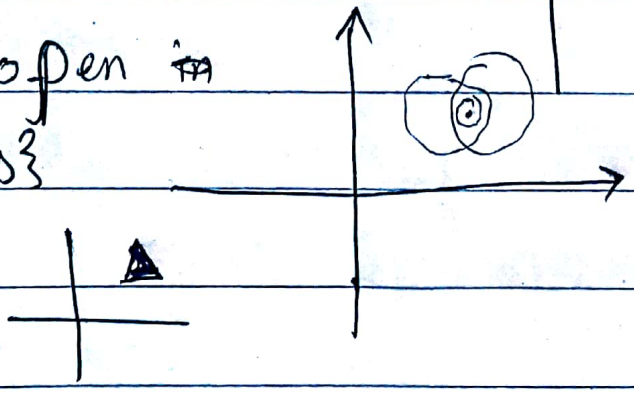
Basis for usual on $\mathbb{R}^2$:-

Consider $\mathbb{R}^2_u$ i.e., $\mathbb{R}^2_u$ with usual topology. The usual basis for $\mathbb{R}^2_u$ is the set of all open disks.

$$B = \{(x',y') \mid \sqrt{(x-x')^2+(y-y')^2} < r\}$$



$B_\Delta = \{$All open triangles$\}$



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Basis for usual on $\mathbb{R}^n$:-

Consider $\mathbb{R}^n_u$ i.e., $\mathbb{R}^n$ with usual topology. The usual basis for $\mathbb{R}^n_u$ is the set of all open n-Balls.

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Topic 49:-
Topology on Real line:-

We have already discussed usual topology on $\mathbb{R}$. Now we see ~~that~~ more topologies on $\mathbb{R}$ named as

→ Lower limit topology, $\mathbb{R}_l$.
→ Upper limit topology, $\mathbb{R}_{up}$
→ K-topology, $\mathbb{R}_K$.

Lower limit topology:-

The lower limit topology on $\mathbb{R}$ is denoted by $\mathbb{R}_l$.

$$B_l = \{[a,b) \mid a,b \in \mathbb{R}\}$$

$x \in \mathbb{R}$ $\quad a \leq x < b$.

$x \in [a,b)$

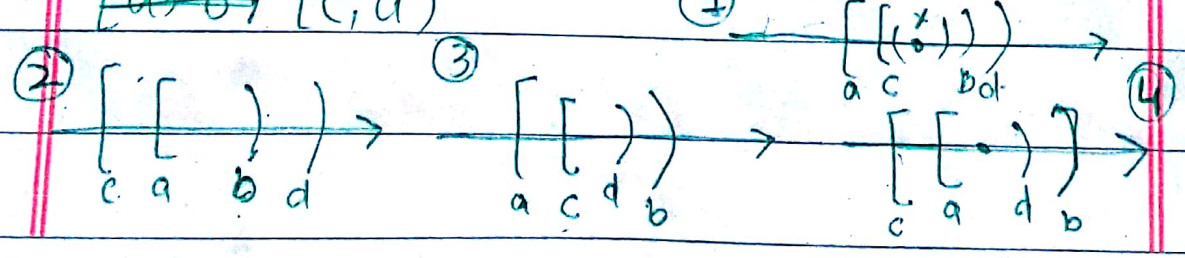
$B_1, B_2 \in B_l$ $\quad x \in B_1 \cap B_2$

$\exists\ B_3 \subset B_l$

Such that

$[a,b)$ $\quad x \in B_3 \subset B_1 \cap B_2$

$[a,b)$ $[c,d)$



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Upper-limit topology:-

The upper limit topology on $\mathbb{R}$ is denoted by $\mathbb{R}_{up}$

$$B_{up} = \{(a,b] \mid a,b \in \mathbb{R}\}$$

K-Topology on $\mathbb{R}$:-

Let

$$K = \{1, 1/2, 1/3, \dots\}$$

be a subset of $\mathbb{R}$ then the K-topology on real lines is denoted as $\mathbb{R}_K$ and its basis as

$$B_K = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{(a,b) \setminus K \mid a,b \in \mathbb{R}\}$$

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Topology Topic 50

Basis for subspace:-

Recall subspace Topology:-

Topology on a subset A of a topological space $(X, \mathcal{T})$ i.e., the collection of subsets of A defines as

$$\mathcal{T}_A = \{V = U \cap A \mid U \in \mathcal{T}\}$$

The sub collection

$$\mathcal{B}_A = \{B_A = B \cap A \mid B \in \mathcal{B}\}$$

forms a basis for $\mathcal{T}_A$, where $\mathcal{B}$ be a basis for $\mathcal{T}$.

Example:-

Let $X = \{a, b, c, d\}$ and

$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, X\}$

be a topology on X. Now

$\mathcal{B} = \{\{a\}, \{b\}, \{a, c, d\}\}$ be a basis for $\mathcal{T}$. Let $A = \{a, b, c\}$

Solution:-

$\mathcal{T}_A = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, A\}$

$\mathcal{B}_A = \{B_A = B \cap A \mid B \in \mathcal{B}\}$.

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Recall subspace Topology :-

Topology on a subset A of a topological space $(X, \mathcal{T})$ i.e., the collection of subsets of A defines as

$$\mathcal{T}_A = \{V = U \cap A \mid U \in \mathcal{T}\}$$

The sub collection

$$\mathcal{B}_A = \{B_A = B \cap A \mid B \in \mathcal{B}\}$$

forms a basis for $\mathcal{T}_A$, where $\mathcal{B}$ be a basis for $\mathcal{T}$.

Example :-

Let $X = \{a, b, c\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Now $\mathcal{B} = \{\{a\}, \{b\}, \{a, c, d\}\}$ be a basis for $\mathcal{T}$. Let $A = \{a, b, c\}$.

Solution :-

$$\mathcal{T}_A = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, A\}.$$

$$\mathcal{B}_A = \{\{a\}, \{b\}, \{a, c\}\}$$

$$\therefore \mathcal{T}_A = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, A\}$$

Theorem :-

Let A be a subset of

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a topological space $(X, \mathcal{T})$. Then the collection

$$\mathcal{B}_A = \{B_A = B \cap A \mid B \in \mathcal{B}\}$$

of subsets of A forms a basis for $\mathcal{T}_A$, where $\mathcal{B}$ be a ~~subspace~~ basis for $\mathcal{T}$.

Proof:-

We know that

$$\mathcal{T}_A = \{V = U \cap A \mid U \in \mathcal{T}\}$$

$\rightarrow \forall \quad V \in \mathcal{T}_A$

$$V = \bigcup_{i \in I} B_{Ai} = \bigcup_{i \in I} (B \cap A)_i$$

$$V = U \cap A \qquad U \in \mathcal{T}$$

$\because \mathcal{B}$ is a basis of Topology

But $\quad U = \bigcup_{i \in I} B_i$

$$V = \left(\bigcup_{i \in I} B_i\right) \cap A$$

$$= \bigcup_{i \in I} (B_i \cap A) \qquad B_i \in \mathcal{B}$$

$$\in \mathcal{B}_A$$

So, $\mathcal{B}_A$ forms a basis of Topology.

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Topic 51:-

Basis: Comparison of Topologies

Theorem:-

Let X be a nonempty set with topologies $\mathcal{T}$ and $\mathcal{T}'$ having basis $\mathcal{B}$ and $\mathcal{B}'$ respectively. Then following are equivalent.

(i) $\mathcal{T}' \subset \mathcal{T}$

(ii) For every $B' \in \mathcal{B}'$ and for every $x \in B'$ there exists an element $B \in \mathcal{B}$ such that $x \in B \subset B'$.

Proof:-

1) $\Longrightarrow$ 2)

Let $\mathcal{T}' \subset \mathcal{T}$

Now, let $B' \in \mathcal{B}' \subset \mathcal{T}' \subset \mathcal{T}$

$\Rightarrow B' \in \mathcal{T}$ ~~but~~

But $\mathcal{B}$ is a basis ~~for~~ of topology.

So by basis defination

$\forall\ x \in B'\ \exists\ B \in \mathcal{B}$

s.t. $x \in B \subset B'$

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2) $\Rightarrow$ 1)

Let $U' \in T'$

$T' \subset T$

we have basis for T' and B'

$\forall \; x \in U' \; \exists \; B' \in \mathcal{B}'$

such that

$x \in B' \subset U' \rightarrow (1)$

But given that

$\forall \; x \in B' \; \exists \; B \in \mathcal{B}$

such that $x \in B \subset B' \subset U'$

$x \in B \subset U'$

$\Rightarrow U' = \bigcup_{i=1} B_i$

$\Rightarrow U' \in T$

$\Rightarrow T' \subset T.$

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Topic 52
Fineness of $\mathbb{R}_\ell$.

The lower limit topology $\mathbb{R}_\ell$ is finer than the usual topology $\mathbb{R}_u$.

Let us prove this fact.

Proof:-

Let us recall

$B_\ell = \{[a,b) \mid a,b \in \mathbb{R}\}$

$B_u = \{(a,b) \mid a,b \in \mathbb{R}\}$

Now for any $(a,b) \in B_u$

$\forall\ x \in (a,b) \quad a < x < b$

We can find an element

$[x,b) \in B_\ell$.

$x \in [x,b) \subset (a,b)$

$\Rightarrow \mathbb{R}_u \subset \mathbb{R}_\ell$

$\mathbb{R}_\ell \not\subset \mathbb{R}_u$

Let $[x,b) \in B_\ell$

$\nexists\ (a,b) \in B_u$

such that $x \in (a,b) \subset [x,b)$

$\Rightarrow \mathbb{R}_\ell \not\subset \mathbb{R}_{up}$.

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Topic 53:- Finesness of $\mathbb{R}_K$

The K-topology $\mathbb{R}_K$ is finer then the usual topology $\mathbb{R}_u$. Let us proove this fact.

Proof:-

$B_u = \{(a,b) \mid a,b \in \mathbb{R}\}$

$B_K = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{(a,b) - K \mid a,b \in \mathbb{R}, K = 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

$\mathbb{R}_u \subset \mathbb{R}_K$, $(a,b) \in B_u$

$x \in (a,b)$ $\exists$ $B \in \mathcal{B}_K$

such that $x \in (a,b) \subseteq B$.

$\rightarrow$ $\mathbb{R}_K \not\subset \mathbb{R}_u$

$B = (-1,1) \mid K \in B_K)$

$0 \in B$? no open interval (a,b) such that

$0 \in (a,b) \subset B$.

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Topic 54:-

Subbasis

Let (X, T) a topological space with basis B. A sub collection $S \subset T$ is called a subbasis for T iff every element of B can be written as finite intersection of elements of S.

i.e., $\forall\ B \in B$

$\exists\ S_1, S_2, \ldots, S_n \in S$.

$$B = \bigcap_{i=1}^{n} S_i$$

Remark:-

If S is a ~~subspace~~ subbasis for a topology T on X then every element of T can be written as union of finite intersection of elements of S.

Proof:-

By defination of subbasis

$\forall\ B_j \in B$, $B_j = \bigcap_{i=1}^{n_j} S_{j_i}$, $S_{j_i} \subset S$

Now by defination of basis

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$\forall\ U \in T$

$$U = \bigcup_{j \in J} B_j$$

where $B_j \in B$

$$= \bigcup_{j \in J} B_j$$

$$= \bigcup_{j \in J} \left(\bigcap_{i=1}^{n_j} S_{j_i} \right)$$

Example:-

Let $X = \{a, b, c, d\}$ and

$T = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$

be a topology of X. Now

$B = \{\{b\}, \{a, b\}, \{b, c\}, X\}$

($X = \cap$ {empty class})

The subcollection

$S = \{\{a, b\}, \{b, c\}\}$.

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Topic 55

Topology Generated by a collection:-

As we have already seen that any given class A, of subsets of a non-empty set X, may or may not be a basis for a topology on X. But A always able to generate a basis for a topology on X. Because by defination of subbasis, basis element can be generated by taking the intersection of finite number of elements of subbasis.

Theorem:-

Any class A of subsets of a nonempty set X is a subbasis for a unique topology T on X. i.e., finite intersection of members of A generates a basis B for a topology T on X.

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Proof :-

$\mathcal{B} = \{B / B = \bigcap_{i=1}^{n} A_i \quad A_i \in A\}$

We have to show that the following two conditions.

(i) $\forall\, x \in X \;\exists\; B \in \mathcal{B}$ such that $x \in B$ i.e., $X = \cup B_i$

(ii) For $B_1, B_2 \in \mathcal{B}$ and $\forall\, x \in B_1 \cap B_2 \;\exists\; B_3 \subset \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

Note that $x \in \mathcal{B}$

$\because X = \cap\{\text{empty class}\}$

So (1) is proved

→ Let $B_1, B_2 \in \mathcal{B}$

Then $B_1 = \bigcap_{i=1}^{n} A_i$,

$B_2 = \bigcap_{i=1}^{n} A_i' \qquad A_i', A_i \in A.$

Now $B_1 \cap B_2 = \left(\bigcap_{i=1}^{n} A_i\right) \cap \left(\bigcap_{i=1}^{n} A_i'\right)$

$B_3 = B_1 \cap B_2 \in \mathcal{B}$

A generates a unique topology on X.

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Topic 56:-

Example: Topology Generated by a Collection:-

Example 1:-

Let $X = \{a, b, c, d\}$ and ~~$A = \{\{c\},$~~

$A = \{\{c\}, \{a, b\}, \{b, c\}\}$

Following is the basis B generated by A.

$B = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$

Now B will generate J.

$J = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$

Example 2:-

Let $X = \{a, b, c, d, e\}$ and

$A = \{\{c, d\}, \{d, e\}, \{a, b, c\}\}$.

Following is the basis B generated by A.

$B = \{\phi, \{c\}, \{d\}, \{c, d\}, \{d, e\}, \{a, b, c\}, X\}$

And following is the topology J generated by B

$J = \{\phi, \{c\}, \{d\}, \{c, d\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, c, d\}, X\}$

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Topic 57:-
Example: Topology Generated by a Collection on $\mathbb{R}$:-

Example 1:-

Consider $\mathbb{R}$ and

$A = \{(-\infty, b), (a, \infty) \mid a, b \in \mathbb{R}\}$

Following is the basis $\mathcal{B}$ generated by A.

$\mathcal{B} = \{\phi, \{(-\infty, b), (a, \infty), (a, b), \mathbb{R} \mid a, b \in \mathbb{R}\}$

Now $\mathcal{B}$ is the basis for usual topology on $\mathbb{R}$.

So, A generates usual topology on $\mathbb{R}$.

Example 2:-

Consider $\mathbb{R}$ and

$A = \{[n, n+1] \mid n \in \mathbb{R}\}$.

Following is the basis $\mathcal{B}$ generated by A.

$\mathcal{B} = \{\phi, \{n\}, [n, n+1], \mathbb{R} \mid n \in \mathbb{R}\}$

Now, $\mathcal{B}$ is the basis for discrete topology on $\mathbb{R}$.

So, A generates discrete

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topology on $\mathbb{R}$.

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Topic 58
Subbasis for subspace

Recall Subspace Topology:-

Topology on a subset A of a topological space (X,T) i.e., the collection of subsets of A defines as

$$T_A = \{U \cap A \mid U \in T\}$$

The collection

$$S_A = \{S \cap A \mid S \in S\}$$

is a subbasis for T_A on A. where S be a subbasis for T.

Theorem:-

Let A be a subset of a topological space (X,T) and S be a subbasis for T. Then the collection

$$S_A = \{S \cap A \mid S \in S\}$$

is a subbasis for the relative topology T_A on A.

Proof:-

Let $V \in T_A$

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$\Rightarrow \exists\ U \in T$ such that $V = U \cap A$

$\because S$ is a subbasis for T. So,

$U = \bigcup_i \{S_{i_1} \cap S_{i_2} \cap \dots \cap S_{in_i}\}$ where

$S_{ij} \in S$

Now,

$$V = [\bigcup_i (S_{i_1} \cap S_{i_2} \cap \dots \cap S_{in_i}] \cap A$$

$$= \bigcup_i [(S_{i_1} \cap A) \cap (S_{i_2} \cap A) \cap \dots \cap (S_{in_i} \cap A)]$$

$\Rightarrow S_A$ is a subbasis for T_A

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Topic 59:-
Smallest Topology Containing Topologies:-

Let T_1 and T_2 be two topologies on a nonempty set X. We have already seen that $T_1 \cup T_2$ may or may not be a topology on X.

Moreover this topology is the smallest containing both T_1, T_2.

Example:-

Let $X = \{a, b, c, d\}$

$T_1 = \{\phi, \{a\}, \{a, b\}, X\}$

$T_2 = \{\phi, \{c\}, X\}$

Now

$T_1 \cup T_2 = \{\phi, \{a\}, \{c\}, \{a, b\}, X\}$

The topology generated by $T_1 \cup T_2$ is the following

$T_3 = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$.

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Topic 60:- Largest Topology Contained in Topologies

Let T_1 and T_2 be two topologies on a non-empty set X.

Question:-

Find the largest topology on X contained in T_1 and T_2?

Answer:-

$T_1 \cap T_2$ is the largest topology on X contained in T_1 and T_2.

Example:-

Let $X = \{a, b, c, d\}$

$T_1 = \{\phi, \{a\}, \{a, b\}, \{X\}\}$

$T_2 = \{\phi, \{b\}, \{a, b\}, X\}$

The topology

$T_1 \cap T_2 = \{\phi, \{a, b\}, X\}$.

is the largest topology contained in T_1 and T_2.

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Topic 61:- Local Base

Local Base at a Point:

Let p be any point of a topological space (X, T). A class B_p of open sets containing p is called a local base at p iff for every $U_p \in T$ (open set containing p) there exists $B_p \in B_p$ such that $p \in B_p \subset U_p$.

Remark:-

Let B be a basis for a topology T on X and $p \in X$. Then the elements of B which contain p form a local base at the point p.

Proof:-

Given B is a basis for T on X.

$\forall \; U \in T \quad \forall \; x \in U \; \exists \; B \in B$ such that $x \in B \subset U$

Let $p \in X$.

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Now consider the following subclass of B.

$B_p = \{B_p \in B \mid p \in B_p\}$

Now consides any element $U_p \in J$ such that $p \in U_p \Rightarrow B_p \subseteq B_p$ such that $p \in B_p \subset U_p$

$\Rightarrow B_p$ form a local base at p.

Remark:-

$$B = \bigcup_{p \in X} B_p$$

B is a basis for J on X.

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Topic 62:-

Examples: Local Base

Let us see some examples of local base at a point for the better understanding of the defination.

Example 1:-

Let $X = \{a, b, c, d\}$ and

$T = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$

be a topology of X.

$d \in \{a, b, c\} \notin \{c, d\}$

Solution:-

'a' $B_a = \{\{a\}\}$

'b' $B_b = \{\{a, b\}\}$

'c' $B_c = \{\{c, d\}\}$

'd' $B_d = \{\{a, c, d\}\}$

$$B = \bigcup_{x \in X} B_x$$

$$= \{\{a\}, \{a, b\}, \{c, d\}\}$$

Example 2:-

Let p be a point of a nonempty set X with discrete topology. Then a local base at p is

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$$\mathcal{B}_p = \{\{p\}\}$$

Example 3:-

Consider $\mathbb{R}$ with usual topology. Let x be any real number then a local base at x is

$$\mathcal{B}_x = \{(x-\epsilon, x+\epsilon) \mid \epsilon \in \mathbb{R}\}$$

Solution:-

$$x \in (a,b) \subset \mathbb{R} \qquad a < x < b$$

$$a < x-\varepsilon < x < x+\varepsilon < b$$

$$x \in (x-\varepsilon, x+\varepsilon) \subset (a,b)$$

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Topic 63:- Continuous Function

In this topic we will discuss about the "Continuous Function" between two topological spaces.

Defination:-

Let $(X, T_x), (Y, T_y)$ be' two topology space.

A function $f: X \to Y$ is continuous iff $\forall\ U \in T_y\ \exists\ V \in T_x$ such that $f^{-1}(U) = V$

i.e., the inverse of every open subset of Y is open in X.

Remark:-

If f is continuous function (X, T_x) to (Y, T_y). Then we say that f is a $T_x - T_y$ continuous. f is continuous relative to T_x and T_y.

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Topic 64

Examples: Continuous Function:-

Let us see some examples to understand the defination of "continuous Function."

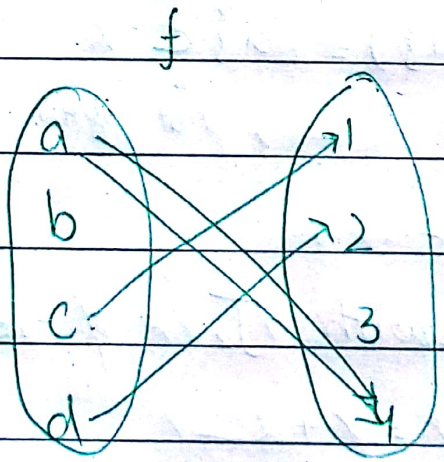
Example:-

Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$ with topologies

$T_X = \{\phi, \{a, b\}, \{c, d\}, X\}$

$T_Y = \{\phi, \{4\}, \{1, 2\}, \{1, 2, 4\}, Y\}$

Solution:-



$f^{-1}(\phi) = \phi$

$f^{-1}(\{4\}) = \{a, b\}$

$f^{-1}(\{1, 2\}) = \{c, d\}$

$f^{-1}(\{1, 2, 4\}) = X$

$f^{-1}(Y) = X$

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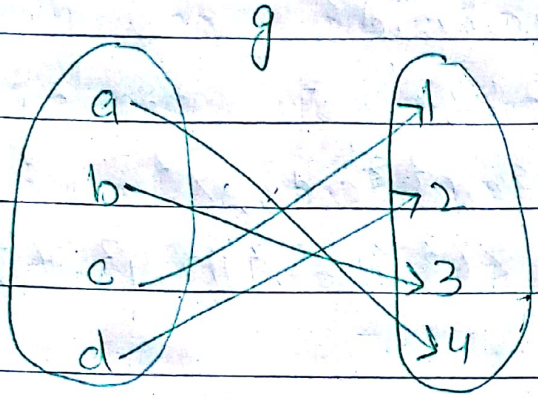
Example 2:-

Let $X=\{a,b,c,d\}$ and $Y=\{1,2,3,4\}$ with topologies

$T_x=\{\phi,\{a,b\},\{c,d\},X\}$

$T_y=\{\phi,\{4\},\{1,2\},\{1,2,4\},Y\}$.

Solution.



$g^{-1}(\{4\})=\{a\}\subset X$

↓ Not open.

Example 3:-

Let $X=\mathbb{R}$ with T_x, $Y=\mathbb{R}$ with T_y.

So $f(x)=x+2$

Solution:-

$f(x)=x+2$

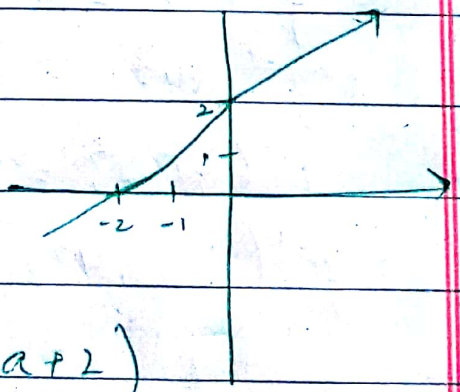
$(a,b)\subset Y=\mathbb{R}$

~~$f^{-1}(a,b)=(a-b)$~~,

$f^{-1}(a,b)=(a-2,a+2)$

$\subset X=\mathbb{R}$

open



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f is continuous function from $X \longrightarrow Y$.

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Topic 65
Constant Function:-

Constant Function:-

A function whose range is singleton i.e., whose output is the same for every point.

Continuity of Constant function:-

Constant function from any topological space (X, T_x) to any topological space (Y, T_y) is always continuous

Proof:-

Let $f: X \longrightarrow Y$ is constant function i.e.,

$$f(x) = c \quad \forall\ x \in X, c \in Y$$

Now take $U \subseteq Y$ (open)

either $c \in U$ or $c \notin U$

$\Rightarrow f^{-1}(U) = X$ open in X.

$\Rightarrow f^{-1}(U) = \phi$ open in X.

Constant function is continuous from (X, T_x) to (Y, T_y).

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Topic 66
Functions into discrete Space :-

In this topic we will discuss the continuity of functions into discrete space.

Theorem :-

Let (X, T_X) be any topological space and (Y, T_{ind}) be an indiscrete space. Then any function $f: X \longrightarrow Y$ is continuous.

Proof :-

Given that Y is indiscrete space

i.e; $T_Y = \{\phi, Y\}$

Let us consider any function

$$f: X \longrightarrow Y$$

$\Rightarrow f^{-1}(\phi) = \phi$ open in X.

$\Rightarrow f^{-1}(Y) = X$ open in X.

f is ~~cont~~ continuous function.

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Topic 67:- Function from discrete space:-

Theorem:-

Let (X, T_{dis}) be discrete space and (Y, T_Y) be any topological space. Then any function $f: X \to Y$ is continuous:-

Proof:-

Given that (X, T_{dis}) is a discrete space

$\Rightarrow T_{dis} = P(X)$

$f: X \to Y$

$U \in T_Y$

$f^{-1}(U) = V \subset X$ $\quad \because T_{dis} = P(X)$

$\Rightarrow V \in T_{dis}$

$\Rightarrow f^{-1}(U)$ is open in X

$\Rightarrow f$ is continuous function.

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Topic 68

Continuity and open sets:-

Continuity of a function depends on the topology of the domain space and the range space i.e., "The collection of open subsets of both sets."

The more rich space on domain side provides more room from continuity and more rich space on the range side ~~restriction~~ restricts more a function to be continuous.

In our previous three topics we have seen three things

(i) Constant function $f: X \to Y$ is always continuous.

(ii) Any function

$g: X \longrightarrow$ Indiscrete Space

is continuous.

(iii) Any function.

$h:$ Discrete Space $\longrightarrow Y$

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is continuous.

Let (X, T_x) be strictly finer than (X, T_x'). Then identity function

$$I:(X, T_x') \to (X, T_x)$$

is not continuous.

Proof:-

$$I : X \to X$$

$$I(x) = x \quad \forall x$$

$T_x' \subset T_x \quad \exists \ U \in T_x$ such that $U \notin T_x'$

Now let $U \in T_x$

$$I(U) = U \notin T_x'$$

$\Rightarrow I:(X, T_x') \to (X, T_x)$ is not continuous.

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Topic 69

Examples: Continuity and open Sets

Let us see some examples to under-stand the role of open collection in continuity of a function.

Example:-

Let $X = \{a, b, c, d\}$

$T_X = \{\phi, \{b\}, X\}$

$Y = \{1, 2, 3, 4\}$

$T_Y = \{\phi, \{2\}, \{1, 2\}, \{2, 3, 4\}, Y\}$

$f : X \to Y$ defined as

$f(a) = 2$

$f(b) = 4$

$f(c) = 2$

$f(d) = 3$

Now f is not continuous

$\because f^{-1}(\{1, 2\}) = \{a, c\} \notin T_X$

$T_X' = \{\phi, \{b\}, \{a, c\}, \{a, b, c\}, X\}$

$g : X \to Y$

$T_Y = \{\phi, \{2\}, \{1, 2\}, \{2, 3, 4\}, Y\}$

$g(a) = 2$.

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$g(b) = 4$

$g(c) = 2$

$g(d) = 3$

g is not continuous

(T_x - T_y continuous)

Example 2:-

~~$X = \mathbb{R}_m$~~ $X = \mathbb{R}_u$, $Y = \mathbb{R}_u$

$f : \mathbb{R}_u \to \mathbb{R}_u$

$f(x) = x + 1$

$f(x)$ is continuous.

$g : \mathbb{R}_u \to \mathbb{R}_l$

$g(x) = x + 1$

g is not ~~constant~~ continuous.

$\because [0,2)$ open $\subset \mathbb{R}_l = Y$

$g^{-1}([0,2)) = [-1,1) \notin \mathbb{R}_u = \tau$.

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Topic 70:-
Continuity and Compasiability

Proposition 1:-

Let $f:(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous function and $\mathcal{T}_X'$ is finer than $\mathcal{T}_X$. Then

$f:(X, \mathcal{T}_X') \to (Y, \mathcal{T}_Y)$ is continuous.

Proposition 2:-

Let

$g:(X:\mathcal{T}_X) \to (Y:\mathcal{T}_Y)$ is not continuous and $\mathcal{T}_X''$ is weaker than $\mathcal{T}_X$. Then

$g:(X, \mathcal{T}_X'') \to (Y, \mathcal{T}_Y)$ is not continuous.

Proof:- (Proposition 1):-

$f:(X, \mathcal{T}_X) \xrightarrow{\text{cont.}} (Y, \mathcal{T}_Y)$

So $\forall \; U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$

Given that $\mathcal{T}_X \subset \mathcal{T}_X'$

$\Rightarrow f^{-1}(U) \in \mathcal{T}_X'$

$\Rightarrow f:(X, \mathcal{T}_X') \to (Y, \mathcal{T}_Y)$

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is continuous.

Proposition 2:-

$g : (X, \mathcal{T}_X) \xrightarrow{\text{Not cons}} (Y, \mathcal{T}_Y)$

This means that

$\exists \cdot u \in \mathcal{T}_Y$ such that

$g^{-1}(u) \notin \mathcal{T}_X$

Now,

$\mathcal{T}_X'' \subset \mathcal{T}_X$

$\Rightarrow g^{-1}(u) \notin \mathcal{T}_X''$

$\Rightarrow g : (X, \mathcal{T}_X'') \to (Y, \mathcal{T}_Y)$ is ~~cons~~ not continuous.

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Topic 71:- Continuity and Basis

Proposition:-

A function

$f : (X, T_X) \to (Y, T_Y)$

is continuous iff the inverse image of each member of a basis B_Y for T_Y is an open subset of X.

$\forall\ B_Y \in B_Y \Rightarrow f^{-1}(B_Y) \in T_X$

Proof:-

Given $\forall\ B_Y \in B_Y$

$\Rightarrow f^{-1}(B_Y) \in T_X$

let $U \in T_Y$

$f^{-1}(U) \in T_X$

Now $\because B_Y$ is a basis for T_Y so,

$\forall\ U \in T_Y$, $U = \bigcup_i B_{Yi}$, $B_{Yi} \in B_Y$

$f^{-1}(U) = f^{-1}(\bigcup_i B_{Yi})$

$= \bigcup_i (f^{-1}(B_{Yi}))$

Now, $f^{-1}(B_{Yi}) \in T_X\ \forall i$

and T_X is a topology.

$\Rightarrow f$ is continuous.

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Let

$f : X \to Y$ is continuous

$\Rightarrow \forall \cdot u \in T_y \Rightarrow f^{-1}(u) \in T_x$

But we know that

$B_y \subset T_y$

$\Rightarrow \forall \quad B_y \subset B_y \subset T_y$

$\Rightarrow f^{-1}(B_y) \in T_x$

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Topic 72:- Continuity and subbasis

Theorem :-

A function $f:(X,T_X)\to(Y,T_Y)$ is continuous iff the inverse image of each member of a subbasis S_Y for T_Y is an open subset of X.

Proof :-

Given $\forall\ S_Y \in S_Y$

$\Rightarrow f^{-1}(S_Y) \in T_X$

Let $U \in T_Y$

$\Rightarrow f^{-1}(U) \in T_X$

$\because S_Y$ is a subbasis for T_Y

$\Rightarrow \forall\ U \in T_Y,\ U = \bigcup_i \left(\bigcap_{j=1}^{n_i} S_{Y_{ij}}\right)$

$f^{-1}(U) = f^{-1}\left(\bigcup_i \left(\bigcap_{j=1} S_{Y_{ij}}\right)\right)$

$\Rightarrow f^{-1}(U)$ is open in X.

$\Rightarrow f$ is continuous function

Let $f:X\to Y$ is continuous

$\Rightarrow \forall\ U \in T_Y$

$\Rightarrow f^{-1}(U) \in T_X$

$S_Y \subset T_Y$

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$\Rightarrow \forall \quad S_Y \in \mathcal{S}_Y$

$f^{-1}(S_Y) \in \mathcal{T}_X.$

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Topic 73
Absolute Value Function

In this topic we will discuss the continuity of Absolute Value function.

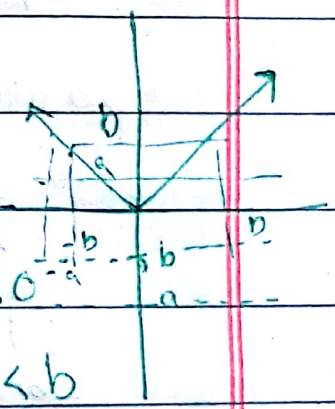
$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Show that $f : \mathbb{R}_u \longrightarrow \mathbb{R}_u$ is continuous when $f(x) = |x|$

Sol:-

Let $(a,b) \in \mathbb{R}_u = (\mathbb{R}, \mathcal{J}_u)$



Now

$$f^{-1}((a,b)) = \begin{cases} \phi & \text{if } a < b \leq 0 \\ (-b,b) & \text{if } a < 0 < b \\ (-b,-a) \cup (a,b) & \text{if } 0 \leq a < b \end{cases}$$

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Topic 74
Composition of Continuous Function:-

In this topic we will prove the fact

"Composition of two continuous functions is a continuous function."

$f : X \to Y$

$g : Y \to Z$

$h(x) = gof(x) = g(f(x))$

$h : X \to Z$

Proposition:-

Let $f : X \to Y$

$g : Y \to Z$ are two continuous function. Then

$gof : X \to Z$ is continuous.

Proof:-

Let $U \in T_Z$

$(gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \in T_X$

$\Rightarrow (gof)^{-1}(U)$ is open in X.

gof is a continuous function

$X \to Z$.

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Topic 75

Continuous Function:

Another Definition:-

Proposition:-

A function

$f:(X, T_X) \to (Y, T_Y)$

is continuous iff the inverse image of every closed subset of Y under is closed in X.

Proof:-

f is a continuous function from $X \to Y$.

$\forall\ U \in T_Y \Rightarrow f^{-1}(U) \in T_X$.

Let $A \underset{\text{closed}}{\subset} Y \Rightarrow A \underset{\text{open}}{\subset} Y\ (A^c \in T_Y)$

$\because$ f is continuous.

$\Rightarrow f^{-1}(A^c)$ is open in X.

But $f^{-1}(A^c) = (f^{-1}(A))^c \to$ open in X.

$\Rightarrow (f^{-1}(A)^c)^c$ is closed in X.

$\Rightarrow f^{-1}(A)$ is closed in X.

(Given) We have $\forall\ A \subset Y \Rightarrow f^{-1}(A) \underset{\text{closed}}{\subset} X$

Let $U \underset{\text{open}}{\subset} Y \Rightarrow U^c \underset{\text{closed}}{\subset} Y$

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$f^{-1}(U^c) = (f^{-1}(U))^c$

$\therefore f^{-1}(U) \subseteq_{open} X$ or $f^{-1}(U) \in T_X$

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Topic 76
Restriction of Function

Defination:-

Let $f: X \to Y$ be a function and $A \subset X$.

The restriction of f to A is a function $f_A: A \to Y$ defined as

$f_A(a) = f(a) \quad \forall \ a \in A$

Proposition:-

Let $f: X \to Y$ be a continuous function. Then $f_A: A \to Y$ is continuous. when $A \subset X$.

Proof:-

For any $B \subset Y$ $\quad f_A^{-1}(B) = A \cap f^{-1}(B)$

Let $U \in T_Y$

$\Rightarrow f_A^{-1}(U) = A \cap f^{-1}(U)$

Given f is continuous $\Rightarrow$ $f^{-1}(U) \in T_X$.

$\Rightarrow f_A^{-1}(U) \in T_A$.

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Topic 77:- Continuous Function and Arbitrary Closeness

Recall closure point:-

$$\bar{A} = A \cup A'$$

A point that is arbitrary close to a set A.

Continuity preserve arbitrary closeness.

Theorem :-

A function $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is continuous iff for every subset A of X.

$$f(\bar{A}) = \overline{f(A)}$$

Proof:-

Let $f: X \to Y$ is continuous.

Let $A \subset X$.

Now we know that $f(A) \subset \overline{f(A)}$

So

$$A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)}) \to (1)$$

But $\overline{f(A)}$ is closed in Y.

$\Rightarrow f^{-1}(\overline{f(A)})$ is closed in X.

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$\because f$ is continuous

Now ① $\Rightarrow A \subset f^{-1}(\overline{f(A)})$

i.e., $A \subset \bar{A} \subset f^{-1}(\overline{f(A)})$

$\Rightarrow f(\bar{A}) \subset f(f^{-1}(\overline{f(A)})) = \overline{f(A)}$

$\Rightarrow f(\bar{A}) \subset \overline{f(A)}$

For the other side

let $f(\bar{A}) \subset \overline{f(A)}$

let $B \overset{\text{closed}}{\subset} Y$

let $f^{-1}(B) = A$

we have to show that A is closed in X. i.e., $\bar{A} = A$

we know that

$f(\bar{A}) \subset \overline{f(A)}$

$\Rightarrow f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} = \bar{B} = B$

$\Rightarrow f(\bar{A}) \subset B$

$\Rightarrow f^{-1}(f(\bar{A})) \subset f^{-1}(B)$

$\Rightarrow \bar{A} \subset f^{-1}(B) = A$

$\therefore \bar{A} \subset A \quad , A \subset \bar{A}$

$\Rightarrow \bar{A} = A$

$\Rightarrow A$ is closed ~~on~~ in X.

$\Rightarrow f$ is continuous.

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Topic 78
Continuity at a point

We have discussed continuity in global sense ~~i.e~~ i.e., continuity on a set.

Now let us discuss locally i.e., continuity at a point.

Defination:-

Let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a function. f is continuous at a point $x\in X$ iff for every $U_{f(x)}$ (open set containing $f(x)$) There exist V_x (open set of X containing x) such that

$$V_x \subset f^{-1}(U_{f(x)}) \quad \text{or}$$

$$f(V_x)\subset U_{f(x)}.$$

For a given $x\in X$.

Fin $f(x)$.

Collect all the open subsets of Y containing $f(x)$ i.e.,

$$U_{f(x)}=\{U_{f(x)} \mid f(x)\in U_{f(x)}\}\subset \mathcal{T}_Y$$

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For each element of $U_{f(x)}$ (say $U_{f(x)}$)
check is $\exists$ some V_x such that $V_x \subset f^{-1}(U_{f(x)})$

If your answer is yes for all $U_{f(x)}$. Then you can say f is continuous at x.

But if $\exists$ some $U_{f(x)}$ such that $\nexists U_x$ such that $V_x \subset f^{-1}(U_{f(x)})$. Then you say that f is not continuous at x.

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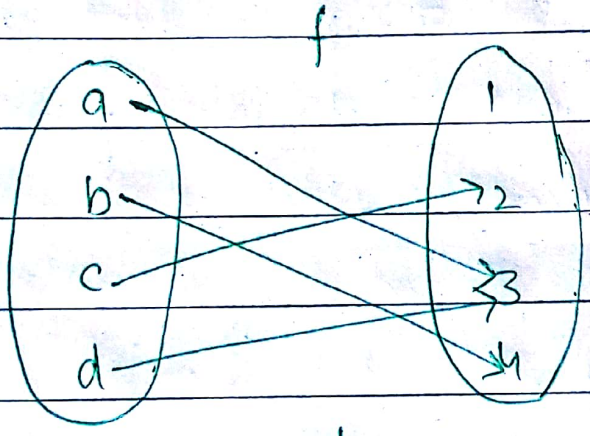
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Topic 79
Examples: Continuity at a point!

Let us see some ~~eqn~~ examples of continuity at a point.

Example 1:-

Let $X = \{a, b, c, d\}$ with
$T_X = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, X\}$
and $Y = \{1, 2, 3, 4\}$ with
$T_Y = \{\phi, \{2\}, \{1, 2\}, \{2, 3, 4\}, Y\}$



f is continuous at 'a'
$\forall\ U_3 \in T_Y\ \exists\ V_a \in T_X$
such that $V_a \subset f^{-1}(U_3)$
$U_3 = \{\{2, 3, 4\}, Y\}$
$f^{-1}\{2, 3, 4\} = X$ open in X.

f is not continuous at 'c'
$U_2 = \{\{2\}, \{1, 2\}, \{2, 3, 4\}, Y\}$

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$f^{-1}(\{2\}) = \{c\} \not\ni V_c$

such that

$V_c \subset \{c\}$

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Topic 80:- Continuity at every and continuous Function

Theorem :-

A function $f: X \to Y$ is continuous iff it is continuous at every point of X.

Proof :-

Let $f: X \to Y$ is continuous

Let $x \in X$

Now take $U_{f(x)} \in T_Y$

(open subset of containing $f(x)$)

$\because$ f is continuous. so,

$f^{-1}(U_{f(x)}) \in T_X$

So, $V_x = f^{-1}(U_{f(x)}) \subset f^{-1}(U_{f(x)})$

$\Rightarrow$ f is continuous at x.

For the other side

$f: X \to Y$ is continuous at every $x \in X$. i.e., $\forall\ x \in X$ and $\forall\ U_{f(x)} \in T_Y$ $\exists\ V_x \in T_X$ such that $V_x \subset f^{-1}(U_{f(x)})$

$\because$ V_x $f^{-1}(U_{f(x)})$ can be written

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as the union of such V_x.

i.e.,

$$f'(U_{f(x)}) = \cup \{ V_p \mid p \in f'(U_{f(x)}) \}$$

$\Rightarrow f'(U_{f(x)})$ is open in X.

$\Rightarrow f$ is continuous function from (X, T_x) to (Y, T_y).

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Topic 81:- Topology induced by function:-

Defination:-

Let $f: X \to Y$ be a function and $\mathcal{T}_Y$ be a topology on Y. The class of subsets of X.

$\mathcal{T}_{xf} = \{U = f^{-1}(V) \mid V \in \mathcal{T}_Y\}$

forms a topology on X.

This topology is called "Topology on X induced by f."

Task:-

Show that $\mathcal{T}_{xf}$ is a topology on X and also show that

$f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$

is continuous.

Proof:-

$\phi \in \mathcal{T}_{xf}$ $\because \phi = f^{-1}(\phi)$, $\phi \in \mathcal{T}_Y$

$X \in \mathcal{T}_{xf}$ $\because X = f^{-1}(Y)$, $Y \in \mathcal{T}_Y$

Let $U_i \in \mathcal{T}_{xf}$ $i \in I$

$U_i = f^{-1}(V_i)$ $(V_i \in \mathcal{T}_Y)$ $\forall$ $i \in I$

$\bigcup_i U_i = \bigcup_i (f^{-1}(V_i)) = f^{-1}(\bigcup_i (V_i)) \in \mathcal{T}_Y$

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$\Rightarrow \bigcup U_i \in T_{xf}$

let $U_1, U_2 \in T_{xf}$

$\Rightarrow \exists\ V_1, V_2 \in T_Y$

such that

$U_1 = f^{-1}(V_1)$,

$U_2 = f^{-1}(V_2)$

$U_1 \cap U_2 = f^{-1}(V_1) \cap f^{-1}(V_2)$

$= f^{-1}(V_1 \cap V_2)$

$\Rightarrow U_1 \cap U_2 \in T_{xf}$

$f : (X, T_{xf}) \longrightarrow (Y, T_Y)$

f is continuous iff $\forall\ V \in T_Y$

$\Rightarrow f^{-1}(V) \in T_{xf}$

T_{xf} so, f is continuous.

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Topic 82
Examples: Topology Induced by Function:-

Example 1:-

Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3\}$

$T_Y = \{\phi, \{2\}, \{1, 2\}, \{2, 3\}, Y\}$

$f : X \to Y$ defined as

$f(a) = 1$, $f(b) = 2$

$f(c) = 2$, $f(d) = 3$

$T_{X_f} = \{\phi, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$

Example 2:-

$g : X \to Y$ (X, Y are same as in ex. 1)

$g(a) = 1$, $g(b) = 1$

$g(c) = 3$, $g(d) = 2$

$\therefore T_{X_g} = \{\phi, \{d\}, \{a, b, d\}, \{c, d\}, X\}$

Question:-

What is T_X on X such that $f : X \to Y$, $g : X \to Y$ are both continuous.

Answer:-

The topology with the subbasis

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$$\mathcal{S}_{x_{f,g}} = \{f^{-1}(V) \mid V \in \mathcal{T}_Y\} \cup \{g^{-1}(V) \mid V \in \mathcal{T}_Y\}$$

The topology generated by $\mathcal{S}_{x_{f,g}}$ (subbasis) is induced by f and g.

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Topic 83
Topology and on real line and linear function

In this topic we will show that the topology on real line induced by the set of all linear functions from real line to real line with lower limit topology is discrete.

$f : X \to Y$

$g : X \to Y$

$\mathcal{T}_Y$ is a topology on Y.

Then $\mathcal{T}_{X_{f,g}}$ has basis

$$\mathcal{S}_{X_{f,g}} = \{f^{-1}(v) \mid v \in \mathcal{T}_Y\} \cup \{g^{-1}(v) \mid v \in \mathcal{T}_Y\}$$

Let

$f : \mathbb{R} \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

$f(x) = x - p$, $g(x) = p - x$ $\quad p \in \mathbb{R}$.

$A \subset \mathbb{R}$, $A \in \mathcal{T}_{\mathbb{R}}$

$A = [0, 1)$

$f^{-1}([0,1)) = [p, p+1)$

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$$g^{-1}([0,1)) = (p-1, p]$$

$$f^{-1}([0,1)) \cap g^{-1}([0,1)) = [p, p+1) \cap (p-1, p].$$

$$= \{p\}$$

$\{p\} \in \mathcal{B}_x$

$\Rightarrow$ The topology on $\mathbb{R}$ induced by set of all linear function is discrete.

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Topic 84
Open Mapping

Defination:-

A map

$f:(X,T_X)\longrightarrow(Y,T_Y)$

is called an open mapping iff the image of each open subset of X is open in Y.

$\forall\ U\in T_X,\ f(U)\in T_Y$

Example:-

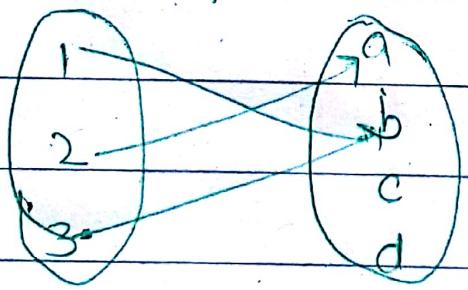
$X=\{1,2,3,4\}$ and with

$T_X=\{\phi,\{1\},\{2\},\{1,2\},X\}$

$Y=\{a,b,c,d\}$ with

$T_Y=\{\phi,\{a\},\{b\},\{a,b\},\{b,c,d\},Y\}$

f



f is open map.

$f(\phi)=\phi$

$f(\{1\})=\{b\}$

$f(\{2\})=\{a\}$

$f(\{1,2\})=\{a,b\}$

$f(X)=\{a,b\}$

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Topic 85
Open Mapping and Basis

Proposition:-

A map

$f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$

is an open map iff the image of each element of a basis B_X for $\mathcal{T}_X$ is an open subset of Y.

Proof:-

Let f is open

$\Rightarrow$ The image of every element of $\mathcal{T}_X$ is an element of $\mathcal{T}_Y$.

Now $B_X \subset \mathcal{T}_X$

$\Rightarrow f(B_X) \in \mathcal{T}_Y$

$B_X \in \mathcal{B}_X$.

For other side

Let $U \in \mathcal{T}_X$

$\mathcal{B}_X$ is a basis of $\mathcal{T}_X$.

So $U = \bigcup_i B_i \qquad B_i \in \mathcal{B}_X$

Now $f(U) = f(\bigcup_i B_i) = \bigcup_i f(B_i)$

Now $f(B_i)$ is open in Y.

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$\Rightarrow \bigcup_i f(B_i)$ is open in Y.

So $f(U)$" is open in Y.

$\Rightarrow f$ is an open map.

———.

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