Mth634

"Final Term preparation"

"Topology"

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Open Mapping:

⇒ def:

A map $f: (X, T_X) \to (Y, T_Y)$ is called an open map iff the image of each open subset of X is open in Y.

Closed mapping:

A map $f: (X, T_X) \to (X, T_Y)$ is called a closed map iff the image of each closed subset X is closed in Y.

=> Projection Mapping:-

Let (X, T_X) and (Y, T_Y) be two topological Spaces. The maps;

$$\pi_1 : (X \times Y, T_{X \times Y}) \to (X, T_X)$$
$$(x, y) \longrightarrow x$$
$$\pi_2 : (X \times Y, T_{X \times Y}) \to (Y, T_Y)$$
$$(x, y) \to (y)$$

are called projection maps, where $T_{X \times Y}$ is product topology on $X \times Y$. π_1 is First projection & π_2 is Second projection.

=> Fact; projection may or may not be closed.

$$\pi_1(A) = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \subset \mathbb{R}$$

Not closed.

=> Continuous Map may or may not be open map.

=> open map may or may not be Continous maps.

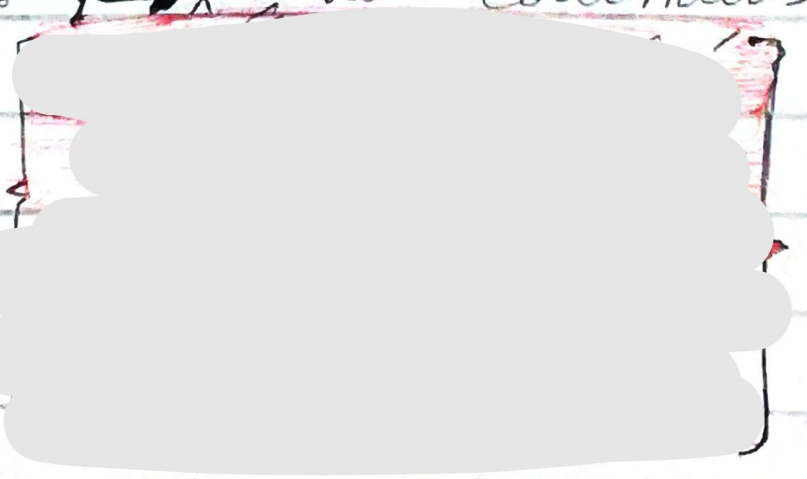
=> Restriction of an open map may or may not be an open map.

★ Homeomorphism;

A map $f: X \to Y$ is called a homeomorphism between two topological spaces; (X, T_X) and (Y, T_Y) iff

(1) f is bijective (2) f is continuous.

(3) $f^{-1}: Y \to X$ is continuous. OR f is bicontinuous



★ Homeomorphic Spaces;

Two topological spaces (X, T_X) and (Y, T_Y) are called homeomorphic iff there exists a Homeomorphism $F \to X \to Y$ it is also called topologically equivalent spaces.

➡ Closed Mapping:

$f: (X, T_X) \to (Y, T_Y)$

is called a closed map iff the Image of each closed subset of X is closed in Y.

➡ Projection Mapping:-

➡ projection mapping is not closed in General.

★ Def:-

Let (X, T_X) and (Y, T_Y) be two topological spaces.

★ Equivalence Relation:-

A binary relation $\sim$ on a set X is said to be an equivalence relation iff $\sim$ satisfies the following three properties for all $a, b, c \in X$.

1. Reflexive: $a \sim a$
2. Symmetric; $a \sim b$ iff $b \sim a$.

(3) Transitive; if $a \sim b$ and $b \sim c$ then $a \sim c$.

→ Homeomorphism of topological Spaces is an equivalence relation.

→ Topological property also called topological Invariant.

⇒ Some Non-Topological properties;

* Area
* length * Boundedness.

"Metric on a Set"

"A metric is a distance function"

"A function that defines a distance between each pair of elements of a set."

Metric :-

A metric on a nonempty set X is a real valued function d defines on $X \times X$.

$$d : X \times X \longrightarrow R.$$

Such that for all $x, y, z \in X$. the following conditions are satisfied;

(1) $d(x,y) \geq 0$ equality holds iff $x = y$

(2) $d(x,y) = d(y,x)$ symmetry

(3) $d(x,z) \leq d(x,y) + d(y,z)$ (Triangle Inequality)

★ **Metric Space:-**

A set x with a metric d defined on it is called a metric space.

⇒ Example:-

Let $X = \mathbb{R}$ and d is defined as $d(x,y) = |x-y|$ where $x, y \in \mathbb{R}$.

Sol;

(1) $d(x,y) \geq 0$

$\Rightarrow |x-y| \geq 0$

$\Rightarrow x-y = 0$

$\Rightarrow x = y$

(2) $d(x,y) = d(y,x)$

$d|x-y| = |-(y-x)|$

$d(x-y) = d(y,x)$

(3) $d(x,z) \leq d(x,y) + d(y,z)$

$|x-z| = |x-y+y-z| \leq |x-y| + |y-z|$

(usual Metric:)

$d(x,y) = |x-y|$ where $x, y \in \mathbb{R}$.

This is the usual metric in $\mathbb{R}$.

we can also write it in the following way:

$$d(x,y) = \sqrt{(x-y)^2}$$

* **Open ball:**

consider X with a metric d. For any $x \in X$ and any real number $r > 0$, the set of all points $y \in X$, whose distance from x is less than r is called the open ball of radius r centered at x.

$$B(x,r) = \{y \in X \mid d(x,y) < r\}$$

Continuous Functions :-

A Function ;

$f : (X, T_X) \to (Y, T_Y)$

is Continuous iff for every Subset A of X.

$$F(\bar{A}) = \overline{f(A)}$$

Continuity at a point :-

Theorem ;

A function $f : X \to Y$ is Continuous iff it is continuous at every point of X.

Topology Induced by a Function :-

Def :-

Let $f : X \to Y$ be a Function and T_Y be a topology on Y. The class of Subsets of X,

$T_X := \{U = F^{-1}(V) \mid V \in T_Y\}$

Forms a topology on X. This topology is Called "Topology on X Induced by f"

Open Ball as Open set;

Lemma;

Let d be a Metric defined on a nonempty set X. for $x \in X$ and a positive real Number r, consider open ball $B(x,r)$. Then for every point $y \in B(x,r)$ there exist an open ball $B'(y,r')$ such that; $B'(y,r') \subset B(x,r)$.

(Open Ball Centered at one point;)

fact :-

Let $B_1(x,r_1)$ and $B_2(x,r_2)$ be two open balls centered at x with Radius r_1 and r_2 respectively;
$B_1(x,r_1) \subset B_2(x,r_2)$ or $B_2(x,r_2) \subset B_1(x,r_1)$.

➡ (Metric Topology:-)

Let X be a nonempty set with metric d. The topology τ on X generated by the set of all open balls in X with respect to 'd' is called "Metric Topology".

➡ (Metric Space)

Consider X with a metric d, this 'd' induced a topology the "metric topology". (X, d) is called a 'Metric Space'.

(IMP)

⇒ Topology Generated by the set of all open Intervals is usual topolgy.

⇒ Topology generated by the set of all singletone is discrete topology.

"Metrizable Spaces:-"

Def:-

"A topological Space (X, T) is called Metrizable Space if there exists a Metric d on X that induces the Topology T."

MCQs

- Not all Spaces are Metrizable.
- Every Metrizable Space is topological Space.
- $\mathbb{R}$ with lower limit topology is not metrizable Space.
- Separable Metric Spaces are Second Countable.

First Countable Spaces:-

Def:-

A topological Space X is said to be first countable if every $x \in X$ has countable Neighbourhood base.

★ "2nd Countable prove krne k liye Jo Topology di hui hoge us Main sy 1 set bases ka layna hai, Base o ky Sub elements ka union lyna hai ak dusry k sth Agr wo topology Main exist krty hain sb toh us Set ko 2nd Countable bolen gy."

★ 1st Countable:-

Jo Set dia hoga us Ki Ak Topology banani hai. Hr ~~set~~ element ka local base Find krna hai. (local base Main ~~topology~~ set k hr element ko lyna hain or topology Main Jis Jis set Main wo element hoga woo sb ly lyny hai)

wo sb set es element
x local base hoon gy}
sb elements x local base
Agr countable hain toh
usko 1st countable bhi
bolen gy.

★ Second Countable Space:-
A topological space X is said to be second Countable space if it has Countable open base.

★ Theorem:-
"Every second Countable Space is first Countable but converse may not be true."

⇒ Any uncountable set X with cofinite topology is not First Countable so is not second Countable.

➡ Every Subspace of a First Countable Space is First Countable.

➡ Discrete Space First Countable.

➡ A local base at a point p in discrete space.
$B_p = \{\{p\}\}$. and it is finite.

(Open Cover)

Cover ⇒ A cover of a set X is a collection of sets;

$$U = \{U_i : i \in I\}$$

where I is Index set, such that;

$$X \subset \bigcup_{i \in I} U_i.$$

Cover IN Topology

Let (X, T) be a Topological Space. A cover of a X is a collection of Subsets of X,

$$U = \{U_i : U_i \subset X, i \in I\}$$

where I is Index set, such that;

$$X = \bigcup_{i \in I} U_i$$

(★) open cover find krne k liye jo set dia hua hoga us par Ak topology define ~~krn~~ krni hai, us topology main sy kuch sets lyny hain, Ak Alg set define kna hai, jo New set bny ga us k Sb elements ka aps mai union lyna hai. Agr wo union lyny k bhd given set aa jay toh wo uska open cover hoga.

(Def)

Let (X, T) be topological space. An open cover of a X is a collection of open subsets of X,

$U = \{U_i \mid U_i \in T, i \in I\}$
where I is Index set,
Such that;

$$X = \bigcup_{i \in I} U_i$$

(Open Subcover :-)
Let (X, T) be a topological Space. Consider C be an open cover of X,

$$C = \{C_i \mid C_i \in T\}$$

Such that $X = \bigcup_i C_i$.
A Subclass D of C that also cover X is called an open Subcover of C.

Lindelöf Space

A Topological Space (X, τ) is called a lindelöf Space iff every open cover of X has a Countable Subcover.

Examples

(1) Let X be a finite set with any topology then it is lindelöf.

(2) A Set X with Indiscrete topology then It is lindelöf.

(3) $X = \mathbb{R}$ with usual topology is lindelöf.

Remark:-

A Subspace of a Lindelöf Space needs not be Lindelöf.

Theorem:-

"Every Second Countable Space is Lindelöf Space."

(Method)

AK Set or topology di hoge, us topology Main sy AK Set lyna hai (subset) us subsets k Sb elements ka Aps Main union lyna, Set X bn jayga. To yeh subset open cover hua. Ab open cover Main sy koi se collection lyni hai uska union lyny say bo given Set ayga toh uska Sub open cover hoga. Isi ko Lindelof Space bolty gy.

(Separable Spaces)

Def :-

A topological space $(X, \mathcal{T})$ is said to be "Separable" if there exists a countable dense subset A of X.

$\exists\ A \subset X$ such that :

(1) A is countable

(2) $\bar{A} = X$.

(OR)

A topological space $(X, \mathcal{T})$ is said to be "Separable" if there exists a sequences of points

$$\{x_n\}_{n=1}^{\infty}$$

in X such that every nonempty subset of X contains at least one point of the sequence.

Remark:

Every second countable space is separable. The converse of this is not true in General.

Example:

→ $\mathbb{R}$ with lower limit topology is separable but not second countable.

→ Let X be a finite set. Then X with any topology is separable.

→ Consider a set X with indiscrete topology. Then X is separable.

→ $\mathbb{R}$ with usual topology is separable.

→ $\mathbb{R}$ with discrete topology is not separable.

Proposition ①

⇒ A discrete space is separable iff it is countable.

② (X, T_{cof}) is separable.

(Theorem)

* A separable metric space is 2nd countable.

EXP:-

* An infinite set X with trivial metric is not second countable and not separable.

(Method)

(Separation Axioms)

(1) T_0 Spaces (2) T_1 Spaces
(3) T_2 (Hausdorff Spaces)
(4) Regular Spaces
(5) T_3 Spaces
(6) Normal Spaces.
(7) T_4 Spaces:-

(T_0 Space)

Given Set ki AK topology create krni hai, T_0 Space ki property hai that k us topology main At least 1 open set hona chaye, Zyda jtne be ho jaen, Agr us topology main open set hain toh usko T_0 Space bolen gy;
(open set; Topolgy ka wo JO element given set mai bhi mojod hoga wo open set hoga).

Def:-

A topological Space (X, T) is Said to be "T_0 Space" iff for each $x, y \in X$ such that $x \neq y$ there exist an open set $U_x \subset X$ containting x such that $y \notin U_x$

OR

there exists an open Set $V_y \subset X$ containting y such that $x \notin V_y$.

(REMARK)

T_0 property is a topological property.

(EXAMPLE)

Every discrete Space is T_0.

(T_1 Space)

Def:

A topological space (X, T) is said to be "T_1 Space" IOD for each $x, y \in X$ such that $x \neq y$ there exist open subsets U_x, U_y of X containing x, y respectively such that $y \notin U_x$ and $x \notin U_y$.

(Method)

To Space main At least one open Set hona chaye tha.

T_1 Space main At least two open set hony chaye topology main Tab usko T_1 Space kahen gy.

Example:

$X = \{1, 2\}$, $T = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$

$1, 2 \in X$, $1 \neq 2$

$1 \in \{1\}$, $2 \notin \{1\}$

$2 \in \{2\}$, $1 \notin \{2\}$

So, (X, T) is T_1-Space.

* Every discrete space is T_1.
* Every Metric Space is T_1.
* Every Cofinite Space is T_1.

⇒ Every T_1 space is T_0, but not every T_0 space is T_1.

⇒ Every subspace of a T_1 space is also a T_1 sp.

➡ Theorem:

"A topological space (X, T) is T_1 iff every singleton subset $\{x\} \subset X$ is closed".

(Properties)

⇒ Every Finite Subset of T_1 Space is closed.

⇒ Let X be a Finite set. Then $(X, \mathcal{T})$ is a T_1 Space iff $(X, \mathcal{T})$ is discrete Space.

⇒ Let A be a Finite Subset of a T_1 Space X then A' is empty.

(T_2 - Hausdorff Space)

Def;

A topological Space $(X, \mathcal{T})$ is said to be "T_2 Space" iff for every distinct pair of elements $x, y \in X$ there exist open subsets U_x, U_y of X containing x, y respectively such that $U_x \cap U_y = \phi$.

T_2 Space is also called Hausdorff Sp

① Every discrete space is Hausdorff.
② A nonempty set $X \neq \{x\}$ with Indiscrete topology is not Hausdorff.
③ $\mathbb{R}$ with lower limit topology is Hausdorff.

Method.

Same procedure hai To Ep T_1 jaisa, Sirf To 2 open set lyny hain, unka $\cap$, ϕ k equal Ana chaye.

(Example)

$X = \{1, 2\}$, $\mathcal{T} = \{\phi, X, \{1\}, \{2\}\}$

$1, 2 \in X$, $1 \neq 2$

$1 \in \{1\}$, $2 \in \{2\}$

$\{1\} \cap \{2\} = \phi$

So topological space is T_2 Space.

(Regular Space)

A Topological space $(X, \mathcal{T})$ is said to be regular Space if for every $x \in X$ and for any closed subsets A of X with $x \notin A$.

Then there exist two open set

U and V such that
$x \in U$ and $A \subseteq V$ and
$U \cap V \neq \phi$

Example :- $X = \{1, 2\}$

$\mathcal{T} = \{\phi, X, \{1\}, \{2\}\}$

$1 \in X$

(for Finding closed set
Find complement of each
set of topology)

$\phi' = X - \phi = X$

$X' = X - X = \phi$

$\{1\}' = X - \{1\} = \{2\}$

$\{2\}' = X - \{2\} = \{1\}$

$1 \notin \{2\} \Rightarrow$ closed set.

$\{1\}$ and $\{2\}$ are two
open sets.

$1 \in \{1\}, \{2\} \subseteq \{2\}$

$\{1\} \cap \{2\} = \phi$

(T_3 - Space)

A Topological Space $(X, \mathcal{T})$ is said to be "T_3 Space" iff it is either

* Regular + T_0

or

* Regular + T_1

or

* Regular + T_2.

⇒ Discrete Space is a regular Space.

⇒ Indiscrete Space is a regular Space.

⇒ An Infinite set with Cofinite topology is not regular. so it is not T_3.

(Normal Spaces)

A topological Space (X, T) is said to be Normal Space for any two closed disjoints subsets A and B for X. There are open sets U and V in $x \subseteq t$. $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

Example;

$X = \{a, b\}$, $T = \{\phi, X, \{a\}, \{b\}\}$

Closed sets,

$\phi' = X$; $\{a\}' = \{b\}$

$X' = \phi$, $\{b\}' = \{a\}$

X, ϕ, $\{b\}$, $\{a\}$ are closed sets.

$\{a\} \subseteq \{a\}$

$\{b\} \subseteq \{b\}$, $\{a\} \cap \{b\} = \phi$

So T is Normal Space.

T_4 Space:-

A Topological Space $(X, \mathcal{T})$ is said to be "T_4 Space" iff it is either Normal + T_1 or Normal + T_2.

(REMARK)

Subspace of a Normal Space Needs not be a Normal Space.

(Complete Normal Space)

A topological Space $(X, \mathcal{T})$ is said to be completely Normal Space iff every Subspace of X with Subspace topology is Normal.

$\Rightarrow$ Every Metric Space is Normal.

7.
16
19
23
14
7

(Metric Space are Normal)

Definition :-

Let (X, d) be a metric space and A be a nonempty subset of X. Then a continuous function;

$$d_A : X \longrightarrow \mathbb{R}$$

defined as;

$$d_A(x) = \inf \{ d(x, a) \mid a \in A \}$$

provides the distance between x and A. we denote it as

$$\text{dist}(x, A) = d_A(x)$$

(Urysohn's Lemma)

X is a Normal Space iff for every two disjoint closed subsets A & B of X. There exists a

Continuous map;

$$f : X \longrightarrow [a,b]$$

Such that

$$f(x) = a \quad \forall\ x \in A \ \&$$

$$f(x) = b \quad \forall\ x \in B.$$

- One important application of Urysohn's Lemma is the partial solution of metrization problem.

Urysohn's metrization Theorem;

- Every Normal-T_1 Space with countable basis is metrizable.
- Every regular-T_1 Space with Countable basis is metrizable.

Compact Space:-

A topological Space (X,T) is said to be compact if every open cover C of X contains a Finite Subcover S of X.

(EXAMPLES)

➡ A set X with Indiscrete topology $\mathcal{T}$ is compact.

Reason:-

Since there is only one open cover of X with respect to Indiscrete topology and that is

$$\{x\}$$

which is obv Finite.

➡ A set X with topology $\mathcal{T}$ containing Finite number subsets of X is compact.

★ REASON:-

Since every open cover C of X is a subclass of $\mathcal{T}$ and $\mathcal{T}$ itself Finite so C is Finite too and any subcover S of C is also Finite.

(Open Intervals In $\mathbb{R}$)

- An open Interval in $\mathbb{R}$ with respect to usual topology is not compact.
- $(\mathbb{R}, T_U)$ is not compact.

- $(\mathbb{R}, T_U)$ is not compact.
  $(\mathbb{R}, T_{indis})$ is compact.
- $(\mathbb{R}, T_{dis})$ is not compact.
- $(\mathbb{R}, T_{cof})$ is compact.

Theorem:-

- "Every closed Subspace of a compact Space is compact."
- Let (A, T_A) be compact Subspace of a Hausdorff Space (X, T).
  Then A is closed in X.
- Every compact Subspace of a Hausdorff Space is closed.

→ Let $(X, \mathcal{T})$ be a compact space and $f: (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a continuous map. Then $f(x)$ is compact.

→ Image of compact space under a continuous map is compact.

(COROLLARY I)

→ Let $(X, \mathcal{T})$ be a compact space and $(Y, \mathcal{T}')$ be a Hausdorff space. Then a continuous map;

$f: (X, \mathcal{T}) \to (Y, \mathcal{T}')$

is a closed mgp.

(II)

→ Let (X, T) be a compact space and (Y, T') be a Hausdorff space. Then a bijective continuous map;

$f: (X, \mathcal{T}) \to (Y, T')$

is a homeomorphism.

Theorem:

* Let $(X, \mathcal{T})$ be a topological space, And $\{(X_i, \mathcal{T}_i) / i \in I\}$ be a finite Family of compact Subspaces of X.

Then;

$$\bigcup_{i \in I} X_i \text{ is compact.}$$

* Let $(X, \mathcal{T})$ be a compact Hausdorff space. Then $(X, \mathcal{T})$ is normal.

"Separated Set"

Two Subsets A & B of a metric Space (X, d) are said to be Separated iff;

$$\bar{A} \cap B = \phi$$

$$A \cap \bar{B} = \phi$$

if $A = (1, 2)$ the $\bar{A} = [1, 2]$

(Connected Spaces)

A Subset A of a topological Space (X, T) is said to be Connected iff there exists "no pair of nonempty open Subsets" U & V of X Such that $A \cap U$ and $A \cap V$ are nonempty disjoint sets and;

$A = (A \cap U) \cup (A \cap V)$

(Disconnected set)

A subset of a topological Space (X, T) is said to be disconnected if it is not Connected. If there exist a "pair of nonempty open Subsets U & V of X" Such that $A \cap U$ & $A \cap V$ are Nonempty disjoint sets and.

$A = (A \cap U) \cup (A \cap V)$.

(Connected Space:-)

A topological space (X, T) is said to be connected if it can not be written as union of two nonempty open disjoint subsets U and V of X.

(Disconnected Space)

A topological space (X, T) is said to be disconnected if it can be written as union of two nonempty open disjoint subsets U and V of X.

(Fixed Point)

A point $x \in X$ is said to be a Fixed point of a map;

$f: X \to X$ iff $f(x) = x$.

Path Connected Spaces:-

"Path"

Let (X, T) be a topological space and $x, y \in X$. A path form x to y in X is a continuous map;

$$\rho : [0,1] \to X$$

Such that

$$\rho(0) = x$$

&

$$\rho(1) = y.$$

Path Connected Space:-

A topological space (X, T) is said to be path connected iff for every $x, y \in X$ there exists a path ρ from x to y in X.

Examples:

⇒ $(\mathbb{R}, T_u)$

⇒ $(\mathbb{R}^2, T_u)$.

Theorem;

Let $(X, \mathcal{T})$ be a path connected topological space then $(X, \mathcal{T})$ is connected.

proposition;

Let $(X, \mathcal{T})$ be a path connected topological space and;

$$f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$$

be a continuous map. Then $f(X)$ is path connected.