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MTH 641

FUNCTIONAL ANALYSIS

MODULE # 60 To 113

(FINAL TERM SYLLABUS)

Don't look for someone who can solve your problems,
Instead go and stand in front of the mirror,
Look straight into your eyes,
And you will see the best person who can solve your problems!
Always trust yourself.

(BY ABU SULTAN)

MODULE NO. 60

LINEAR OPERATORS

➤ **Bounded Linear Operator**

Norms spaces are generalization of distances. By using Norm spaces we are going to discuss Bounded Linear Operator.

Bounded Linear Operator (Definition):

Let X and Y be normed spaces and $T : D(T) \rightarrow Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$.

$$\|Tx\| \leq c\|x\|$$

If this condition satisfied then we call T to be a bounded linear operator. Bounded function mean range is bounded but here bounded set is mapping over a bounded set so we call this a bounded linear operator. c is fix.

$$\frac{\|Tx\|}{\|x\|} \leq c, \quad x \in D(T) - \{0\}$$

The smallest possible value of c is supremum of left hand side. Then the value of c is called

$$c = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \quad \text{as}$$

We call the value as T norm

$$c = \|T\|$$

$$\text{If } D(T) = \{0\}, \quad \|T\| = 0$$

$$c = \|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

$$\|Tx\| \leq \|T\|\|x\|$$

This is the formula that we use for bounded linear operator.

MODULE NO. 61

BOUNDED LINEAR OPERATORS

➤ **Lemma (Norm)**

First we define the norm (equivalent definition) and then prove that the norm defined on T satisfies all four properties of Norm i.e. (N1) to (N4).

Lemma (Statement):

Let T be a bounded linear operator as defined before then an alternate formula for the norm of T is

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\|=1}} \|Tx\|$$

The norm defined on T satisfies (N1) to (N4).

Proof: Part (a)

$$\|Tx\| \leq c\|x\|$$

$$c = \|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in D(T), \\ \|x\|=1}} \|Tx\|$$

We have to prove

$$\sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in D(T), \\ \|x\|=1}} \|Tx\|$$

Let $\|x\| = a$; set $y = \frac{x}{a}$, $x \neq 0$,

$$\|y\| = \frac{\|x\|}{a} = 1$$

$$\|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

as T is linear so, we take constant a inside the norm

$$\|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \left\| T \left(\frac{1}{a} x \right) \right\| = \sup_{\substack{y \in D(T), \\ \|y\|=1}} \|Ty\| \quad \text{as } \frac{1}{a} x = y$$

Here variable is y which can be any other. Part (a) of lemma is proved.

Part (b):

$$\|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in D(T), \\ \|x\|=1}} \|Tx\|$$

N1: $\|T\| \geq 0$ is obvious.

N2: $\|T\| > 0 \Rightarrow T=0$,

$$\|T\| = 0 \Rightarrow Tx = 0, \quad \forall x \in D(T) \Rightarrow T = 0$$

N3: $\|\alpha T\| = \sup_{\substack{x \in D(T), \\ \|x\|=1}} \|\alpha Tx\| = \sup_{\|x\|=1} |\alpha| \|Tx\| = |\alpha| \sup_{\|x\|=1} \|Tx\| = |\alpha| \|T\|$ as $\sup_{\|x\|=1} \|Tx\| = \|T\|$

N4: $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

$$\begin{aligned} \|T_1 + T_2\| &= \sup_{\substack{x \in D(T), \\ \|x\|=1}} \|(T_1 + T_2)x\| \\ &\leq \sup_{\|x\|=1} \|T_1x + T_2x\| \leq \sup_{\|x\|=1} (\|T_1x\| + \|T_2x\|) \\ &= \sup_{\|x\|=1} \|T_1x\| + \sup_{\|x\|=1} \|T_2x\| = \|T_1\| + \|T_2\| \end{aligned}$$

First we define a $T \times T$ norm and then prove the four properties of norm.

MODULE NO. 62

EXAMPLES BOUNDED LINEAR OPERATORS

- Identity Operator
- Zero Operator
- Differentiation Operator
- Integral Operator

Identity operator:

$$I : X \rightarrow X \quad \Rightarrow \quad I_x = x \quad \{x \neq \{0\} \text{ normed space}\}$$

$$\|I\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|x\|}{\|x\|} \quad \text{as} \quad Tx = x$$

$$\|I\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} 1 = 1$$

Zero operator:

The norm space $O : X \rightarrow Y$, $O_x = 0$ $x \in X$

$$\|O\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = 0, \quad \|0\| = 0$$

Differentiation operator:

This is defined on normed space of all polynomial on $J=[0, 1]$

$$\|x\| = \max \{ |x(t)|, \quad t \in J \}$$

Value of t varies from 0 to 1 and where the value is maximum, that maximum value is norm of x .

applying operator the derivative. Differentiation operator is.

$$Tx(t) = x'(t)$$

Derivation is itself a linear operator.

Now we check that it is bounded or not. $\|Tx(t)\| \leq c \|x(t)\|$. If it is bounded then what is the value of c .

Let $x_n(t) = t^n$ $n \in \mathbb{N}$, what is the norm of $x_n(t)$

$$\|x_n(t)\| = \max \{ |x(t)| \mid t \in [0,1] \} = 1$$

Using operator $Tx_n(t) = nt^{n-1}$

define the norm $\|Tx_n(t)\| = \max |nt^{n-1}| = 1$

$$\|Tx_n(t)\| = \max(|nt^{n-1}|; t \in [0,1]) = n.1 = n$$

$$\frac{\|Tx_n\|}{\|x_n\|} = \frac{n}{1} = c, \quad n \in \mathbb{N}$$

As n had no bound so, there does not exist any c such that $\frac{\|Tx\|}{\|x_n\|} \leq c$ hold.

Now c is fixed number which does not depend upon N but in this case it depends on N , if we take c as n then next value $n+1$ will not satisfy this equation. It means that there does not exist

any c that this condition $\frac{\|Tx\|}{\|x_n\|} \leq c$ hold hence derivative operator is not bounded.

Integral Operator

Defined as $T : C[0,1] \rightarrow C[0,1]$,

$$y = Tx \quad y(t) = \int_0^1 k(t, \tau) x(\tau) d\tau$$

k is integral of T it is fix for different integral operator,

T is linear as integration is linear, also derivation is a linear operator same as integral is linear operator.

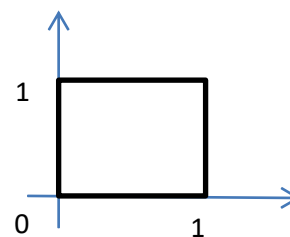
K is continuous on $J \times J$. We have two variables t and τ , $k(t, \tau)$

Whatever the value of k is, it should be in the square

$k(t, \tau)$ is bounded. And if it is bounded then

$k(t, \tau) \leq k_o$, $t, \tau \in J \times J$, $k_o \in \mathbb{R}$ where $J \times J$ is this square box.

$$|x(t)| \leq \max_{t \in J} |x(t)| = \|x\|$$



Now example,

$$\begin{aligned} \|y\| &= \|Tx\| = \max_{t \in J} \left| \int_0^1 k(t, \tau) x(\tau) d\tau \right| \\ &\leq \max_{t \in J} \int_0^1 |k(t, \tau)| |x(\tau)| d\tau \\ &\leq k_o \|x\| \end{aligned}$$

$\|Tx\| \leq k_o \|x\|$ it has k and k_o is fix so integral operator is a linear operator.

MODULE NO. 63

EXAMPLES BOUNDED LINEAR OPERATORS

➤ Matrix

Identity operator:

$$\begin{array}{ccc} T : \mathbb{R}^n \rightarrow \mathbb{R}^r & & \\ \begin{bmatrix} a_{11} & \cdot & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{r1} & \cdot & a_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \cdot \\ \xi_n \end{bmatrix} & = & \begin{bmatrix} x_1 \\ \cdot \\ x_n \end{bmatrix} \\ r \times n & n \times 1 & r \times 1 \\ A & x & = y \end{array}$$

The entries are $x = (\xi_j)$, $y = (\eta_j)$

And the matrix is $A = (\alpha_{ij})$, $1 \leq i \leq r$, $1 \leq j \leq n$

$$\eta_j = \sum_{k=1}^n \alpha_{jk} \xi_k$$

T is linear because the properties of matrices is it bounded?

$$\|x\| = \left(\sum_{m=1}^n \xi_m^2 \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n$$

and

$$\|y\| = \left(\sum_{j=1}^r \eta_j^2 \right)^{\frac{1}{2}}, \quad y \in \mathbb{R}^r$$

for bounded we have to check norm of T “T(x)”.

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$$\|Tx\| = \left(\sum_{j=1}^r \eta_j^2 \right)^{\frac{1}{2}}$$

$$\|Tx\|^2 = \sum_{j=1}^r \eta_j^2$$

$$\|Tx\|^2 = \sum_{j=1}^r \left(\sum_{k=1}^n \alpha_{jk} \xi_k \right)^2$$

Where $\eta_j = \sum_{k=1}^n \alpha_{jk} \xi_k$

Cauchy Schwarz inequality on above $\|Tx\|^2$

$$\leq \sum_{j=1}^r \left(\sum_{k=1}^n \alpha_{jk}^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^n \xi_m^2 \right)^{\frac{1}{2}} = x^2 \left(\sum_{j=1}^r \sum_{k=1}^n \alpha_{jk}^2 \right)$$

$$\|Tx\|^2 \leq c^2 \|x\|^2$$

Here is a c which depends upon T.

We can write as

$$\|Tx\| \leq c \|x\|$$

T is already linear and with this value of c we can say matrices is a linear bounded operator. in last four examples three are linear operator but differential was not linear operator.

MODULE NO. 71

LINEAR FUNCTION (EXAMPLES):

➤ Space $C[a, b]$

➤ Space l^2

Space $C[a, b]$:

We have define a linear function on space $C[a, b]$ that we have fixed an element t_0 from the set J as $t_0 \in J$. Now define a functional operator $f(x)$ which is operating on x which is element from $C[a, b]$. $x \in C[a, b]$

This x is not a variable, it is a function. So f_1 which is defined on $C[a, b]$ linear as it is a linear operator. f_1 is bounded. To find the norm

$$|f_1| = |x(b)| \leq \|x\|$$

$$\|x\| = 1 \Rightarrow \|f_1\| \leq 1 \dots\dots (i)$$

If we take $x_0 = 1$ and substitute in this equation we get

$$|f_1(x_0)| \leq \|f_1\| \cdot \|x\|$$

$$1 \leq \|f_1\| \cdot 1 \Rightarrow \|f_1\| \geq 1 \dots\dots (ii) \quad \text{From i) and ii)}$$

$$\|f_1\| = 1$$

So the function defined on C is linear, bounded and Norm is 1.

Space l^2

We choose a fix say $a = (a_j) \in l^2$

$$f(x) = \sum_{j=1}^{\infty} \xi_j a_j \quad x \in l^2, \quad x = (\xi_j)$$

This sequence is linear, converging and bounded.

For boundedness

$$|f(x)| = \left| \sum_{j=1}^{\infty} \xi_j a_j \right| \leq \sum_{j=1}^{\infty} |\xi_j a_j| \leq \sqrt{\sum_{j=1}^{\infty} |\xi_j|^2} \sqrt{\sum_{j=1}^{\infty} |a_j|^2} = \|x\| \cdot \|a\|$$

It is the same definition of bounded.

M of a complete metric space X is itself complete if and only if the set M is closed in X .

MODULE NO. 72

LINEAR FUNCTION:

- Algebraic Dual Space
- Second Algebraic Dual Space
- Canonical Mapping

Algebraic Dual Space

Set of all linear function defined on a vector space X is itself a vector space and called Algebraic Dual Space and denoted by X^*

Operation on this vector space are

1st Operation Sum

$$f_1 + f_2 \quad f_1, f_2 \text{ linear functional}$$

$$(f_1 + f_2)x = f_1(x) + f_2(x) \quad x \in X$$

2nd Operation Scalar Multiplication

$$(af)x = af(x)$$

Second Algebraic Dual Space X^{**}

Space	element	Vector at a point
X	$x \in X$	
X^*	g	$f(x)$
X^{**}	G	$g(x)$

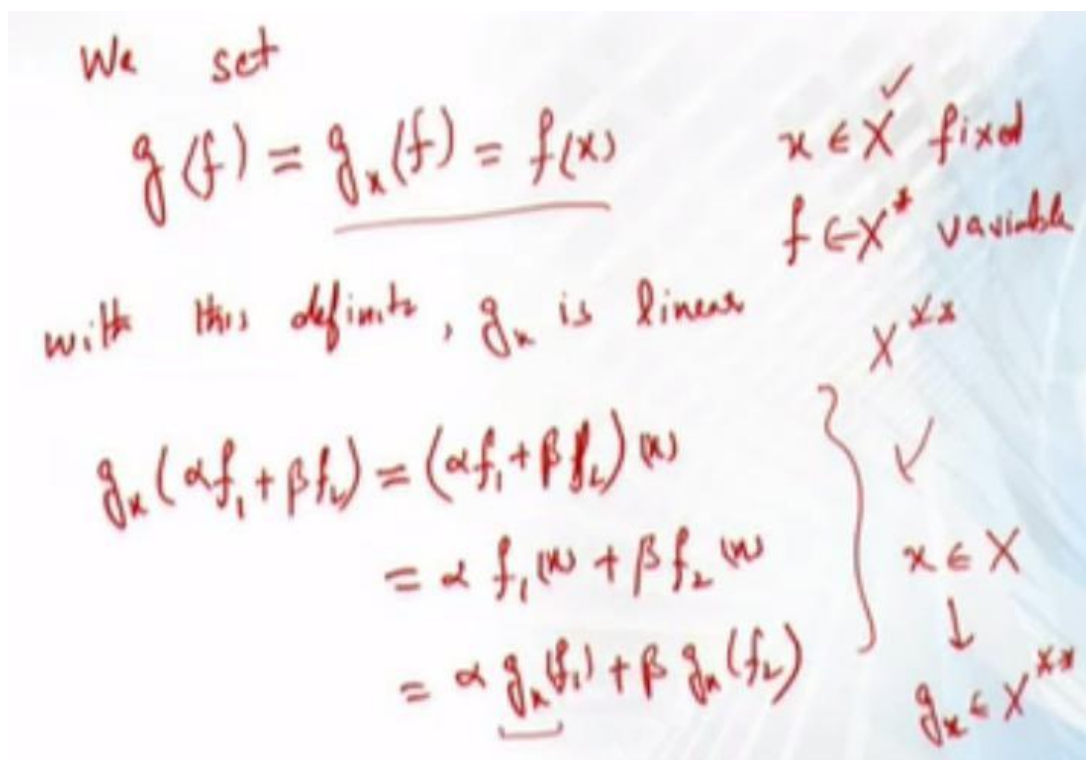
For each $x, g \in X^{**}$

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Conical Mapping:

$C : X \rightarrow X^{**}$ this mapping is called canonical mapping of X into X^{**} defined as $x \mapsto g_x$.

$$\begin{aligned} C(\alpha x + \beta y)(f) &= g_{\alpha x + \beta y}(f) \\ &= f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_y(f) \\ &= \alpha (Cx)(f) + \beta (Cy)(f) \end{aligned}$$

So, this is a linear function as well. Canonical mapping is a relation between X and X^{**} .

MODULE NO. 73

LINEAR FUNCTION:

- Algebraically Reflexive
- Second Algebraic Dual Space
- Canonical Mapping

Isomorphism:

It is one-one and onto map.

Algebraically Reflexive:

$$T : (X, d) \rightarrow (X^{\sim}, d^{\sim}) \text{ bijective}$$

$$d^{\sim}(T_x, T_y) = d(x, y)$$

$$C : X \rightarrow X^{**} \quad x \mapsto g_x.$$

If C is surjective (on b) bijection. $\mathfrak{R}(C) = X^{**}$

We call X to be algebraically reflexive.

Set of all linear function defined on a vector space X is itself a vector space and called

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MODULE NO. 74

LINEAR OPERATORS AND FUNCTIONAL ON FINITE DIMENSIONAL SPACES:

Finite dimensions mean basis which have finite many elements.

Let X and Y be finite dimension vector spaces over the same field.

Let $T : X \rightarrow Y$ be a linear operator. let $E = \{e_1, \dots, e_n\}$ be the basis for X and

$B = \{b_1, \dots, b_r\}$ be the basis for Y .

$$x \in X, \quad x = \xi_1 e_1 + \xi_2 e_2 + \dots + \xi_n e_n$$

$$y = Tx = T \left(\sum_{k=1}^n \xi_k e_k \right) = \sum_{k=1}^n T(\xi_k e_k) = \sum_{k=1}^n \xi_k T(e_k)$$

T is uniquely determined if the image $y_k = Te_k$ of n basis vectors $e_1, \dots, e_n$ are prescribed.

$$y = Tx; y \in Y \quad \{b_1, \dots, b_r\}$$

$$y = \eta_1 b_1 + \eta_2 b_2 + \dots + \eta_r b_r$$

$$Te_k \in Y, \quad Te_k = \tau_{k1} b_1 + \tau_{k2} b_2 + \dots + \tau_{kr} b_r$$

$$Te_k = \sum_{j=1}^r \tau_{kj} b_j$$

$$y = \sum_{j=1}^r \eta_j b_j = \sum_{k=1}^n \xi_k Te_k = \sum_{k=1}^n \xi_k \sum_{j=1}^r \tau_{kj} b_j$$

Combining these two summation

$$y = \sum_{j=1}^r \left(\sum_{k=1}^n \tau_{kj} \xi_k \right) b_j$$

$$\eta_j = \sum_{k=1}^n \tau_{kj} \xi_k$$

The image $y = Tx = \sum \eta_j b_j$ of $x = \sum \xi_k Te_k$ can be obtained from

$$\eta_j = \sum_{k=1}^n \tau_{kj} \xi_k$$

MODULE NO. 75

OPERATORS ON FINITE DIMENSIONAL SPACES:

Remarks:

As in the case of linear operators on a finite dimensional normed space, every linear functional defined on a finite dimensional normed space is bounded and hence continuous. Since for linear functionals range is either $\mathbb{R}$ or $\mathbb{C}$, which are complete. So X^* as the space of all bounded linear functionals defined on X , is also complete and hence is Banach space. This is true even if X is not a Banach space.

“Algebraic Dual Space of X ”: set of all linear functionals defined on X .

“Dual or Conjugate Space of X ”: X^* set of all continuous or bounded linear functionals defined on X .

We take algebraic dual when there is no condition of continuous or bounded linear functions.

Theorem:

Let X be an n -dimensional vector space and X^* be its dual space. Then

$$\dim X^* = \dim X = n.$$

X^* is collection of linear functions or linear operator while X may be any space.

Proof:

Let $\dim X = n$.

Let basis of X be $B = \{e_1, \dots, e_n\}$

We define a function.

$$f_j(e_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j: i, j = 1, \dots, n \end{cases}$$

$$\text{e.g. } j=1, \quad f(e_1)=1, f(e_2)=0, f(e_3)=0, \dots, f(e_n)=0$$

$$j=2, \quad f(e_1)=0, f(e_2)=1, f(e_3)=0, \dots, f(e_n)=0$$

but each n -tuples f_j in this case can be extended as linear functions on X .

MODULE NO. 76

OPERATORS ON FINITE DIMENSIONAL SPACES:

Lemma(Zero Vector):

Let X be a finite dimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$ then $x_0 = 0$.

B^* is the basis of X^*

$$\begin{aligned} \{f_1, f_2, \dots, f_n\} \quad i=j \\ \Rightarrow f_j(e_i) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \\ = \delta_{ij} \end{aligned}$$

Proof:

For all $x_0 = 0$,

$$x_0 = \sum_{i=1}^n x_i e_i \quad ; \quad f \in X^* \quad ,$$

$$f(x_0) = 0 \quad \Rightarrow \quad \sum_{i=1}^n f\left(\sum_{i=1}^n x_i e_i\right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^n x_i f(e_i) = 0 \quad , \quad j=1, \dots, n$$

$$\Rightarrow \quad x_j = 0 \quad , \quad \forall j=1, \dots, n$$

$$x_0 = \sum_{i=1}^n x_i e_i = 0 \quad \Rightarrow \quad x_0 = 0$$

MODULE NO. 77

OPERATORS ON FINITE DIMENSIONAL SPACES:

Theorem(Reflexivity):

A normed space X is said to be algebraically reflexive if there is an isometric isomorphism between X and X^{**} .

Ordinarily a normed space may not be reflexive.

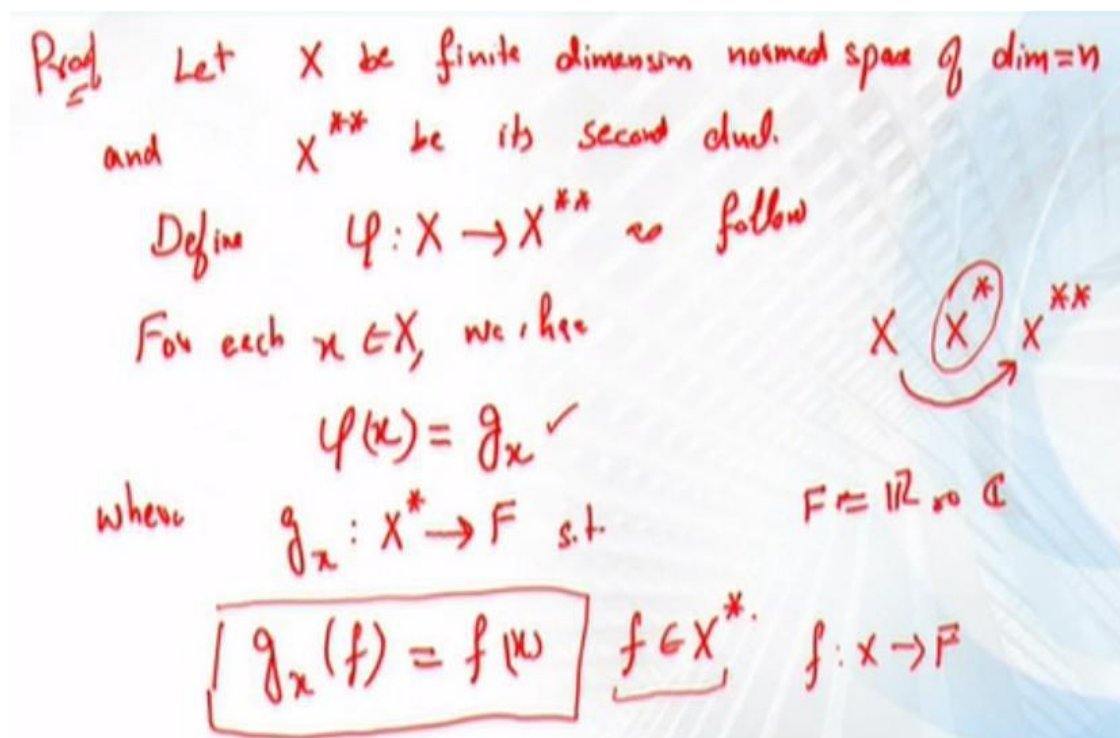
If X is an incomplete normed space even then X^* and X^{**} are Banach spaces. So in this case X cannot be a reflexive space.

However there are Banach spaces which are not reflexive.

Theorem:

A finite dimensional vector space is reflexive.

Equivalently, A finite dimensional normed space is isomorphic to its second dual.



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1) φ is linear $\varphi: X \rightarrow X^{**}$

$$\varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y) \quad x, y \in X$$

$$\Rightarrow \varphi(\alpha x + \beta y) = g_{\alpha x + \beta y}$$

for $f \in X^*$, $g_{\alpha x + \beta y}(f) = f(\alpha x + \beta y)$

$$= \alpha f(x) + \beta f(y)$$

$$= \alpha g_x(f) + \beta g_y(f)$$

$$g_{\alpha x + \beta y}(f) = (\alpha g_x + \beta g_y)(f)$$

$\varphi(x) = g_x \leftarrow$
 $\underline{\underline{g_x(f) = f(x)}}$

$$g_{\alpha x + \beta y} = \alpha g_x + \beta g_y$$

$$\varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y)$$

2) φ is injective (1-1)

Let $\forall x, y \in X$ s.t. $\varphi(x) = \varphi(y)$ (we have to show $\boxed{x=y}$)

$$\Rightarrow g_x = g_y$$

$$\Rightarrow g_x - g_y = 0 \leftarrow \text{operator}$$

For $f \in X^*$ $\Rightarrow (g_x - g_y)(f) = 0(f) = 0 \quad \underbrace{\forall f \in X^*}_{\text{v. Imp}}$

$$\Rightarrow g_x(f) - g_y(f) = 0$$

$$\Rightarrow f(x) - f(y) = 0 \Rightarrow f(x-y) = 0 \quad \forall f \in X^*$$

$\Rightarrow$ by zero lemma
 $x-y=0 \Rightarrow \boxed{x=y}$
 φ -linear. $\varphi: X \rightarrow X^{**}$
 φ -1-1 $\Rightarrow X \cong R(\varphi)$
 It remains to prove $\boxed{R(\varphi) = X^{**}}$
 Now by theorem $\varphi(x)=0 \Rightarrow x=0 \Leftrightarrow \varphi^{-1}$ exist
 $\Rightarrow \bar{\varphi}: R(\varphi) \rightarrow X^{**}$ exist

$\Rightarrow \boxed{\dim(R(\varphi)) = \dim X}$ by the same theorem
 if X^* is dual of X , X -f.d
 $\dim X = \dim X^*$
 applying again $\Rightarrow \dim X^* = \dim X^{**}$
 $\Rightarrow \dim X = \dim X^* = \dim X^{**} = \dim(R(\varphi))$
 $\dim(X^{**}) = \dim(R(\varphi))$ — (1)
 being V.S and (1), $\Rightarrow R(\varphi)$ is not a proper subspace of X^{**}

$R(\varphi) = X^{**}$ φ is onto
 $X \cong X^{**}$ X reflexive

MODULE No. 78

LINEAR TRANSFORMATION:

Q No.1:

Find the null space of $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ represented by

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + x_2 \end{bmatrix}$$

$2 \times 3 \quad 3 \times 1 \quad 2 \times 1$

What is meant by null space, it means we have to find those values of $x \in \mathbb{R}^3$ say

$x = (x_1, x_2, x_3)$ such that we operate T the answer is zeros as

All those x are element of null space.

$$\begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Also we can also say that

$$x_1 + 3x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 = 0$$

We can solve it by using any linear algebra method that will give us solution like echelon form or reduced echelon form and the base of that solution is called basis of null space. Basis mean when apply the element of $\mathbb{R}^3$ the answer should be zero and get a system of linear equation. Find the solution of this system of linear equation. And after finding the solution find the basis that basis are basis of null space.

Example.

Q.NO2

Find the null space of $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $(\xi_1, \xi_2, \xi_3) \leftrightarrow (\xi_1, \xi_2, -\xi_1 - \xi_2)$

- 1) Basis of $\mathbb{R}(T)$
- 2) Basis of $N(T)$
- 3) Matrix representation.

MODULE NO.79

EXERCISES

DUAL BASIS

Example 1:

a): Find the dual basis of X when basis of X are $B = \{(1, -1, 3), (0, 1, -1), (0, 3, -2)\}$,

Find $B^* = ?$, $X^* = ?$ do it yourself

b): let $\{f_1, f_2, f_3\}$ be basis of dual space for X and if X is given by

$$e_1 = (1, 1, 1), e_2 = (1, 1, -1), e_3 = (1, -1, 1)$$

Find $f_1(x), f_2(x), f_3(x)$ when $x = (0, 1, 0)$

MODULE NO.80

NORMED SPACES OF OPERATORS

- Examples of Dual Spaces
- $\mathbb{R}^n$

Isometric Isomorphism

A linear operator $\phi : X \rightarrow Y$. X, Y normed spaces, is said to be Isometric Isomorphism if

ϕ is bijective.

ϕ preserve norms.

That is for any

$x \in X$, $\|\phi(x)\| = \|x\|$ is

MODULE NO.81

EXAMPLES SPACES OF OPERATORS

- Examples of Dual Spaces
- l^1

Space l^1

The dual space of l^n is l^∞ means that it is bijective, it is linear and it preserve norm. After defining the map we shall prove these properties one by one.

Proof:

MODULE NO.82

BOUNDED LINEAR OPERATORS

Quiz: Complete norm spaces are called Banach spaces.

Theorem

Let $B(X, Y)$ be the set of all bounded linear operators from a normed space X to a normed space Y .

If Y is a Banach space, then $B(X, Y)$ is also a Banach.

Proof:

Let $\{T_n\}$ be an arbitrary Cauchy seq. in $B(X, Y)$.

We will show that $\{T_n\}$ converges to an operator T in $B(X, Y)$. Since $\{T_n\}$ is Cauchy for every

$\varepsilon > 0 \quad \exists \quad N$ such that $\|T_n - T_m\| < \varepsilon \quad (m, n > N)$

For all $x \in X$ and $(m, n > N)$ we have

$$\begin{aligned} \|T_n(x) - T_m(x)\| &= \|(T_n - T_m)(x)\| \\ &\leq \|T_n - T_m\| \|x\| < \varepsilon \|x\| \end{aligned}$$

Thus for a fixed x and given ε

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Thus for a fixed x and given $\bar{\epsilon}$ we may choose $z = z_n$ so that

$$\underline{z_n \|x\|} < \bar{\epsilon}$$

$$\Rightarrow \|T_n(x) - T_m(x)\| < \bar{\epsilon} \|x\| = z_n \|x\| < \bar{\epsilon}$$

$\Rightarrow \{T_n(x)\}$ is a Cauchy seq. in Y .

Since Y is complete

Let $T_n(x) \xrightarrow{\text{converge}} y$ for some $y \in Y$

We can define a map

$$T: X \rightarrow Y$$

$$T(x) = y$$

We will show that T is the required bounded linear operator.

to show (1) T is bounded (2) $T_n \rightarrow T$

T is linear $T(\alpha x + \beta z) : x, z \in X$
 $T(x) = y, T(z) = u$

$$\begin{aligned} \underline{T(\alpha x + \beta z)} &= \lim_{n \rightarrow \infty} T_n(\alpha x + \beta z) \\ &= \lim_{n \rightarrow \infty} T_n(\alpha x) + \lim_{n \rightarrow \infty} T_n(\beta z) \\ &= \alpha \lim_{n \rightarrow \infty} T_n(x) + \beta \lim_{n \rightarrow \infty} T_n(z) \\ &= \alpha y + \beta u \\ &= \underline{\alpha T(x) + \beta T(z)} \Rightarrow \underline{T \text{ is linear.}} \end{aligned}$$

1) T is bounded

$$\|T_n(x) - T_m(x)\| < \varepsilon \|x\|$$

$$T_n(x) \rightarrow y; \quad T: X \rightarrow Y \quad y = \underline{T(x)}$$

$$T_m(x) \rightarrow y = \underline{T(x)}$$

$$\underline{T(x)} = T(x)$$

Thus by continuity of norm we have

$$\begin{aligned} \|T_n(x) - T(x)\| &= \|T_n(x) - \lim_{m \rightarrow \infty} T_m(x)\| \\ &= \lim_{m \rightarrow \infty} \|T_n(x) - T_m(x)\| \leq \varepsilon \|x\| \end{aligned}$$

$\Rightarrow \underline{(T_n - T)}$ with $n > N$ is bounded

Also T_n is bounded.

$$\Rightarrow T = \overset{\checkmark}{T_n} - (\overset{\checkmark}{T_n - T})$$

$\Rightarrow T$ is also bounded

$$\Rightarrow T \in B(X, Y) \quad \xleftarrow{\quad} T: X \rightarrow Y$$

$\{T_n\}; \quad T_n \rightarrow T \nearrow$

$$\begin{aligned}
 & \|T_n(x) - T(x)\| \leq \varepsilon \|x\| \\
 & \frac{\|T_n(x) - T(x)\|}{\|x\|} \leq \varepsilon \\
 & \text{for } \|x\| = 1 \\
 & \Rightarrow \sup_{\|x\|=1} \|T_n(x) - T(x)\| < \varepsilon \\
 & \Rightarrow \|T_n - T\| < \varepsilon \quad \swarrow \quad \|T_n - T\| \rightarrow 0 \\
 & \Rightarrow T_n \rightarrow T \Rightarrow \{T_n\} \text{ converges in } B(X, Y).
 \end{aligned}$$

Hence $B(X, Y)$ is complete and Banach space.

MODULE No.83

FINITE HILBERT SPACES

Functional analysis course consist of three major parts parts

1. Metric space is set and we define a space on it that has a certain properties. If it is complete then it is complete space means it should converge within the space
2. Normed Spaces: Norm is a vector space and we define a norm on vector space. Norm is a generalization of distance function.
3. Finite Hilbert Spaces (Inner Product Space)

Hilbert Space

Quiz: Complete inner product space is called a Hilbert Space.

In inner product the generalization is dot product.

Inner product Space

Let V be a vector space over a field F where F is $\mathbb{R}$ or $\mathbb{C}$.

An inner product in V is a function $\langle \cdot, \cdot \rangle: V \times V \rightarrow F$ satisfying the following conditions:

Quiz:

Let $x, y, z \in V$; $\alpha \in F$ where α may be real or complex.

- $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$; but not true for second value as $\langle x, \alpha y \rangle \neq \alpha \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$

$\langle \cdot, \cdot \rangle: V \times V \rightarrow F$ inner product.

Inner Product Space

The pair $(V, \langle \cdot, \cdot \rangle)$ is called an inner product space.

a): $\langle ax+by, z \rangle$ where $x, y, z \in V$, $a, b \in F$

Using (iii) property $\langle ax + by, z \rangle = \langle ax, z \rangle + \langle by, z \rangle$

Using (ii) property $a \langle x, z \rangle + b \langle y, z \rangle$
 $\langle 0, z \rangle = \langle 0, x \rangle = 0 \langle x, z \rangle = 0$

b): **Quiz:**

for all $x, y \in V, a \in F$

$$\begin{aligned} \langle x, ay \rangle &= \overline{\langle ay, x \rangle} = \overline{a \langle y, x \rangle} \\ &= \overline{a} \overline{\langle y, x \rangle} = \overline{a} \langle x, y \rangle \end{aligned}$$

MODULE No.84

CAUCHY SCHWARZ INEQUALITY

Theorem:

For any two elements x, y in an inner product space V ,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|, \text{ the define norm is } \|x\| = \sqrt{\langle x, x \rangle}, \quad x, y \in V$$

Proof:

If $x=y=0$ then $0=0$

Let at least one of x and y is not equal to zero

Let $|\langle x + \lambda y, x + \lambda y \rangle| \geq 0$ by definition

$$\langle x, x + \lambda y \rangle + \langle \lambda y, x + \lambda y \rangle$$

$$\langle x, x + \lambda y \rangle + \lambda \langle y, x + \lambda y \rangle$$

MODULE No.85

NORM ON INNER PRODUCT SPACE

Theorem:

In an inner product space V , the function $\| \cdot \| : V \rightarrow \mathbb{R}^+$ given by

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad x \in V \text{ defines a norm in } V.$$

Proof:

$$N1: \|x\| \geq 0$$

$$\text{For a } x \in V, \|x\| = \sqrt{\langle x, x \rangle} \geq 0 \text{ as } \langle x, x \rangle \geq 0$$

N2:

$$\|x\| = 0 \Leftrightarrow \sqrt{\langle x, x \rangle} = 0 \Leftrightarrow \langle x, x \rangle = 0 \Leftrightarrow x = 0$$

$$N3: \|\alpha x\| = |\alpha| \|x\|$$

$$\text{now } \|\alpha x\| = \sqrt{\langle \alpha x, \alpha x \rangle} \Rightarrow \|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle$$

$$\Rightarrow \|\alpha x\|^2 = \alpha \overline{\alpha} \langle x, x \rangle = |\alpha|^2 \|x\|^2$$

$$N4: \|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$$

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$$\begin{aligned}
 \|x+y\|^2 &= \langle x+y, x+y \rangle \\
 &= \langle x, x+y \rangle + \langle y, x+y \rangle \\
 &= \overline{\langle x+y, x \rangle} + \overline{\langle x+y, y \rangle} \\
 &= \overline{\langle x, x \rangle} + \overline{\langle y, x \rangle} + \overline{\langle x, y \rangle} + \overline{\langle y, y \rangle} \\
 &= \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle \\
 \text{Now} \quad &= \langle x, x \rangle + \langle x, y \rangle + \overline{\langle x, y \rangle} + \langle y, y \rangle \\
 &= \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2 \quad \square \operatorname{Re}(z) \leq |z| \\
 &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\
 &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \square |\langle x, y \rangle| \leq \|x\|\|y\| \\
 &= (\|x\| + \|y\|)^2 \\
 \|x+y\|^2 &\leq (\|x\| + \|y\|)^2
 \end{aligned}$$

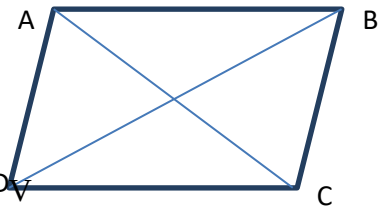
MODULE NO.86

PARALLELOGRAM LAW

$$\overline{AC}^2 + \overline{BD}^2 = 2(\overline{AB}^2 + \overline{BC}^2) \quad \text{Quiz}$$

Theorem:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \text{for all } x, y \in V$$



Proof:

$$\begin{aligned}
 \|x+y\|^2 &= \langle x+y, x+y \rangle \\
 &= \langle x, x \rangle + \langle x, y \rangle + \overline{\langle x, y \rangle} + \langle y, y \rangle \\
 &= \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2 \quad \dots(i)
 \end{aligned}$$

Replace $y=-y$

$$\begin{aligned}
 \|x-y\|^2 &= \langle x-y, x-y \rangle \\
 &= \langle x, x \rangle - \langle x, y \rangle - \overline{\langle x, y \rangle} + \langle y, y \rangle \\
 &= \|x\|^2 - 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2 \quad \dots(ii)
 \end{aligned}$$

Adding (i) and (ii)

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

That we have to prove.

Special Case:

Another result from above equations is

Subtracting (ii) from (i)

$$\|x+y\|^2 - \|x-y\|^2 = 4 \operatorname{Re} \langle x, y \rangle$$

If V is a real inner product space

$\operatorname{Re}(z)=z$ or $\operatorname{Re}\langle x, y \rangle = \langle x, y \rangle$

$$\langle x, y \rangle = \frac{1}{4} \left\{ \|x+y\|^2 - \right.$$

$$\|x - y\|_2 \}$$

The above form is when V is a real inner product space not complex space.

MODULE No.87

➤ POLARIZATION IDENTITY

➤ APPOLONIUS IDENTITY

Polarization Identity

For any x, y in complex inner product space

$$\langle x, y \rangle = \frac{1}{4} \left\{ \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \right\}$$

We have to prove this complex inner product space.

Proof:

Let $x, y \in V$

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \|x\|^2 + 2\operatorname{Re} \langle x, y \rangle + \|y\|^2 \\ &= \|x\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2 \\ &= \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 \end{aligned} \quad \dots (i)$$

If we replace $y=-y$

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + \langle x, -y \rangle + \langle -y, x \rangle + \|-y\|^2 \\ &= \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2 \end{aligned} \quad \dots (ii)$$

Replace $y = iy$ in eq(i)

$$\begin{aligned} \|x+iy\|^2 &= \|x\|^2 + \langle x, iy \rangle + \langle iy, x \rangle + \|iy\|^2 \\ &= \|x\|^2 + i\langle x, y \rangle + i\langle y, x \rangle + \|y\|^2 \quad \square \|iy\|^2 = \langle iy, iy \rangle = ii^{-1} \langle y, y \rangle = -i^2 \langle y, y \rangle \\ &= \|x\|^2 - i\langle x, y \rangle + i\langle y, x \rangle + \|y\|^2 \end{aligned} \quad \dots (iii)$$

Replace $y = -iy$ in eq(i)

$$\begin{aligned} \|x-iy\|^2 &= \|x\|^2 + \langle x, -iy \rangle + \langle -iy, x \rangle + \|-iy\|^2 \\ &= \|x\|^2 + i\langle x, y \rangle - i\langle y, x \rangle + \|y\|^2 \end{aligned} \quad \dots (iv)$$

Subtracting (ii) from (i)

$$\|x+y\|^2 - \|x-y\|^2 = 4\operatorname{Re} \langle x, y \rangle \quad \dots (v)$$

Subtracting (iv) from (iii)

$$\begin{aligned} \|x+iy\|^2 - \|x-iy\|^2 &= 2\{i\langle y, x \rangle - i\langle x, y \rangle\} \\ &= -2i\{\langle x, y \rangle - \overline{\langle x, y \rangle}\} = -2i\{\langle x, y \rangle - \overline{\langle x, y \rangle}\} \\ &= -2i(2i)\operatorname{Im} \langle x, y \rangle = 4\operatorname{Im} \langle x, y \rangle \end{aligned} \quad \dots (vi)$$

Now we solve $4\operatorname{Re} \langle x, y \rangle + 4\operatorname{Im} \langle x, y \rangle$

$$\|x+y\|^2 - \|$$

Appolonius Identity

$$\|x - y\|^2 + i\|x + y\|^2 - i\|x - y\|^2 = 4\{ \langle x, y \rangle \}$$

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2, \quad x, y, z \in V$$

Using parallelogram law

$$\|x' + y'\|^2 + \|x' - y'\|^2 = 2\|x'\|^2 + 2\|y'\|^2 \quad \text{put} \quad x' = z-x, \quad y' = z-y$$

Self-assignment

MODULE NO.88

➤ SPACE $C\left[0, \frac{\pi}{2}\right]$

➤ SPACE l^p

Counter example 1:

Space $C\left[0, \frac{\pi}{2}\right]$

Inner product define a norm and under this norm

Every inner product space is a norm space.

Every norm space is not an inner product space. This is not true always.

If a space is inner product then it satisfied the parallelogram law otherwise it is not an inner product space.

We take a norm and built an inner product space and then prove that this inner product space does not satisfy the parallelogram law.

The given set is $C\left[0, \frac{\pi}{2}\right]$ real valued continuous function defined on $C[a, b]$.

The norm of function $f \in C\left[0, \frac{\pi}{2}\right]$, is

$$\|f\| = \sup_{x \in \left[0, \frac{\pi}{2}\right]} |f(x)|,$$

Let $f, g \in C\left[0, \frac{\pi}{2}\right]$; $f(t) = \sin t$, $g(t) = \cos t$

We know that sin and cos are continuous functions. Let $C\left[0, \frac{\pi}{2}\right]$ is an inner product space

where the inner product $\langle \bullet, \bullet \rangle$ define by

$$\|f\| = \sqrt{\langle f, f \rangle} \Rightarrow \langle f, f \rangle = \|f\|^2$$

$$\|f\| = \sup_{x \in \left[0, \frac{\pi}{2}\right]} |f(x)|$$

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

As $f(t) = \sin t$, $g(t) = \cos t$

$$\|f\| = \sup_{x \in \left[0, \frac{\pi}{2}\right]} |\sin(x)| = 1 = \|g\|$$

$$\begin{aligned}\|f + g\| &= \sup_{x \in [0, \frac{\pi}{2}]} |f(x) + g(x)| \\ &= \sup_{x \in [0, \frac{\pi}{2}]} |\sin x + \cos x| = \sqrt{2}\end{aligned}$$

$$\|f - g\| = 1$$

Now

$$\begin{aligned}\|f + g\|^2 + \|f - g\|^2 &= 2\|f\|^2 + 2\|g\|^2 \\ (\sqrt{2})^2 + (1)^2 &= 2 \times 1^2 + 2 \times 1^2 \\ 2 + 1 &= 2 + 2 \\ 3 &= 4\end{aligned}$$

But $3 \neq 4$ so our supposition is wrong. This inner product space does not satisfy the parallelogram law. Hence every norm space is not an inner product space.

Counter example2: Space l^p

l^p Collection of all bounded sequences,

$p > 1, p \neq 2$ if $p=2$ then it will give inner product space

$$\{x_i\}, \quad \|x\| = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$$

We will see that $\langle x, x \rangle = \|x\|^2$ is an inner product space or not. We will check this if it satisfies the parallelogram or not.

Let

$$x = (1, 1, 0, 0, \dots) ; y = (1, -1, 0, 0, \dots)$$

$$\|x\| = \sqrt[p]{1^p + 1^p + 0 + 0 + \dots} = \sqrt[p]{2} = 2^{\frac{1}{p}}$$

$$\|y\| = \sqrt[p]{1^p + (-1)^p + 0 + 0 + \dots} = \sqrt[p]{2} = 2^{\frac{1}{p}}$$

$$x + y = (2, 0, 0, 0, \dots) \Rightarrow \|x + y\| = \sqrt[p]{2^p} = 2^{p \times \frac{1}{p}} = 2$$

$$x - y = (0, 2, 0, 0, \dots) \Rightarrow \|x - y\| = \sqrt[p]{2^p} = 2^{p \times \frac{1}{p}} = 2$$

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$2^2 + 2^2 = 2 \times 2^{\frac{1}{p}} + 2 \times 2^{\frac{1}{p}}$$

$$8 = 4 \times 2^{\frac{2}{p}} \quad \text{as } p > 1, p \neq 2$$

The values on both sides are also not equal so this does not satisfy the parallelogram law. Contradict to our supposition. So norm space is not an inner product space.

MODULE NO.90

➤ THEOREM (CONTINUITY OF INNER PRODUCT)

Theorem:

Let V be any inner product space. For any sequences $\{x_n\}$ and $\{y_n\}$ in V

$$x_n \rightarrow x, \quad y_n \rightarrow y \text{ implies } \langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$$

Proof:

$$\begin{aligned} & |\langle x_n, y_n \rangle - \langle x, y \rangle| \\ &= |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle| \\ &= |\langle x_n, y_n - y \rangle + \langle x_n - x, y \rangle| \\ &\leq |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle| \end{aligned}$$

Now from Cauchy Swarzinequality

$$\begin{aligned} |\langle x, y \rangle| &\leq \|x\| \|y\| \\ &\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \end{aligned}$$

Given that $x_n \rightarrow x$, $y_n \rightarrow y$ so,

$$\|y_n - y\| = \|y - y\| = 0, \quad \|x_n - x\| = \|x - x\| = 0 \text{ as } n \rightarrow \infty$$

As $n \rightarrow \infty$

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &\leq 0 \\ \langle x_n, y_n \rangle &\rightarrow \langle x, y \rangle \text{ as } n \rightarrow \infty \end{aligned}$$

Theorem:

If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in V , then the inner product $\langle x_n, y_n \rangle$ is a Cauchy sequence in F .

Proof:

$\{x_n\}, \{y_n\}$ are Cauchy sequence

To show $\langle x_n, y_n \rangle$ is also Cauchy Sequence.

$$\begin{aligned} \Rightarrow & \|x_n - x_m\| \rightarrow 0 \quad ; \quad \|y_n - y_m\| \rightarrow 0, \quad m, n \rightarrow \infty \\ & |\langle x_n, y_n \rangle - \langle x_m, y_m \rangle| = |\langle x_n, y_n \rangle - \langle x_n, y_m \rangle + \langle x_n, y_m \rangle - \langle x_m, y_m \rangle| \\ &= |\langle x_n, y_n - y_m \rangle + \langle x_n - x_m, y_m \rangle| \\ &\leq |\langle x_n, y_n - y_m \rangle| + |\langle x_n - x_m, y_m \rangle| \\ &\leq \|x_n\| \|y_n - y_m\| + \|x_n - x_m\| \|y_m\| \\ \Rightarrow & |\langle x_n, y_n \rangle - \langle x_m, y_m \rangle| \rightarrow 0, \text{ as } n, m \rightarrow \infty \\ \Rightarrow & \langle x_n, y_n \rangle \text{ is a Cauchy Sequence} \end{aligned}$$

MODULE NO.91

Examples of Inner product spaces

➤ SPACE $\mathbb{R}^n$

➤ SPACE $\mathbb{C}^n$

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- SPACE $\square [a, b]$
- SPACE l^n
- SPACE P_n (Collection of all polynomials of degree n)

Proof:

1. $\mathbb{R}^n$, the elements are of the form

$$x = (x_1, x_2, \dots, x_n) ; y = (y_1, y_2, \dots, y_n)$$

The inner product form is $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ (Note: check all axiom self-assignment)

The Norm is $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^n x_i x_i} = \sqrt{\sum_{i=1}^n x_i^2}$

2. $\square^n$

The elements are $z = (z_1, z_2, \dots, z_n) ; z' = (z'_1, z'_2, \dots, z'_n)$ if conjugate does not define then it does not satisfied the second or third axiom of inner product space.

The inner product form is $\langle z, z' \rangle = \sum_{i=1}^n z_i \overline{z'_i}$ (Note: check all axiom self-assignment)

3. $\square[a, b]$ be the space of all continuous function defined on $[a, b]$.

$$\langle f, g \rangle = \int_a^b \overline{f(t)} g(t) dt \quad \text{define an inner product on } C[a, b]$$

(Note: complex function can also be including. In previous example the $C[a, b]$ was not inner product space with define function definition).

$$\langle \bullet, \bullet \rangle : V \times V \rightarrow F$$

We will check all four properties of inner product as

- i): $\langle f, f \rangle = 0 \Leftrightarrow f = 0$
- ii): $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$
- iii): $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$
- iv): $\langle g, f \rangle = \overline{\langle f, g \rangle}$

it define inner product and is define inner product space.

4. l^n is a space of sequences.

$$l^2 : x \{x_i\}$$

The condition or norm is

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty$$

Let defined the inner product of $y = \{y_i\}$ is

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

Check all four axioms as exercise for inner product.

5. P_n

Let P_n be the collection of all polynomial of degree n (or less than n).

We can write this as $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ e.g $3x^2 - 2x + 1$ of degree two.

Let $u(x), v(x) \in P_n$

The inner product is

$$\langle u(x), v(x) \rangle = \int_a^b u(x)v(x)dx, \quad x \in [a, b]$$

with this define P_n is an inner product space.

We have not defined conjugate of $v(x)$ as the interval defined is a real valued so its conjugate is also real valued.

MODULE NO.92

Orthogonal Systems

➤ PYTHAGOREAN THEOREM

The dot product of two vectors when they are perpendicular is zero. Similarly in inner product if two vectors are perpendicular then their inner product is zero.

Theorem:

In an inner product space V and x, y in V if $x \perp y$ then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Proof:

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \end{aligned}$$

As x and y are perpendicular so $\langle x, y \rangle = 0, \langle y, x \rangle = 0$

$$\|x + y\|^2 = \langle x, x \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2$$

Generalized form:

$\{x_1, x_2, \dots, x_n\}$ be nonzero vectors in V inner product space such that

$$\langle x_i, x_j \rangle = 0, \quad i \neq j$$

This system $\{x_1, x_2, \dots, x_n\}$ is called orthogonal system as all vectors inside it are perpendicular to each other.

The generalized statement is $\|x_1 + x_2 + \dots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_n\|^2$

The idea of proof is

$$\begin{aligned} \left\| \sum_{i=1}^n x_i \right\|^2 &= \left\langle \sum_{i=1}^n x_i, \sum_{i=1}^n x_i \right\rangle \\ &= \langle x_1 + \dots + x_n, x_1 + \dots + x_n \rangle \\ &= \langle x_1, x_1 + \dots + x_n \rangle + \dots + \langle x_n, x_1 + \dots + x_n \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \langle x_i, x_j \rangle \\ \langle x_i, x_j \rangle &= \|x_i\|^2, \quad \text{if } i \neq j \quad \langle x_i, x_j \rangle = 0 \text{ and for } i = j \text{ then } \langle x_i, x_j \rangle = \|x_i\|^2 \end{aligned}$$

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2$$

MODULE NO.93

Orthogonal Systems

➤ THEOREM (LINEARLY INDEPENDENCE)

Any sequence $\{x_n\}$ of non-zero mutually orthogonal vectors in an inner product space V is linearly independent.

Proof: do it yourself

Let $x = (x_1, x_2, \dots, x_n)$ be the orthogonal sequence.

Remark:

$$\text{If } \langle x, x_i \rangle = 0, \quad \forall i=1, 2, \dots, n \quad \Rightarrow \quad \left\langle \sum_{i=0}^n \alpha_i x_i, x \right\rangle = 0$$

$$\left\langle \sum_{i=0}^n \alpha_i x_i, x \right\rangle = \langle \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n, x \rangle = \alpha_1 \langle x_1, x \rangle + \dots + \alpha_n \langle x_n, x \rangle = 0$$

**All Subjects Notes, Past Quizzes &
Past Papers**

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