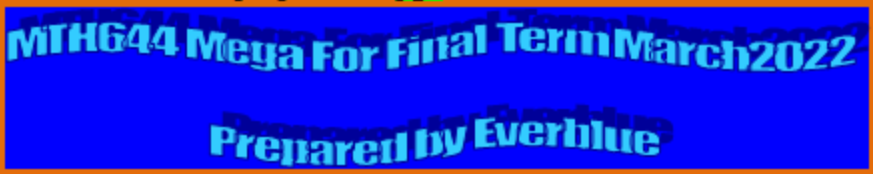


Lec (22-45), Topic (47-93)

Objective (2 files)

- Mth644 by Tanveer abbas pg4
- MTH 644 MCQ topic72+61 pg5



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Lec (22-45), Topic (47-93)

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Mth644 by tanveer abbas

Q1: in $\{ I_n \}_{n=1}^{\infty}$ for $A \subseteq R$ interval are measurable.

a) (a, ∞)

Q2: A subset $\phi \neq E \subseteq R$ is measurable for given $\varepsilon > 0$ there exist an open set $u \supseteq E$ with.

a) $m^*(u - E) < \varepsilon$

Q3: in closed set if $F \subseteq E$ for given $\varepsilon > 0$ then,

a) $m^*(E - F) < \varepsilon$

Q4: let $\phi \neq E \subseteq R$ then G_δ set G with $E \subseteq G$.

a) $m^*(G - E) = 0$

Q5: let $\phi \neq E \subseteq R$ then F_δ set F with $F \subseteq E$.

a) $m^*(E - F) = 0$

Q6: 1st dropped length are mean n=1. Solve $m = 2^{n-1} = 2^{1-1} = 2^0 = 1$ and $l(I_{nm}) = \frac{1}{3^n} = l(I_{11}) = \frac{1}{3^1} = \frac{1}{3}$

a) $\frac{1}{3}$

Q7: 2nd dropped length are mean n=2. Solve $l(I_{22}) = \frac{1}{3^2} = \frac{1}{9}$ short and mcqz

a) $\frac{1}{9}$

Q8: let $\{ E_n \}_{n=1}^{\infty}$ be an increasing sequence are.

a) $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$

Q9: let $\{ E_n \}_{n=1}^{\infty}$ be an decreasing sequence are.

a) $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$

Q10: Every Borel set are,

a) Measurable note: measurable pr 3 mcqz aty hy.

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Q11: of lebesgue measure m to the class of borel set B is said to Borel space (R, β, m).

a) Restriction

Q12: Countable sets are Borel sets with measure.

a) Zero

Q13: let $x, y \in A = [0,1)$ and in sum modulo 1 of x and y if $x + y$ then,

a) $x + y < 1$

Q14: let $x, y \in A = [0,1)$ and in sum modulo 1 of x and y if $x + y - 1$ then,

a) $x + y \geq 1$

Q15: if $x \in A$ then $f(x)$.

a) 1

Q16: if $x \notin A$ then $f(x)$.

a) 0

Q17: in positive if $f(x) \geq 0$ then f^+?

a) $f(x)$

b) $-f(x)$

Q18: in positive if $f(x) \leq 0$ then f^+?

a) 0

Q19: in nagtive if $f(x) \geq 0$ then f^-?

a) 0

b) F(x)

Q20: in nagtive if $f(x) \leq 0$ then f^-?

a) $-f(x)$

b) 0

Q21:if $\overline{\lim}_{n \in N}$ then limit are,

a) Superior

b) Inferior

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Q22:if $\underline{\lim}_{n \in N}$ then limit are,

a) Inferior

b) Superior

Q23: if $\lim_{n \to \infty} f_n = f$ then f is measurable.

a) $\lim_{n \to \infty} f_n = \overline{\lim}_{n \in N} f_n = \underline{\lim}_{n \in N} f_n = f$

Q24: if $A = \phi$ then χ_A ?

a) 0

b) 1

Q25: if $A = x$ then χ_x ?

a) 1

b) 0

Q26: if $\frac{f + |f|}{2}$ then, is pr 4 mcqz ho gya wo find k boly ga. Short question aur mcqz.

a) f^+

b) f

Q27: if $\frac{|f| - f}{2}$ then,

a) f

b) f^+

Q28: Restriction of a function let $f : X \to y$ be a function and $A \subseteq X$ then $g : A \to y$ defined as

a) $g(x) = f(x)$

b) $f(x) = g(x)$

Q29: if $\{x : f(x) > a\} \in M$ in f are measurable then kf.

a) Measurable

b) Not measurable

Q30: if $f + g$ are measurable then.

a) f measurable

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b) g measurable

c) Both a and b

Q31:let $\mathbb{M}$ be the class of measurable sets in $\delta-algebra$ then lebesgue measure of any set $E \subseteq \mathbb{M}$ is defined to its lebesgue outer measure.

a) $m^* = m$

b) $m^*(f) = m(f)$

Q32: if $f(x) = \cos x$ and $-\pi \leq x \leq \pi$ find f^+ ? Is pr sin bhi ata hy.

a) Cosx $-\pi \leq x \leq -\frac{\pi}{2}$

b) Sinx $-\pi \leq x \leq -\frac{\pi}{2}$

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Topic # 72

(1) MTH-644

A function that can takes values $-\infty$ or $+\infty$ is called Extented Real Value function

$$f : R_n \longrightarrow [-\infty, +\infty]$$

Extented Real value function

$$|f(x)| = \begin{cases} +f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

$$f^+ = \max\{f(x), 0\} = f \vee 0$$

$$f^- = \max\{-f(x), 0\} = -f \vee 0$$

Note

(i) $f^+ = \dfrac{f + |f|}{2}$ (ii) $f^- = \dfrac{|f| - f}{2}$

(iii) $f = f^+ - f^-$ (iv) $|f| = f^+ + f^-$

(v) $\text{Max}(f, g) = \dfrac{(f+g) + |f-g|}{2}$

(vi) $\text{Min}(f, g) = \dfrac{(f+g) - |f-g|}{2}$

Characteristic function

Let $\phi \neq X$ and $A \subseteq X$, then characteristic function

$$\chi_A : X \longrightarrow \mathbb{R}$$

$$\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(zero function)

Note (i) if $A = \phi \Rightarrow \chi_A(x) = 0 = 0(x) \Rightarrow \chi_\phi = 0$

(ii) if $A = X \Rightarrow \chi_A(x) = 1 = I(x)$ (identity fn) $\Rightarrow \chi_X = 1$

$\Rightarrow$ if $A \subseteq B \Rightarrow \chi_A \leq \chi_B$

$\Rightarrow \chi_{A \cap B} = \chi_A \cdot \chi_B$

$\Rightarrow \chi_{A-B} = \chi_A - \chi_{A \cap B}$

$\Rightarrow \chi_{A \times B} = \chi_A \cdot \chi_B$

$\Rightarrow$ if $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subset of X, then $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

$\Rightarrow$ let $\phi \neq A \subseteq X$ and for characteristic function $\chi_A : X \longrightarrow \mathbb{R}$ then χ_A is measurable iff A is measurable.

$\Rightarrow$ **Algebra of measurable Function**

If f & g are two extended real valued ~~function~~ and measurable function define on some domain $\phi \neq D$ then

(i) $f + k$ (ii) kf (iii) $f \pm g$ (iv) fg
(v) $\frac{f}{g}$ (vi) $|f|$ (vii) $f \vee g$ (viii) $f \wedge g$

are measurable functions

$\Rightarrow$ If f is measurable then $\frac{1}{f}$ is also measurable.

$\Rightarrow$ **Restriction of a function**

Let $f : X \longrightarrow Y$ be a function and $A \subseteq X$ then $g : A \longrightarrow Y$ defined as $g(x) = f(x)$ is called restriction of "f" to A and given as $f_A = g$

$\Rightarrow$ Restriction of a function is also measurable

$\Rightarrow$ Extented real value function define on set of measure zero is measurable

⇒ let $\{f_n\}$ be sequence of extended real valued function on "D" then

(i) $\sup f_n$, $\inf f_n$ are measurable

(ii) $\max(f_1, f_2)$ and $\min(f_1, f_2)$ are measurable

(iii) $\limsup f_n$, $\liminf f_n$ are measurable

(iv) If $\lim_{n\to\infty} f_n = f$ then f is measurable

Topic #6

⇒ **Borel Set**

A set E is said to be a borel set if it can be expressed as countable union or intersection of open or closed set.

eg $\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$

Borel measurable Space (a measure that is defined on all open set and thus on all borel sets)

Restriction of Lebesgue measure "m" to the class of borel set B is said to be borel measurable space. (R, B, m)

⇒ Every borel set is measurable

⇒ Countable set are boral sets with zero measure

⇒ **Sum Module (1)** let $x, y \in A = [0,1)$, we defined sum module 1 of x and y

$$x \hat{+} y = \begin{cases} x+y & \text{if } x+y < 1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

Translation Modulo 1

let $E \subseteq A = [0,1)$ and $y \in A$,
we defined a translation modulo 1 of E as

$$E \hat{+} y = \{ x \hat{+} y , x \in E \}$$

↓
Set of all sum modulo "1"

⇒ Set $E \hat{+} y$ is measurable and $m(E) = m(E \hat{+} y)$

Measurable function

Let "M" be the measurable space, a function $f : E \to R$ is measurable function on $\phi \neq E \in M$ if $\{x \in E : f(x) > a\} \in M \quad \forall a \in R$

⇒ let $f : E \to R \cup \{\pm\infty\}$, $\phi \neq E \in M$ then

(i) f is measurable iff $\{x \in E, f(x) \geq a\}$ is measurable

(ii) $\{x \in E, f(x) \geq a\}$ is measurable iff $\{x \in E, f(x) < a\}$ is measurable

(iii) $\{x \in E, f(x) < a\}$ is measurable iff $\{x \in E, f(x) \leq a\}$ is measurable

⇒ let $\phi \neq E \in M$, $f : E \to R \cup \{\pm\infty\}$ then
$\{x \in E : f(x) = a\} \in M, \quad \forall a \in R$

⇒ Interval (a, ∞) is measurable $\forall a \in R$

⇒ open interval $(a, b) \subseteq R$ is measurable

⇒ A subset $\phi \neq E \subseteq R$ is measurable iff for given $\varepsilon > 0$ $\exists$ open set $U \supseteq E$ with
$m^*(U - E) < \varepsilon$

Lebesgue Measure

Let $\mathcal{M}$ be the class of measurable set then the Lebesgue measure is defined as

$m^* = m$; $\forall A$; $m^*(A) = m^*(A \cap E) + m^*(A \cap E')$

⇒ let $\{E_i\}_{i=1}^{n}$ be sequence in measurable space $\mathcal{M}$ then m is

(i) Countably Subadditive

(ii) finitely Subadditive provide that $E_i \cap E_j = \phi$

(iii) Countably additive of E_i are pairwise disjoint

(iv) m is monotonic

⇒ lebesgue measure "m" is translation invariant

⇒ let E_1 & E_2 are measurable sets then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

Cantor set.

$$C = [0,1] - G = G'$$

A set of point lying on a single line segment that has a number of Unintuitive properties.

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M@L!K Hafeez

MTH # 644

Current papers:-

* ——— *

Q.No.1) (2 marks)

if $f(x) = \cos x$, $-\pi \leq x \leq \pi$

Find f^+.

Sol:-

As we know that

$f^+(x) = \max(f(x), 0)$

or

$f^+(x) = \frac{f(x) + |f(x)|}{2}$

So,

$$f^+(x) = \begin{cases} \cos x & -\pi \leq x < -\pi/2 \\ 0 & -\pi/2 \leq x < 0 \\ \cos x & 0 \leq x < \pi/2 \\ 0 & \pi/2 \leq x < \pi \end{cases}$$

Q.No.2:- (3 marks)

If $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of X then show that

$$\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$$

Sol:-

If $x \in A$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow x \in A_k$ (exactly for $n=k$)

$\chi_A(x) = 1$

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$x \notin \bigcup_{n=1}^{\infty} A_n = \{A_k\}$

$\Rightarrow \chi_{A_k}(x) = 1 \quad , \chi_{A_n}(x) = 0, \quad \forall\ n \neq k$

Sol

$\chi_A(x) = 1 = 0 + 0 + \cdots + \underbrace{1}_{n=k} + 0 + \cdots\cdots$

$\chi_A(x) = 1 = \sum_{n=1}^{\infty} \chi_{A_n}(x)$

$\Rightarrow \chi_A(x) = \sum_{n=1}^{\infty} \chi_{A_n}$

Q.No.3):- (5 marks)

A subset $\phi \neq E \subseteq R$ is measurable for given $\varepsilon > 0$ there exist an open set $U \supseteq E$ with

$m^*(U - E) < \varepsilon$ (finite).

Sol:-

suppose 'E' is measurable

AS

$m^*(E) < \infty$ (finite)

$\forall\ \phi \neq E \subseteq R$

there exist

an open set 'U'

such that

$E \subseteq U$

and for any $\varepsilon > 0$

$m^*(U) < m^*(E) + \varepsilon \rightarrow (1)$

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Since

$E \subseteq U$

$\Rightarrow U = E \cup (U - E)$

$\Rightarrow m^*(U) = m^*(E) + m^*(U - E)$

$m^*(U - E) = m^*(U) - m^*(E)$

By (1)

$m^*(U - E) = m^*(U) - m^*(E) < \varepsilon$.

(Q. No. 4) (3 Marks)

Show that if $\{E_i\}_{i=1}^{n}$ be a sequence in measurable space 'M' then 'm' is countably additive.

Sol:-

$\forall \ \phi \neq A \subseteq \mathbb{R}$

and for sequence $\{E_i\}_{i=1}^{n}$ of pairwise disjoint measurable sets.

$$m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

$$\Rightarrow m^*(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i)$$

So,

$$m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$$

So 'm' is countably additive.

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Q. No. 5) (3 Marks)

if $\{E_n\}$ is an increasing sequence then show that

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

Sol:-

Given that

$\{E_n\}$ is increasing sequence such that

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots\cdots$$

Suppose

$$F_1 = E_1, F_2 = E_2 - E_1, \cdots\cdots, F_n = E_n - E_{n-1}$$

where $\{F_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable sets.

$$\Rightarrow \bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} E_k$$

$$\Rightarrow \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k$$

So,

$$m^*\left(\bigcup_{k=1}^{\infty} F_k\right) = m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$= m\left(\bigcup_{k=1}^{\infty} F_k\right)$$

$$= \sum_{k=1}^{\infty} m(F_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} m(F_k)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} F_k\right)$$

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$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$= \lim_{n\to\infty} m^*(E_n)$$

Q. No. 6:- (5 marks)

Let $\phi \neq E \in M$. $f: E \to \mathbb{R} \cup \{+\infty\}$, then

$$\{x \in E : f(x) = a\} \in M \quad \forall\ a \in \mathbb{R}.$$

Sol:-

i) if $a < \infty$

since $a \in \mathbb{R}$

$$\Rightarrow \{x : f(x) = a\} = \{x : f(x) \geq a\} \cap \{x : f(x) \leq a\}$$

As

$$\{x : f(x) \geq a\},\ \{x : f(x) \leq a\} \in M$$

then

$$\{x : f(x) \geq a\} \cap \{x : f(x) \leq a\} \in M$$

So,

$$\{x : f(x) = a\} \in M.$$

ii) If $a = \infty$

Let

$$y \in \{x : f(x) = \infty\}$$

$$\Rightarrow y \in \{x : f(x) > n\}, \quad \forall\ n \in \mathbb{N}$$

$$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f(x) > n\}$$

So,

$$\{x : f(x) = \infty\} \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) > n\}.$$

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Q.No.7) (2 marks)

Define open set in $\mathbb{R}$.

Sol:-

An open set in $\mathbb{R}$ say (U) can be written as countable union of disjoint open intervals.

i.e $U = \bigcup_{n=1}^{\infty} \{I_n\}$.

As each open interval is measurable so countable union of measurable sets is also measurable.

$\Rightarrow$ U is measurable

Q.No.8):- (3 Marks)

Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$, $\phi \neq E \in M$ then f is measurable if $\{x \in E : f(x) \geq a\}$ is measurable. $\forall\ a \in \mathbb{R}$.

Sol:-

here

$$\{x \in E : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \tfrac{1}{n}\}$$

Let $y \in \{x : f(x) \geq a\}$

$\Rightarrow f(y) \geq a$, $\forall\ a \in \mathbb{R}$

$\Rightarrow f(y) > a - \frac{1}{n}$, $\forall\ n \in \mathbb{N}$

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So,

$$y \in \{x : f(x) > a - \tfrac{1}{n}\}, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f(x) > a - \tfrac{1}{n}\}$$

So,

$$\{x : f(x) \geq a\} \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) > a - \tfrac{1}{n}\}$$

So,

f is measurable.

Q.No.9) 5 (marks)

If $\{x : f(x) \leq a\}$ is measurable then $\{x : f(x) < a\}$ is also measurable.

Sol:-

We have to show that

$$\{x : f(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \tfrac{1}{n}\}$$

Let

$$y \in \{x \in E : f(x) \leq a\} \rightarrow (1)$$

$$\Rightarrow f(y) \leq a, \quad \forall a \in \mathbb{R}$$

$$f(y) < a + \tfrac{1}{n}, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \{x : f(x) < a + \tfrac{1}{n}\}, \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f(x) < a + \tfrac{1}{n}\} \rightarrow (2)$$

By (1) and (2)

$$\{x \in E : f(x) \leq a\} \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) < a + \tfrac{1}{n}\}$$

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Q.No.10) (5 marks)

Let $\phi \neq X \supseteq A$ and for characteristic function $\chi_A : X \to \mathbb{R}$ χ_A is measurable iff 'A' is measurable.

Solution:-

Suppose χ_A is measurable

$\Rightarrow \forall\ q \in \mathbb{R}$

The set $\{x : \chi_A(x) > q\}$ is measurable.

for $q = 0$

$\{x : \chi_A(x) \geq 0\}$ is measurable.

So A is measurable.

Conversely:-

Let A is measurable, and we prove 'χ_A' is measurable.

As

$$\{x : \chi_A(x) > q\} = \begin{cases} \phi & , q \geq 1 \\ A & , 0 \leq q < 1 \\ A \cup A^c & , q < 0 \end{cases}$$

If A is measurable
then A^c is also measurable.

So,
$A \cup A^c$ is also measurable.
and we know that

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ϕ is measurable.

$\Rightarrow \{x : \chi_A(x) > a\} \in M.$

So,

χ_A is measurable.

Q. No. 11) (2 marks)

Show that Lebesgue measure 'm' of a sequence $\{E_i\}_{i=1}^{\infty}$ is countable subadditive.

Sol:-

Since All E_i's are measurable

$\Rightarrow \bigcup_{i=1}^{k} E_i$ is also measurable.

$\Rightarrow$ 'M' is space of Lebesgue measurable sets.

$$m\left(\bigcup_{i=1}^{k} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

$$= \sum_{i=1}^{\infty} m(E_i)$$

Q No. 12) (2 Marks)

Determine the measure of 1st step construction of cantor set from [0 1].

Solution:-

we know that

$m = 2^{n-1}$

For 1st step put $n=1$

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$n = 2^{1-1}$

$= 2^0$

$= 1$

So

$G_7 = I_{nm}$

$= I_{11}$

$m(G_7) = m(I_{11}) = l(I_{11}) = \frac{1}{3^1} = \frac{1}{3}$

Also

Cantor set $= G_7' = [0,1] - G_7$

Question No. 13) (2 Marks)

$\phi \neq A, B \subseteq X$ and x is in $A \cup B$, then verify that.

$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$$

Sol:-

Suppose $x \in A \cup B$.

Then there are three cases arises

$x \in A$ and $x \in B$

$x \notin A$ and $x \in B$

$x \in A$ and $x \notin B$

In case (i) – (iii)

$\chi_{A \cup B} = 1 = \chi_A + \chi_B - \chi_A \cdot \chi_B$

$\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$ $\because \chi_A \cdot \chi_B = \chi_{A \cap B}$

Suppose $x \notin A \cup B$

Then $x \notin A$ and $x \notin B$

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Thus

$$\chi_{A\cap B} = 0 = \chi_A + \chi_B - \chi_A \cdot \chi_B$$

Q.No 14) (3 Marks)

Let f be a extended real function define in a set E of $\exists x_0$ measure. Then show that

$$\{x \in E : f(x) > a\} \leq 0 \;,\; \forall a \in \mathbb{R}$$

Sol:-

Let $\phi \neq E \subseteq \mathbb{R}$ such that $m^*(E) = 0$ and

$$f : E \to \mathbb{R} \cup \{\pm\infty\}.$$

To prove 'f' is measurable, we have to show that

$$\{x \in E : f(x) > a\} \in M \;,\; \forall a \in \mathbb{R}$$

Since

$$\{x \in E : f(x) > a\} \subseteq E$$

$$\Rightarrow m^*\{x \in E : f(x) > a\} \leq m^*(E)$$

$$\Rightarrow m^*\{x \in E : f(x) > a\} \leq 0 \quad \text{(given)}$$

But by non-negativity of 'm'

$$\Rightarrow m^*\{x \in E : f(x) > a\} \geq 0$$

$$\Rightarrow m^*\{x \in E : f(x) > a\} = 0 \;,\; \forall a \in \mathbb{R}$$

$$\Rightarrow \{x \in E : f(x) > a\} \in M \;,\; \forall a \in \mathbb{R}$$

So,

f is measurable.

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Q. No. 15) (5 marks)

Prove that $\min(f,g)$ is measurable. If f & g and $(f+g)$ are measurable.

$$\min(f,g) = \frac{(f+g)-(f-g)}{2}$$

Sol:-

As given that f, g and $f+g$ are measurable. First we will show that $f-g$ is measurable.

For this

As g is measurable

$\Rightarrow$ $-g$ is also measurable.

$\Rightarrow$ $f+(-g)$ is measurable.

$\Rightarrow$ $f-g$ is measurable.

As $f, g, f+g$ and $f-g$ are measurable so,

$\min(f,g)$ is also measurable.

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Q. No. 16:- (5 marks)

Show that

$$\min(f,g) = \frac{(f+g) - (f-g)}{2}$$

Sol:-

Since

$$|f-g|(x) = |f(x)-g(x)|$$

$$|f-g|(x) = \begin{cases} f(x)-g(x) & \text{if } f(x) \geq g(x) \\ g(x)-f(x) & \text{if } f(x) \leq g(x) \end{cases}$$

case (i):-

If $f(x) \geq g(x)$

$$f(x) - g(x) \geq 0$$

$$|f(x)-g(x)| = f(x)-g(x)$$

So,

$$\frac{(f+g)-(f-g)}{2}(x) = \frac{f(x)+g(x)-|f-g|(x)}{2}$$

$$= \frac{f(x)+g(x)-(f(x)-g(x))}{2}$$

$$= \frac{f(x)+g(x)-f(x)+g(x)}{2}$$

$$= g(x)$$

$$= \min(f,g)$$

Case ii)

If $f(x) < g(x)$

$$f(x) - g(x) < 0$$

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$|f(x) - g(x)| = g(x) - f(x)$

and

$$\min(f,g) = f(x)$$

So,

$$\frac{(f+g) - (f-g)}{2}(x) = \frac{f(x) + g(x) - (g(x) - f(x))}{2}$$

$$= f(x)$$

$$= \min(f,g)$$

So,

$$\min(f,g) = \frac{(f+g) - (f-g)}{2}$$

Q.No.17) (2 Marks)

If f g and $f+g$ are measurable then show that $f-g$ is also measurable.

Sol:-

As given that

f, g and $f+g$ are measurable.

$\Rightarrow -g$ is also measurable.

So,

$f + (-g) \in M$

$f - g$ is measurable.

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Q. No. 18) (3 Marks)

Let $E \subseteq A = [0,1)$ be a measurable set, then for each $y \in A$, the set $E \dot{+} y$ is measurable and

$$m(E) = m(E \dot{+} y)$$

Then show that m is finitely additive.

Sol:-

Suppose

$$E_1 = E \cap (0, 1-y)$$

and

$$E_2 = E \cap [1-y, 1)$$

Since

E, $[0, 1-y)$, $[1-y, 0)$ are measurable

$\Rightarrow$ E_1 and E_2 are also measurable.

Also

$$E_1 \cap E_2 = \{E \cap (0, 1-y)\} \cap \{E \cap [1-y, 1)\}$$

$$= E \cap \{(0, 1-y) \cap [1-y, 1)\}$$

$$= E \cap \phi$$

$$= \phi$$

and

$$E_1 \cup E_2 = E \cup \phi = E$$

$$\Rightarrow m(E) = m(E_1) + m(E_2)$$

$\Rightarrow$ m is finitely additive.

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Q.No.19) (2 Marks)

$f(x) = |x+2|$

find f^+ if $f(x) \geq 0$.

Sol:

As we know that

if $f(x) \geq 0$

$$f^+ = \frac{f + |f|}{2}$$

here

$f(x) = |x+2| = x+2$

$|f(x)| = ||x+2|| \quad \because f(x) \geq 0$

$= x+2$

So,

$$f^+ = \frac{(x+2) + (x+2)}{2}$$

$= x+2$

$= |x+2|$

$= f(x)$ Ans

Q.No.20:- (5 marks)

Let $\{f_n\}$ be a sequence of extended real valued measurable function on 'D', then

$\sup_{n\in\mathbb{N}} f_n$ is measurable.

Let $f = \sup_{n\in\mathbb{N}} f_n$.

we have to show that

'f' is measurable.

For this we have to show

$\{x : f(x) > q\} \in M$, $\forall q \in \mathbb{R}$.

we claim

$$\{x : f(x) > q\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > q\}$$

Let

$y \in \{x : f(x) > q\}$

$\Rightarrow f(y) > q$

since

$f = \sup_{n\in\mathbb{N}} f_n$

$\Rightarrow f(y) \leq f_n(y)$, $\forall n \in \mathbb{N}$

So,

$f(y) > q$

$\Rightarrow q < f(y) \leq f_n(y)$, $\forall n \in \mathbb{N}$

$\Rightarrow f_n(y) > q$, $\forall n \in \mathbb{N}$

$\Rightarrow y \in \{x : f_n(x) > q\}$, $\forall n \in \mathbb{N}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > q\}$

$\Rightarrow \{x : f(x) > q\} \subseteq \bigcap_{n=1}^{\infty} \{q : f_n(x) > q\}$

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So,

$\{x : f(x) > a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\}$ is measurable.

$\Rightarrow \sup_{n \in \mathbb{N}} f_n$ is measurable.

(Q. No. 21) (5 marks)

Evaluate $m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{mn}\right)$ where these intervals I_{mn} are pairwise disjoint and

$m^* = \ell(I_{mn}) = \frac{1}{3^n}$.

Sol:-

we know that

$$G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} (I_{mn})$$

$$m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{mn}\right)$$

Since m is countably additive.

So,

$$m(G) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{mn})$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} \ell(I_{mn})$$

expand first by 'n'

$$= \sum_{m=1}^{2^{1-1}} \ell(I_{1m}) + \sum_{m=1}^{2^{2-1}} \ell(I_{2m}) + \cdots\cdots$$

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$= l(I_{11}) + l(I_{21}) + l(I_{22}) + l(I_{31}) + l(I_{32})$
$+ l(I_{33}) + l(I_{34}) + \ldots\ldots$

So,

$m(G) = 1$. hs

(for Q. No. 22) (5 marks)

check the measurability of Cantor set in n^{th} steps.

Sol:-

We know that Dropped interval is

$$G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$$

Since 'G' is measurable due to countable union of disjoint intervals of open set.

$\Rightarrow$ Cantor set $(C) = G'$ = closed and measurable

Since

$$G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} (I_{nm})$$

$$m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$$

As m is countably additive.

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{nm})$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} l(I_{nm})$$

first expand by 'n'

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$$= \sum_{m=1}^{2^{1-1}} l(I_{1m}) + \sum_{m=1}^{2^{2-1}} l(I_{2m}) + \sum_{m=1}^{2^{3-1}} l(I_{3m}) + \cdots$$

$$= l(I_{11}) + l(I_{21}) + l(I_{22}) + l(I_{31}) + l(I_{32}) + l(I_{33}) + l(I_{34}) + \cdots$$

As

$$l(I_{nm}) = \frac{1}{3^n}$$

So,

$$= \frac{1}{3} + \left(\frac{1}{3^2} + \frac{1}{3^2}\right) + \left(\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}\right) + \cdots$$

$$= \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^3} + \cdots$$

It is geometric series

$$a_1 = \frac{1}{3} \qquad r = \frac{2}{3}$$

$$S_\infty = \frac{a}{1-r} = \frac{1/3}{1-2/3} = 1$$

So,

$$m(G) = 1$$

Since

$$[0,1] = G \cup C$$

$$m[0,1] = m(G \cup C)$$

$$1 = m(G) + m(C)$$

$$1 = 1 + m(C)$$

$$\Rightarrow m(C) = 0$$

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Q.No.23) (2 mark)

Define Borel set.?

Sol:-

A set E is said to be a Borel set if it can be expressed as countable union or intersection of open or closed sets.

e.g.

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$$

Q.No.24):- (3 marks)

Let f be an extended real valued measurable function on D. Let $\phi \neq A \subseteq D$ then restriction of f to A is also measurable.

Sol:-

Since f is measurable.

$\Rightarrow \{x \in D : f(x) > q\} \in M$, $\forall q \in \mathbb{R}$

To prove 'f_A' is measurable, we show

$\{x \in A : f_A(x) > q\} \in M$.

Note that

$\{x \in A : f_A(x) > q\} = \{x \in D : f(x) > q\} \cap A$

on R.H.S both sets are measurable

$\Rightarrow \{x \in A : f_A(x) > q\} \in M$

As

Intersection of two measurable sets is measurable.

So,

f_A is measurable.

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Q.No.25):- (2 Marks)

if $f(x) = |x+2|$, $f(x) \geq 0$

find $f^{-} = ?$

Sol:-

As we know that

$$f^{-} = \frac{|f| - f}{2} \quad \text{if } f(x) < 0.$$

but here

$$f(x) \geq 0$$

so

$$f^{-}(x) = 0.$$

so there is nothing to prove.

Q.No.26):- (5 marks)

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable function on Ω. then Show that $\sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \right)$ is measurable.

Sol:-

we know that

$$\underline{\lim} f_n = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} f \right)$$

$$\inf_{k \geq n} f_k = \inf \{ f_k, f_{k+1}, \ldots, f_{k+l}, \ldots \}$$

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As given

$\{F_n\}_{n=1}^{\infty}$ is a sequence of measurable functions.

Now

$\sup_{n\in\mathbb{N}} \{F_n\}_{n=1}^{\infty}$ is also measurable.

$\Rightarrow \underline{\lim} f_n = \sup_{n\in\mathbb{N}} \{F_n\} = \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} f\right)$

So,

$\sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} f\right)$ is measurable.

Q. No. 27):- (3 marks)

open interval $(a,b) \subseteq \mathbb{R}$ is measurable.

Sol:-

As we know that

(a,∞) is measurable.

$\Rightarrow (a,\infty)^c = (-\infty, a)$ is also measurable.

$\forall\, n\in\mathbb{N}$

and if $n\to\infty$

$(-\infty, b) = \bigcup_{n=1}^{\infty} \left(-\infty, b-\frac{1}{n}\right)$ is also measurable being countable union of measurable set.

$\Rightarrow (a,\infty) \cap (-\infty, b) = (a,b)$.

also measurable being intersection of two measurable sets.

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Q.No.28) (5 marks)

If f is a real valued measurable function then f^2 is also measurable.

Sol:-

We have to show that

$\{x : f^2(x) > a\} \in M$, $\forall a \in \mathbb{R}$

since

$f^2(x) > a$

$f(x) > \sqrt{a}$ and $f(x) < -\sqrt{a}$

So,

$\{x : f^2(x) > a\} = \{x : f(x) > \sqrt{a}\} \cup \{x : f(x) < -\sqrt{a}\}$

Since

f is measurable.

So,

$\{x : f(x) > \sqrt{a}\} \in M$

and

$\{x : f(x) < -\sqrt{a}\} \in M$

$\Rightarrow \{x : f(x) > \sqrt{a}\} \in M$

$\Rightarrow \{x : f(x) > \sqrt{a}\} \cup \{x : f(x) < -\sqrt{a}\} \in M$

So,

$\{x : f^2(x) > a\} \in M$.

$\Rightarrow f^2$ is measurable.

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Q. No. 29):- (5 marks)

If $\emptyset \neq A, B \subseteq X$ then prove that $\chi_{A-B} = \chi_A - \chi_{A \cap B}$

Sol:-

(Case i):-

if $x \in A-B$

$x \in A$ but $x \notin B$, $x \notin A \cap B$

$\Rightarrow \chi_{A-B}(x) = 1$

$\chi_A - \chi_{A \cap B} = 1 - 0$

$\chi_A - \chi_{A \cap B} = \chi_{A-B}$

Case ii)

if $x \notin A-B$, there are three cases

i) $x \notin A$, $x \notin B$, $x \notin A \cap B$

ii) $x \notin A$, $x \in B$, $x \notin A \cap B$

iii) $x \in A$, $x \in B$, $x \in A \cap B$

In all cases

$\chi_{A-B}(x) = 0$

and

$\chi_A - \chi_{A \cap B} = 0$

So

$\chi_{A-B} = \chi_A - \chi_{A \cap B}$.

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Q.No.30) (5 marks)

For a given $\phi \neq E \subseteq R$ and $\varepsilon > 0$, if there is a closed set $F \subseteq E$ such that $m^*(E-F) < \varepsilon$ then show that there exist an open set $O \supseteq E$ satisfying $m^*(O-E) < \varepsilon$

Sol:-

for $\phi \neq E \subseteq R$

$\Rightarrow E' \subseteq R$.

there exist an open set $O \supseteq E$ and

$m^*(O-E) < \varepsilon \rightarrow (1)$

$O \supseteq E$

$\Rightarrow E = E'' \supseteq O' = F$ (say)

ie closed set.

$\Rightarrow E \supseteq F$

$m^*(E-F) = m^*(E \cap F')$

$= m^*(F' \cap E)$

$= m^*(O - E') < \varepsilon$ Hence

Q.No.31) (2 marks)

If $m^*(G-E) = 0$ then show that $m^*(G-E') = 0$.

Solution:-

As $\phi \neq E \subseteq R$ and $E \subseteq G$

then $E' \subseteq G$. and $\phi \neq E'$

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then $m^*(G-E') = 0$.

Q. No. 32) (2 marks)

If A and B are infinitely measurable sets then show that

$m(A \cup B) + m(A \cap B) = m(A) + m(B)$.

Sol:-

If $m(A) = m(B) = \infty$ (infinite)

then the result is obvious.

So,

$m(A \cup B) + m(A \cap B) = m(A) + m(B)$.

Q. No. 33):- (5 marks)

Lebesgue measure 'm' is translational invariant.

Sol:-

As $\phi \neq E \subseteq M$ is measurable.

Since we know that

'E' is measurable if and only if for given $\varepsilon > 0$, there exist an open set $U \supseteq E'$ such that

$m^*(U - E) < \varepsilon$.

As

'U' is open set in R.

$\Rightarrow U+y$ is also open in R, $\forall y \in R$.

If $E \subseteq U$

$\Rightarrow E+y \subseteq U+y$

$\Rightarrow (U+y) - (E+y) = (U - E) + y$

$m^*[(U+y) - (E+y)] = m^*[(U-E)+y]$

Since

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Lebesgue outer measure is translational invarient.

So,

$$m^*[(U+y) - (E+y)] = m^*(U-E) < \varepsilon$$

But

$$m^* = m$$

$$\Rightarrow \quad m[(U+y) - (E+y)] = m[U-E] < \varepsilon$$

So,

$E+y$ is also measurable.

$\Rightarrow$ m is translational invarient.

Q. No. 34):- (2 marks)

Prove that kf is measurable for $k > 0$.

Sol:-

Since f is measurable.

$$\Rightarrow \{x : f(x) > a\} \in M \quad \forall a \in \mathbb{R}$$

If $k > 0$

$$f(x) > a \quad , \forall a \in \mathbb{R}$$

$$\Rightarrow f(x) > \frac{a}{k}$$

$$\Rightarrow kf(x) > a \quad , \forall a \in \mathbb{R}$$

So,

$$\{x : kf(x) > a\} \in M \quad , \forall a \in \mathbb{R}$$

So, 'kf' is measurable.

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Q.No. 35) (5 marks)

Countable sets are Borel sets with zero measure.

Sol:-

(i) Singleton

$\forall a \in R$

$$\{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a + \frac{1}{n}\right) ; n \in N$$

since

$\left\{a - \frac{1}{n}, a + \frac{1}{n}\right\}$ is a decreasing sequence

with $m(E_n) = \frac{2}{n} < \infty$

where

$$E_n = \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$$

$$\Rightarrow m\{a\} = m\left\{\bigcap_{n=1} E_n\right\}$$

$$= \lim_{n \to \infty} m(E_n)$$

$$= \lim_{n \to \infty} l(E_n)$$

$$= \lim_{n \to \infty} \left(\frac{2}{n}\right)$$

$$= \frac{2}{\infty}$$

$$= 0$$

(ii) Finite set:-

$$\{x_1, x_2, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{n} \{x_i\}\right) = \sum_{i=1}^{n} m\{x_i\}$$

$$= 0$$

Because $\{x\}$ is a single and measure of singleton is zero.

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Case iii) Infinite set:-

$$\{x_1, x_2, \cdots\} = \bigcup_{i=1}^{\infty} \{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) = \sum_{i=1}^{\infty} m\{x_i\}$$

$$= 0.$$

Q.No. 36) (5 marks)

For $\phi \neq E \subseteq \mathbb{R}$ given that G_δ-set $E \subseteq G$ and $m^*(G-E) = 0$ then there exist F_σ-set with $F \subseteq E$ then show that

$$m^*(E-F) = 0.$$

Sol:-

As

$$\phi \neq E \subseteq \mathbb{R}$$

$$\Rightarrow \phi \neq E' \subseteq \mathbb{R}$$

For

$$\phi \neq E',$$

given that

$$E' \subseteq G$$

$$m^*(G-E') = 0$$

$$\Rightarrow m^*(G-E') = 0$$

$$m^*(G \cap (E')') = 0$$

$$m^*(G \cap E) = 0$$

$$m^*(E \cap G) = 0$$

$$m^*(E \cap (G')') = 0$$

$$m^*(E-G') = 0$$

$$m^*(E-F)$$

where

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$$G_{\delta} = G = \bigcup_{n=1}^{\infty} G_n$$

$$\Rightarrow G_{\delta}' = G' = \left(\bigcup_{n=1}^{\infty} G_n\right)'$$

$$= \bigcap_{n=1}^{\infty} G_n'$$

$$= \bigcap_{n=1}^{\infty} F_n'$$

$$= F$$

$$= F_{\sigma}.$$

So

$$m^*(E-F) = 0$$ Ans

Q.No. 37) (5 marks)

Show that

$$\chi_{A\cap B} = \chi_A \cdot \chi_B$$

Sol:-

i) If $x \in A \cap B$

$x \in A$ and $x \in B$

$\chi_{A\cap B}(x) = 1 = \chi_A = \chi_B$

$\chi_{A\cap B} = 1 \cdot 1 = \chi_A \cdot \chi_B$

$$\chi_{A\cap B} = \chi_A \cdot \chi_B$$

ii)

$x \notin A \cap B$

$x \in A$ but $x \notin B$

$\chi_A(x) = 1$, $\chi_B(x) = 0$

$\chi_{A\cap B}(x) = 0$

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$\chi_{A\cap B} = 1 \cdot 0$

$= \chi_A \cdot \chi_B$

iii)

$x \notin A \cap B$

$x \in B$ but $x \notin A$

So,

$\chi_{A\cap B} = 0 \cdot 1$

$= \chi_A \cdot \chi_B$

iv)

$x \notin A \cap B$

$x \notin A, \quad x \notin B$

$\Rightarrow \chi_{A\cap B} = 0 \cdot 0$

$= \chi_A \cdot \chi_B$

So, in all cases

$\chi_{A\cap B} = \chi_A \cdot \chi_B$

(Q.No. 38)

Show that

$\chi_{A\times B} = \chi_A \cdot \chi_B$

i) if $x \in A \times B$

$x \in A$ and $x \in B$

$\chi_A(x) = 1, \quad \chi_B(x) = 1$

$\chi_{A\times B} = 1$

$= 1 \cdot 1$

$= \chi_A \cdot \chi_B$

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ii) if $x \notin A \times B$

a) $x \in A$ but $x \notin B$

b) $x \notin A$ but $x \notin B$

c) $x \notin A$ but $x \in B$

in all cases

$$\chi_{A\times B}(x) = 0$$

So,

$$\chi_{A\times B} = \chi_A \cdot \chi_B$$

Q.No. 39) (3 marks)

Let $\{E_i\}_{i=1}^{n}$ be a seq. in measurable space M, then m^* is monotone.

Sol:-

Let $\phi \neq A, B \in M$ such that $B \subseteq A$.

$$\Rightarrow A = B \cup (A-B)$$

Since m^* is finitely additive

$$\Rightarrow m^*(A) = m^*(B) + m^*(A-B)$$

$$\because m^*(A-B) \geq 0$$

So,

$$m^*(A) \geq m^*(B)$$

$$m^*(B) \leq m^*(A)$$

So,

$$m(B) \leq m(A).$$

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Q. No. 40):- (5 marks)

Let $\{E_n\}$ be a decreasing sequence of measurable sets such that $m(E_k) < \infty$, for some $k \in \mathbb{N}$ then

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

Sol:-

Let $m(E_1) < \infty$

Since $\{E_n\}$ is decreasing sequence.

So,

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$$

$$\Rightarrow \phi = E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \cdots$$

$$\cdots \subseteq E_1 - E_n \subseteq \cdots$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty}(E_1 - E_n)\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$$

Since

$$\bigcup_{n=1}^{\infty}(E_1 - E_n) = \bigcup_{n=1}^{\infty}(E_1 \cap E_n')$$

$$= (E_1 \cap E_1') \cup (E_1 \cap E_2') \cup \cdots$$

$$= E_1 \cap [E_1' \cup E_2' \cup \cdots]$$

$$= E_1 \cap [E_1' \cup E_2' \cup \cdots]$$

$$= E_1 \cap [E_1 \cap E_2 \cap \cdots]'$$

$$= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)'$$

$$= E_1 - \left(\bigcap_{n=1}^{\infty} E_n\right)$$

So,

$$m^*\left(E_1 - \bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty}(E_1 - E_n)$$

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$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$$

$$= \lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

we get

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n).$$ Ans

Q. No. 41):- (2 marks)

Every Borel set is measurable.

Sol:-

As we know that

'Every open and closed set is measurable'

$\Rightarrow$ 'countable union or intersection of closed or open sets is also measurable'

So,

Borel set is measurable being union or intersection of closed or open sets.

Q. No. 42):- (2 marks)

Give examples of Borel set.

Sol:-

Both F_σ and G_δ sets

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are called Borel set.

$G_\delta = \bigcap_{i=1}^{\infty} U_i$

$F_\sigma = \bigcup_{n=1}^{\infty} F_n$.

Q.No.43:- (2 marks)

Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$ and $\phi \neq E \in M$ be measurable space then show that

$\{x \in E : f(x) > 0\} \in M$.

Sol:-

As we know that by definition of measurable function "a function $f: E \to \mathbb{R}$ is lebesgue measurable function on $\phi \neq E \in M$ if

$\{x \in E : f(x) > a\} \in M$

for every $a \in \mathbb{R}$.

Q.No.44):- (5 marks)

Let $\phi \neq E$ be a measurable set and $f: E \to \mathbb{R} \cup \{\pm\infty\}$ then show that

$\{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$.

Sol:-

Let

$y \in \{x : f(x) = -\infty\}$

$\Rightarrow y \in \{x : f(x) < -n\}$, $\forall n \in \mathbb{N}$

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$\Rightarrow t \in \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$

So)

$$\{x : f(x) = -\infty\} \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$$

Conversely

Let

$$t \in \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$$

$\Rightarrow t \in \{x : f(x) < -n\}, \forall n \in \mathbb{N}$

$\Rightarrow f(t) < -n, \forall n \in \mathbb{N}$

$\Rightarrow t \in \{x : f(x) < -n\} \subseteq \{x : f(x) = -\infty\}$

So)

$$\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$$

Q. No. 45):- (5 marks)

If f and g are extended real valued measurable function then prove that

$$\bigcup_{p \in Q} [\{x : f(x) > p\} \cap \{x : a - g(x) < p\}] \subseteq \{x : (f+g)(x) > a\}$$

Sol:-

Let

$$t \in \bigcup_{p \in Q} [\{x : f(x) > p\} \cap \{x : a - g(x) < p\}]$$

$\Rightarrow t \in \{x : f(x) > p\} \cap \{x : a - g(x) < p\}$, For some $p \in Q$

$\Rightarrow t \in \{x : f(x) > p\}$ and $t \in \{x : a - g(x) < p\}$

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$f(t) > p$ and $a - g(t) < p$

$\Rightarrow$ $p > a - g(t)$

So,

$f(t) > p > a - g(t)$

$\Rightarrow$ $f(t) > a - g(t)$

$\Rightarrow$ $(f+g)t > a$

$\Rightarrow t \in \{x : (f+g)(x) > a\}$

So,

$\bigcup_{p \in Q} [\{x : f(x) > p\} \cap \{x : a - g(x) < p\}] \subseteq \{x : (f+g)x > a\}$

Q.No. 46):- (3 marks)

Let f be a real valued and measurable function defined on domain D, then show that $|f|$ is measurable.

Sol:-

Since f is measurable.

$\Rightarrow \{x : f(x) > a\} \in M$, $\forall a \in R$.

We have to show that

$\{x : |f|(x) > a\} \in M$, $\forall a \in R$

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$$\{x : |f|(x) > a\} = \{x : |f(x)| > a\}$$

$$= \begin{cases} \{x : f(x) > a\} \cup \{x : f(x) < -a\}, & \forall a > 0 \\ D \in M & a \leq 0 \end{cases}$$

In R.H.S we get a measurable set.

$\Rightarrow \{x : |f|(x) > a\} \in M \quad \forall a \in \mathbb{R}.$

So,

$|f|$ is measurable.

Q.No.47):- (3 marks)

Show that $f = f^+ - f^-$

Sol:-

if $f(x) \geq 0$

$f^+(x) = f(x)$, $f^-(x) = 0$

$\Rightarrow (f^+ - f^-)x = f^+(x) - f^-(x)$

$= f(x) - 0$

$= f(x)$

So,

$f^+ - f^- = f$

Q.No.48) (2 marks)

Show that $|f| = f^+ + f^-$

Sol:-

if $f(x) \geq 0$

$f^+(x) = f(x)$ and $f^-(x) = 0$

So

$(f^+ + f^-)x = f^+(x) + f^-(x)$

$= f(x) \rightarrow (1)$

If $f(x) \leq 0$

$f^+(x) = 0$, $f^-(x) = -f(x)$

So,

$(f^+ + f^-)x = f^+(x) + f^-(x)$

$= -f(x) \rightarrow (2)$

By (1) and (2)

$f^+ + f^- = |f|$

Remember in your prayers

MTH644 Current Papers Questions (Fall 2021 Final Term)

① If f is measurable and $k>0$ then show that kf is measurable. ~~$\forall k \in \mathbb{R}$~~

Sol: "f" is measurable.

$\Rightarrow \{x : f(x) > a\} \in M, \quad \forall a \in \mathbb{R}$

If $k>0$, $\because f(x) > a, \quad \forall a \in \mathbb{R}$

$\Rightarrow f(x) > \frac{a}{k} \Rightarrow kf(x) > a, \quad \forall a \in \mathbb{R}$

$\therefore \{x : kf(x) > a\} \in M, \quad \forall a \in \mathbb{R}$

So "kf" is measurable.

② If f, g and $f+g$ is measurable then $f-g$ is measurable.

Sol: Given that f, g and $f+g$ is measurable.

We have to prove "$f-g$" is measurable.

$\because g$ is measurable $\Rightarrow (-1)g = -g$ is also M

$\Rightarrow f + (-1)g = f - g$

$\because f+g$ is measurable $\Rightarrow f-g$ is also measurable.

③ If $m(A)$ and $m(B)$ are infinitely measurable then prove $m(A)+m(B) = m(A \cup B) + m(A \cap B)$.

Sol: Given that $m(A) = m(B) = \infty$, then result is obvious.

$\Rightarrow m(A \cup B) = m(A \cap B) = \infty$,

So $m(A)+m(B) = m(A \cup B) + m(A \cap B)$

(proved)

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(4) Find: dropped interval in contor set, for $n=1$

Sol: We know that

$$m = 2^{n-1}$$

for $n=1$, $m = 2^{1-1} = 2^0 = 1$

Hence dropped interval G_1 is

$$G_1 = I_{mm} = I_{11}$$

(5) Define Borel Measure Space.

Sol: Restriction of lebesgue measure "m" to the class of borel set "B" is said to be Borel measure space $(\mathbb{R}, B, m)$.

(6) $f(x) = |x+2|$, find f^+ for $f(x) \geq 0$

Sol: We know that

$$f^+ = \frac{f + |f|}{2}$$

$f = |x+2| \Rightarrow f = x+2 \quad \because f(x) \geq 0$

$|f| = ||x+2|| \Rightarrow |f| = x+2 \quad \because f(x) \geq 0$

So

$$f^+ = \frac{x+2+x+2}{2} = \frac{2x+4}{2}$$

$$f^+ = \frac{2(x+2)}{2}$$

$$f^+ = x+2 \quad \text{Ans.}$$

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$\exists$ there exists

⑦ Lebesgue measure "m" is translation invariant.

Sol: Given that say $\phi \neq E \in M$ (measurable)

$\therefore$ E is measurable $\Leftrightarrow$ for given $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$, such that $m^*(U-E) < \varepsilon$

"U" is open set in $\mathbb{R} \Rightarrow$ "U+y" is also open in $\mathbb{R}$. $\forall\ y \in \mathbb{R}$

If $E \subset U \Rightarrow E+y \subseteq U+y$

$\Rightarrow (U+y)-(E+y) = (U-E)+y$

$m^*[(U+y)-(E+y)] = m^*[(U-E)+y] = m^*(U-E) < \varepsilon$

But $m^* = m \Rightarrow m[(U+y)-(E+y)] = m(U-E) < \varepsilon$

$\Rightarrow$ "E+y" is also measurable.

So "m" is translation invariant. (proved)

⑧ Let $E \subset A = [0,1)$ be a measurable set, then for each $y \in A$. The set $E \hat{+} y$ is measurable and $m(E) = m(E \hat{+} y)$.

Sol: Suppose $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$

The set E, $E \cap [0, 1-y)$ and $E \cap [1-y, 1)$ are measurable

So E_1 and E_2 are also measurable. Further

$E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$

$\Rightarrow m(E) = m(E_1) + m(E_2) \longrightarrow ①$ $\because$ "m" is finitely additive.

Let $y_1 \in E_1$ then $y_1 < 1-y \Rightarrow y_1 + y < 1$

$\Rightarrow y_1 \hat{+} y = y_1 + y \Rightarrow E_1 \hat{+} y = E_1 + y \longrightarrow ②$

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Now if $y_2 \in E_2 \Rightarrow \nexists\, y_2 \geq 1-y \Rightarrow y_1 + y \geq 1$

$\Rightarrow y_2 \hat{+} y = y_2 + y - 1 \Rightarrow E_2 \hat{+} y = E_2 + (y-1) \longrightarrow (b)$

$\because m'$ is translation invariant.

$\Rightarrow E_1 + y$ and $E_2 + (y-1)$ are measurable.

$\Rightarrow E_1 \hat{+} y$ and $E_2 \hat{+} (y-1)$ are measurable

$\Rightarrow m(E_1 \hat{+} y) = m(E_1 + y) = m(E_1)$

and $m(E_2 \hat{+} y) = m(E_2 + (y-1)) = m(E_2)$ $\longrightarrow (a)$

$\because E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$

$\Rightarrow E \hat{+} y = (E_1 \hat{+} y) \cup (E_2 \hat{+} y)$ and $(E_1 \hat{+} y) \cap (E_2 \hat{+} y) = \phi$

$\Rightarrow m(E \hat{+} y) = m(E_1 \hat{+} y) + m(E_2 \hat{+} y)$ (union of two disjoint)

$= m(E_1) + m(E_2)$ by (a)

$m(E \hat{+} y) = m(E)$ by (1) (proved)

(9) A subset $\phi \neq E \subseteq \mathbb{R}$ is measurable $\Leftrightarrow$ for given $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$ with $m^*(U-E) < \varepsilon$ (for finite)

Sol: Suppose E is measurable.

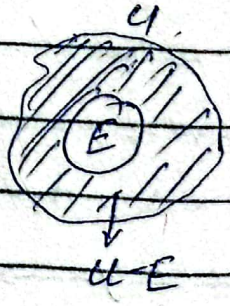
If $m^*(E) < \infty$, $\forall\, \phi \neq E \subseteq \mathbb{R}$, for $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$, $\Rightarrow m^*(U) < m^*(E) + \varepsilon \longrightarrow (a)$

$E \subseteq U \Rightarrow U = E \cup (U-E)$

$\Rightarrow m^*(U) = m^*(E) + m^*(U-E)$

$\Rightarrow m^*(U-E) = m^*(U) - m^*(E) < \varepsilon$ (from eq (a))

$\Rightarrow m^*(U-E) < \varepsilon$ (proved)



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δ Delta σ sigma

(10) Prove that every Borel set is measurable.

Sol: Every open and closed set is measurable.
⇒ countable union or intersection of closed or open set is also measurable.
Therefore Borel set is also measurable being union or intersection of closed or open sets.

(11) Give examples of Borel set.

Sol: Examples of Borel set:

(i) $G_\delta = \bigcap_{n=1}^{\infty} u_n$ (ii) $F_\sigma = \bigcup_{n=1}^{\infty} F_n$

(12) Let $\{f_n\}$ be a sequence of extended real valued measurable function on "D" then prove that $\sup_{n\in N}\left(\inf_{k\geq n} f_k\right)$ is also measurable.

Sol: let $p = \sup_{n\in N}\left(\inf_{k\geq n} f_k\right)$ and

$G_n = \inf\{f_n, f_{n+1}, f_{n+2}, \ldots f_{n+k}, \ldots\}, \forall n\in N$

Each G_n is measurable by the measurability of f_n for each n and $\inf_{n\in N} f_n$ is measurable.
Consequently $p = \sup_{n\in N} G_n$ is a measurable function because $\sup_{n\in N} f_n$ is measurable.

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(13) Let $f: E \to R \cup \{\pm\infty\}$, $\phi \neq E \in M$, where M be measurable space. Then show that:

$$\{x \in E : f(x) < a\} \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$$

Sol:

Let $y \in \{x \in E : f(x) \leq a\} \to (a)$

$\Rightarrow f(y) \leq a, \forall\ a \in R.$

$\Rightarrow f(y) < a + \frac{1}{n}, \forall\ n \in R$

$\Rightarrow y \in \{x \in E : f(x) < a + \frac{1}{n}\}, \forall\ n \in R.$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \to (b)$

From eq. (a) and (b)

$$\{x \in E : f(x) \leq a\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$$

(proved)

(14) Evaluate $m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$ where interval I_{nm} are pairwise disjoint and $m(I_{nm}) = \ell(I_{nm}) = \frac{1}{3^n}$

Sol:

Let $m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{nm}) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} \ell(I_{nm})$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} \left(\frac{1}{3^n}\right)$$

$$= \sum_{m=1}^{2^{1-1}} \left(\frac{1}{3^1}\right) + \sum_{m=1}^{2^{2-1}} \left(\frac{1}{3^2}\right) + \sum_{m=1}^{2^{3-1}} \left(\frac{1}{3^3}\right) + \cdots$$

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$$m(G_1) = \frac{1}{3} + \left(\frac{1}{3^2} + \frac{1}{3^2}\right) + \left(\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}\right) + \cdots\cdots$$

$$m(G_1) = \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^3} + \cdots\cdots$$

$$m(G_1) = 1$$

$a = \frac{1}{3}$, $r = \frac{2}{3}$

$$S_\infty = \frac{a}{1-r} = \frac{1/3}{1-2/3}$$

$$S_\infty = \frac{1/3}{1/3} = 1$$

$$m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right) = 1 \quad \text{Ans}$$

(15) Check the measurability of the cantor set in nth steps.

Sol: In general, at nth step, 2^{n-1} interval I_{nm}, $m = 1, 2, 3, \ldots 2^{n-1}$ are removed. Each I_{nm} has obviously length equal $\frac{1}{3^n} = 3^{-n}$. We have a sequence $\{I_{nm}\}$ of pairwise disjoint open sets. Thus the set $G_1 = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$ being the union of a sequence of open sets is open and measurable. The compliment of G_1 with respect to $[0, 1]$ is called cantor set and is denoted by C. The set C is closed and so it must be a measurable.

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(16) If "f" and "g" are two extended real valued and measurable functions defined on same domain ($\phi \neq I$) then prove that $f \wedge g$ is a measurable.

Sol: $\because f \wedge g = \min(f, g) = \frac{1}{2}[(f+g) - |f-g|]$

So $f \wedge g$ is measurable because $(f+g)$ and $|f-g|$ are measurable and "kf" is measurable $\forall\, k \in \mathbb{R}$.

OR

$(f \wedge g)(x) = \min(f(x), g(x))$

Observe that any $a \in \mathbb{R}$, we have

$\{x : (f \wedge g)(x) > a\} = \{x : f(x) > a\} \cap \{x : g(x) > a\}$

$f(x)$ and $g(x)$ are measurable functions and the intersection of two measurable function is also measurable $\Rightarrow (f \wedge g)(x)$ is measurable.

(17) Determine the measure 1st step construction of contor set from $[0, 1]$

Sol: We know that

$$m(G_1) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$$

For 1st step

put $n = 1 \Rightarrow m = 2^{n-1} = 2^{1-1} = 2^0 = 1$

$$m(G_1) = m(I_{11}) = \ell(I_{11})$$

$$m(G_1) = m(I_{11}) = \ell(I_{11}) = \frac{1}{3^1} = \frac{1}{3} \qquad \because \ell(I_{nm}) = \frac{1}{3^n}$$

Ans

(18) Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$ and $\phi \neq E \in M$, be the measurable space then prove that $\{x \in E : f(x) > 0\}$ is also measurable

Sol: By the definition of measurable function $f: E \to \mathbb{R}$ is lebesgue measurable function on $\phi \neq E \in M$ if $\{x \in E : f(x) > a\} \in M$ (measurable) for every $a \in \mathbb{R}$.

So $\{x \in E : f(x) > 0\}$ is also measurable because $0 \in \mathbb{R}$.

(19) Let $\phi \neq E \subset \mathbb{R}$, show that there exist F_σ set $F \subset E$.

Sol: For each n, let F_n be a closed set with $F_n \subseteq E$ and $m^*(E - F_n) < \frac{1}{n}$. Then $E = \bigcup_{n=1}^{\infty} F_n$ is an F_σ set $F_n \subseteq \bigcup_{n=1}^{\infty}$

$= F_n \subseteq E$ proved

(20) If $\phi = A, B \subset X$ and same member x are in $A \cup B$ Then verify that $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$

Sol: Given that $A, B = \phi \Rightarrow \chi_A = \chi_B = 0$

$\Rightarrow A \cup B = \phi$ and $A \cap B = \phi \Rightarrow \chi_{A \cup B} = 0$ & $\chi_{A \cap B} = 0$

Hence prove that

$\chi_{A \cup B} = 0 = \chi_A + \chi_B = \chi_{A \cap B}$

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(20) Let $\phi \neq E \subseteq \mathbb{R}$, If given that ① G_δ set "G" with $E \subseteq G$ and $m^*(G-E)=0$ then prove that ② F_σ set "F" with $F \subseteq E$ and $m^*(E-F)=0$

Sol: ① ⇒ ② ∵ $\phi \neq E \subseteq \mathbb{R} \Rightarrow \phi = E' \subseteq \mathbb{R}$

∴ For $\phi \neq E'$ given $E' \subseteq G$, $m^*(G-E')=0$

$\Rightarrow m^*(G \cap (E')') = 0$

$\Rightarrow m^*(G \cap E) = 0 \Rightarrow m^*(E \cap G) = 0$

$\Rightarrow m^*(E \cap (G')') = 0 \Rightarrow m^*(E - G') = 0$

$\Rightarrow m^*(E-F)$ where F is an F_σ set being the complement of G_δ set G. i.e. $G_\delta' = F_\sigma$

(21) Let $\phi \neq E$ be a measurable set, $f: E \to \mathbb{R} \cup \{\pm\infty\}$ then prove that $\{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

Sol: Let $z \in \{x \in E : f(x) = -\infty\}$

$\Rightarrow z \in \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

$\Rightarrow z \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$, "

$\Rightarrow \{x \in E : f(x) = -\infty\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\} \to (a)$

Now let $s \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

$\Rightarrow s \in \{x \in E : f(x) < -n\}$, "

$\Rightarrow f(s) < -n \Rightarrow s \in \{x \in E : f(x) = -\infty\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\} \subseteq \{x \in E : f(x) = -\infty\} \to (b)$

From (a) and (b)

$\therefore \{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$ (proved)

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(22) If f and g are extended real valued (and measurable) functions. Then prove $\bigcup_{p\in Q}\left[\{x: f(x)>p\}\cap\{x: a-g(x)<p\}\right]\subseteq\{x:(f+g)(x)>a\}$

Sol: Let $t\in\bigcup_{p\in Q}\left[\{x: f(x)>p\}\cap\{x: a-g(x)<p\}\right]$

$\Rightarrow t\in\{x: f(x)>p\}\cap\{x: a-g(x)<p\}$ for some $p\in Q$

$\Rightarrow t\in\{x: f(x)>p\}$ and $t\in\{x: a-g(x)<p\}$

$\Rightarrow f(t)>p$ and $a-g(t)<p \Rightarrow p>a-g(t)$

$\Rightarrow f(t)>p>a-g(t) \Rightarrow f(t)>a-g(t)$

$\Rightarrow f(t)+g(t)>a \Rightarrow (f+g)(t)>a$

$\Rightarrow t\in\{x:(f+g)(x)>a\}$

So $\bigcup_{p\in Q}\left[\{x: f(x)>p\}\cap\{x: a-g(x)<p\}\right]\subseteq\{x:(f+g)(x)>a\}$ (proved)

(23) If "f" is real valued and measurable function. Then f^2 is also measurable.

Sol: We need to prove $\{x: f^2(x)>a\}\in M$, $\forall a\in R$

$$\{x: f^2(x)>a\}=\begin{cases}\{x: f(x)>\sqrt{a}\}\cup\{x: f(x)<-\sqrt{a}\}; & a>0\\ \text{Domain}; & a<0\end{cases}$$

$\because f$ is measurable $\Rightarrow \{x: f(x)>\sqrt{a}\}\in M$ and $\{x: f(x)<-\sqrt{a}\}\in M$

$\Rightarrow \{x: f(x)>\sqrt{a}\}\cup\{x: f(x)<-\sqrt{a}\}\in M$ and Domain is given to be measurable. i.e. Domain $\in M$

$\therefore \{x: f^2(x)>a\} \Rightarrow$ "f^2" is also measurable.

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(24) If $\phi \neq A, B \subseteq X$ Then prove that $\chi_{A-B} = \chi_A - \chi_{A\cap B}$

Sol: Case I: If $x \in A-B$

$\Rightarrow x \in A$ but $x \in B$, $x \notin A\cap B \Rightarrow \chi_{A-B} = 1$

$\chi_A - \chi_{A\cap B} = 1 - 0 = 1 = \chi_{A-B}$

Case II: If $x \notin A-B$

$$x \notin A-B = \begin{cases} x \notin A, x \notin B, x \notin A\cap B \longrightarrow (a) \\ x \notin A, x \in B, x \notin A\cap B \longrightarrow (b) \\ x \in A, x \in B, x \in A\cap B \longrightarrow (c) \end{cases}$$

$\Rightarrow \chi_{A-B} = 0$

$$\chi_A - \chi_{A\cap B} = \begin{cases} (a) \Rightarrow 0-0 = 0 \\ (b) \Rightarrow 0-0 = 0 \\ (c) \Rightarrow 1-1 = 0 \end{cases} = \chi_{A-B}$$

$\therefore$ For all cases $\chi_{A-B} = \chi_A - \chi_{A\cap B}$ (proved)

(25) Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$ and $\phi \neq E \in \mathcal{M}$, where $\mathcal{M}$ be the measurable space. Then show that

$$\{x \in E : f(x) \geq a\} \subseteq \bigcap_{n=1}^{\infty} \left\{x \in E : f(x) > a - \frac{1}{n}\right\}, \forall\, n \in \mathbb{N}$$

Sol: Let $y \in \{x \in E : f(x) \geq a\} \Rightarrow f(y) \geq a, \forall\, a \in \mathbb{R}$

$\Rightarrow f(y) > a - \frac{1}{n}, \forall\, n \in \mathbb{N}$

$\Rightarrow y \in \left\{x \in E : f(x) > a - \frac{1}{n}\right\}, \forall\, n \in \mathbb{N}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \left\{x \in E : f(x) > a - \frac{1}{n}\right\}$

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$\Rightarrow \{x \in E : f(x) \geq a\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$ (proved)

(26) For a given $\phi \neq E \subset R$ and $\varepsilon > 0$, there is a a closed set $F \subset E$ such that $m^*(E-F) < \varepsilon$, then show that $\exists$ an open set satisfying $m^*(O-E) < \varepsilon$.

Sol: For $\phi \neq E \subset R \Rightarrow E' \subset R$ .. $\exists$ a closed set "F" and $\varepsilon > 0$ such that $F \subset E'$ and $m^*(E'-F) < \varepsilon \rightarrow (1)$

$F \subset E' \Rightarrow E = E'' \subseteq E' = O$ (open set) say

$\Rightarrow m^*(O-E) = m^*(O \cap E') = m^*(E' \cap O)$

$\Rightarrow m^*(O-E) = m^*(E'-O') = m^*(E'-F) < \varepsilon$ from (1)

$\Rightarrow m^*(O-F) < \varepsilon$ (proved)

(27) If $m^*(G-E) = 0$, then show that $m^*(G-E^c) = 0$ (2)

Sol: $\because m^*(G-E) = 0$, $\phi \neq E \subset R \Rightarrow \phi \neq E' \subseteq R$

$\therefore$ for $\phi \neq E'$ gives $E' \subseteq G$ $\quad \therefore E \subseteq G$

Hence $m^*(G-E^c) = 0$ (proved)

(28) Let f be an extended real valued measurable function on D and $\phi \neq A \subset D$. Then show that $\{x \in A : f(x) > a\}$ is measurable. $\forall a \in R$ (3)

Sol: Refer ... Measurable Function ...

For Solution See Question No. 36

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(29) Let "f" be an extended real valued and measurable functions defined on some domain ($\phi \neq D$) the prove that $|f|$ is measurable function.

Sol: ∵ "f" is measurable $\Rightarrow \{x : f(x) > a\} \in M, \forall a \in \mathbb{R}$

To prove $|f|$ is measurable, we need to show that

$$\{x : |f|(x) > a\} \in M, \forall a \in \mathbb{R}$$

$$\{x : |f|(x) > a\} = \{x : |f(x)| > a\}$$

$$\{x : |f|(x) > a\} = \begin{cases} \{x : f(x) > a\} \cup \{x : f(x) < -a\} ; & a > 0 \\ D ; & a \leq 0 \end{cases}$$

In all cases on R.H.S, we get a measurable sets.

$\Rightarrow \{x : |f|(x) > a\}$ is measurable $\forall a \in \mathbb{R}$

Therefore $|f|$ is also measurable.

(30) Countable set are Borel sets with zero measure.

Sol: Case-I Singleton

$$\forall a \in \mathbb{R} ; \{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a + \frac{1}{n}\right) ; n \in \mathbb{N}$$

∵ $\left\{\left\{a - \frac{1}{n}, a + \frac{1}{n}\right\}, \downarrow\right\}$ is decreasing sequence.

with $m(E_n) = \frac{2}{n} < \infty$ ∴ where $E_n = \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$

$$\Rightarrow m\{a\} = m\left\{\bigcap_{n=1}^{\infty} E_n\right\} = \lim_{n \to \infty} m(E_n) = \lim_{n \to \infty} m(E_n)$$

$$\Rightarrow m\{a\} = \lim_{n \to \infty} \left(\frac{2}{n}\right) = 0$$

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Case II : Finite set

let $\{x_1, x_2, x_3, \ldots x_n\} = \bigcup_{i=1}^{n} \{x_i\}$

$\Rightarrow m\left(\bigcup_{i=1}^{n} \{x_i\}\right) = \sum_{i=1}^{n} m\{x_i\} = 0$

Case III : Infinite set

Let $\{x_1, x_2, x_3, \ldots\} = \bigcup_{i=1}^{\infty} \{x_i\}$

$\Rightarrow m\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) = \sum_{i=1}^{\infty} m\{x_i\} = 0$

(proved)

(31) Open interval $(a,b) \subset \mathbb{R}$ is measurable.

Sol: (a,∞) is measurable.

$\Rightarrow (a,\infty)^c = (-\infty, a]$ is also measurable.

and $(-\infty, b-\frac{1}{n}]$ is also measurable $\forall n \in \mathbb{N}$, as $n \to \infty$

and $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b-\frac{1}{n}]$ is also measurable.

because being countable union of measurable set

$\Rightarrow (a,\infty) \cap (-\infty, b) = (a,b)$ is also measurable.

because being intersection of two measurable set.

(32) If $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of X. Then shows that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Sol: Case I: If $x \in A \Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow x \in A_k$ exactly for $n=k$

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$x \notin \bigcup_{n=1}^{\infty} A_n = \{A_k\}$

$\Rightarrow \chi_{A_k}(x) = 1 \;,\; \chi_{A_m}(x) = 0, \quad \forall\ n \neq k$

$x \in A \Rightarrow \chi_A(x) = 1$

$\therefore \chi_A(x) = 1 = 0+0+\cdots+\underset{n=k}{1}+0+0+\cdots$

$\chi_A(x) = 1 = \sum_{n=1}^{\infty} \chi_{A_n}(x) \Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Case II: $x \notin A = \bigcup_{n=1}^{\infty} A_n \Rightarrow x \notin A_n, \forall n \in \mathbb{N}$

$\Rightarrow \chi_A(x) = 0 \quad | \quad \Rightarrow \chi_{A_n}(x) = 0, \forall n \in \mathbb{N}$

$\chi_A(x) = 0+0+0+\cdots+0+\cdots = \sum_{n=1}^{\infty} \chi_{A_n}(x)$

$\Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$ (proved)

Ex: (33) Let $\{E_n, \uparrow\}_{n=1}^{\infty}$ be an increasing sequence of measurable sets. Then show that

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

Sol: Given that $(E_n, \uparrow)$, such that

$\Rightarrow E_1 \subset E_2 \subset E_3 \subset \cdots$ of measurable sets.

Defining: $F_1 = E_1,\ F_2 = E_2 - E_1,\ F_3 = E_3 - F_2, \ldots$

$F_n = E_n - E_{n-1}, \ldots$ where $\{F_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable sets

$\Rightarrow \bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} E_k \Rightarrow \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k$

$\therefore m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} F_k\right) = m\left(\bigcup_{k=1}^{\infty} E_k\right)$

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$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k) = \lim_{n\to\infty} \sum_{k=1}^{\infty} m(E_k)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$= \lim_{n\to\infty} m^*(E_n) \qquad \text{(proved)}$$

(34)

Let $\{E_n, \downarrow\}_{n=1}^{\infty}$ be a decreasing sequence of measurable sets such that $m(E_k) < \infty$ for some $k \in \mathbb{N}$. Then shows that $m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$

Sol: Let $m(E_1) < \infty$, say

$\because \langle E_n \downarrow \rangle \Rightarrow E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots\cdots$

$\Rightarrow \phi = E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \cdots\cdots E_1 - E_n \subseteq \cdots\cdots$

$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} (E_1 - E_n)\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$

$$\bigcup_{n=1}^{\infty} (E_1 - E_n) = \bigcup_{n=1}^{\infty} (E_1 \cap E_n')$$

$$= (E_1 \cap E_1') \cup (E_2 \cap E_2') \cup \cdots\cdots$$

$$= E_1 \cap [E_1' \cup E_2' \cup E_3' \cup \cdots]$$

$$= E_1 \cap [E_1 \cap E_2 \cap E_3 \cap \cdots]'$$

$$= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)'$$

$$\bigcup_{n=1}^{\infty} (E_1 - E_n) = E_1 - \left(\bigcap_{n=1}^{\infty} E_n\right)$$

$$\lim_{n\to\infty} m^*(E_1 - E_n) = m^*\left(E_1 - \bigcap_{n=1}^{\infty} E_n\right)$$

$$\lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n) = m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right)$$

$$m^*(E_1) - \lim_{n\to\infty} m^*(E_n) = m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right)$$

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$\Rightarrow m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$ (proved)

(3a) If "f" and "g" are two extended real valued and measurable functions defined on same domain ($\phi \neq D$) then prove that $f \vee g = \max(f, g)$ is measurable.

Sol: $\because f \vee g = \max(f, g) = \frac{1}{2}[(f+g) + |f-g|]$

So $f \vee g$ is measurable because $(f+g)$ and $|f-g|$ are measurable and kf is measurable for all $k \in R$.

OR

$(f \vee g)(x) = \max(f(x), g(x))$

For any $a \in R$, we have

$\{x : (f \vee g)(x) > a\} = \{x : f(x) > a\} \cup \{x : g(x) > a\}$

$f(x)$ and $g(x)$ are measurable functions and the Union of two measurable functions is also measurable.

$\Rightarrow (f \vee g)(x)$ is measurable.

(3b) Let f be an extended real valued measurable function on D and $\phi \neq A \subset D$. Then show that restriction of "f" to A is also measurable and $A \in M$. (OR). $\{x \in A : f_A(x) > a\}$ is measurable. $\forall a \in R$.

Sol: $\because$ "f" is measurable

$\rightarrow \{x \in D : f(x) > a\} \in M$, $\forall a \in R$.

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To prove "f_A" is measurable we have
$\{x \in A : f_A(x) > a\} \in \mathcal{M}$.

Note: $\{x \in A : f_A(x) > a\} = \{x \in D : f(x) > a\} \cap A$

On R.H.S both sets are measurable.

∴ Intersection of two measurable sets is measurable

⇒ "f_A" is measurable; $\{x \in A : f_A(x) > a\} \in \mathcal{M}$

(proved)

(37) Show that $f = f^+ - f^-$

Sol: If $f(x) \geq 0 \Rightarrow f^+(x) = f(x)$, $f^-(x) = 0$

$\therefore (f^+ - f^-)(x) = f^+(x) - f^-(x) = f(x) - 0 = f$

If $f(x) < 0 \Rightarrow f^+(x) = 0$, $f^-(x) = -f(x)$

$\therefore (f^+ - f^-)(x) = f^+(x) - f^-(x) = 0 - (-f(x)) = f$

$\Rightarrow f^+ - f^- = f$ (proved)

(38) Show that $f^+ = \frac{f + |f|}{2}$

Sol: If $f(x) \geq 0 \Rightarrow f^+(x) = f(x) \Rightarrow f^+ = f$

$|f(x)| = |f|(x) = f(x)$

$\therefore \frac{f + |f|}{2}(x) = \frac{f(x) + |f|(x)}{2} = \frac{f(x) + f(x)}{2} = f(x) = f^+(x)$

If $f(x) < 0 \Rightarrow f^+(x) = 0$, $\therefore |f|(x) = |f(x)| = -f(x)$

$\therefore \frac{f + |f|}{2}(x) = \frac{f(x) + |f|(x)}{2} = \frac{f(x) - f(x)}{2} = 0 = f^+(x)$

$\Rightarrow \frac{f + |f|}{2} = f^+$ (proved)

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Solved by Mahar Afaq Safdar Muhammadi

دعاؤں میں یاد رکھیے گا۔ جزاک اللہ

(39) Define Borel set.

Sol: A set "E" is said to be a Borel set if it can be expressed as countable union or intersection of open or closed sets.

e.g. $\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ is Borel set.

↳ G_δ

(40) Define Sum Modulo 1 and Translation Modulo 1.

Sol: Sum Modulo 1:

Let $x, y \in A = [0, 1)$ and we define sum modulo 1 of x and y as

$$x \hat{+} y = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{if } x + y \geq 1 \end{cases}$$

Translation Modula 1.

Let $E \subseteq A = [0, 1)$ and $y \in A$, we define a translation modula 1 of "E" as

$$E \hat{+} y = \{x \hat{+} y, x \in E\}$$

$\overline{E}$:

Set of all Sum Modula 1.

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MTH644 Current Papers Questions (Fall 2021 Final Term)

① If f is measurable and $k>0$ then show that kf is measurable. ~~$\forall k \in \mathbb{R}$~~

Sol: "f" is measurable.

$\Rightarrow \{x : f(x) > a\} \in M, \quad \forall a \in \mathbb{R}$

If $k>0$, $\because f(x) > a, \quad \forall a \in \mathbb{R}$

$\Rightarrow f(x) > \frac{a}{k} \Rightarrow kf(x) > a, \quad \forall a \in \mathbb{R}$

$\therefore \{x : kf(x) > a\} \in M, \quad \forall a \in \mathbb{R}$

So "kf" is measurable.

② If f, g and $f+g$ is measurable then $f-g$ is measurable.

Sol: Given that f, g and $f+g$ is measurable.

We have to prove "$f-g$" is measurable.

$\because g$ is measurable $\Rightarrow (-1)g = -g$ is also M

$\Rightarrow f + (-1)g = f - g$

$\because f+g$ is measurable $\Rightarrow f-g$ is also measurable.

③ If $m(A)$ and $m(B)$ are infinitely measurable then prove $m(A)+m(B) = m(A \cup B) + m(A \cap B)$.

Sol: Given that $m(A) = m(B) = \infty$, then result is obvious.

$\Rightarrow m(A \cup B) = m(A \cap B) = \infty$,

So $m(A)+m(B) = m(A \cup B) + m(A \cap B)$

(proved)

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(4) Find: dropped interval in contor set, for $n=1$

Sol: We know that

$$m = 2^{n-1}$$

for $n=1$, $m = 2^{1-1} = 2^0 = 1$

Hence dropped interval G_1 is

$$G_1 = I_{mm} = I_{11}$$

(5) Define Borel Measure Space.

Sol: Restriction of lebesgue measure "m" to the class of borel set "B" is said to be Borel measure space $(\mathbb{R}, B, m)$.

(6) $f(x) = |x+2|$, find f^+ for $f(x) \geq 0$

Sol: We know that

$$f^+ = \frac{f + |f|}{2}$$

$f = |x+2| \Rightarrow f = x+2 \quad \because f(x) \geq 0$

$|f| = ||x+2|| \Rightarrow |f| = x+2 \quad \because f(x) \geq 0$

So

$$f^+ = \frac{x+2+x+2}{2} = \frac{2x+4}{2}$$

$$f^+ = \frac{2(x+2)}{2}$$

$$f^+ = x+2 \quad \text{Ans.}$$

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$\exists$ there exists

⑦ Lebesgue measure "m" is translation invariant.

Sol: Given that say $\phi \neq E \in M$ (measurable)

$\therefore$ E is measurable $\Leftrightarrow$ for given $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$, such that $m^*(U-E) < \varepsilon$

"U" is open set in $\mathbb{R} \Rightarrow$ "U+y" is also open in $\mathbb{R}$. $\forall\ y \in \mathbb{R}$

If $E \subset U \Rightarrow E+y \subseteq U+y$

$\Rightarrow (U+y)-(E+y) = (U-E)+y$

$m^*[(U+y)-(E+y)] = m^*[(U-E)+y] = m^*(U-E) < \varepsilon$

But $m^* = m \Rightarrow m[(U+y)-(E+y)] = m(U-E) < \varepsilon$

$\Rightarrow$ "E+y" is also measurable.

So "m" is translation invariant. (proved)

⑧ Let $E \subset A = [0,1)$ be a measurable set, then for each $y \in A$. The set $E \hat{+} y$ is measurable and $m(E) = m(E \hat{+} y)$.

Sol: Suppose $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$

The set E, $E \cap [0, 1-y)$ and $E \cap [1-y, 1)$ are measurable

So E_1 and E_2 are also measurable. Further

$E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$

$\Rightarrow m(E) = m(E_1) + m(E_2) \longrightarrow ①$ $\because$ "m" is finitely additive.

Let $y_1 \in E_1$ then $y_1 < 1-y \Rightarrow y_1 + y < 1$

$\Rightarrow y_1 \hat{+} y = y_1 + y \Rightarrow E_1 \hat{+} y = E_1 + y \longrightarrow ②$

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Now if $y_2 \in E_2 \Rightarrow \nexists\, y_2 \geq 1-y \Rightarrow y_1 + y \geq 1$

$\Rightarrow y_2 \hat{+} y = y_2 + y - 1 \Rightarrow E_2 \hat{+} y = E_2 + (y-1) \longrightarrow (b)$

$\because m'$ is translation invariant.

$\Rightarrow E_1 + y$ and $E_2 + (y-1)$ are measurable.

$\Rightarrow E_1 \hat{+} y$ and $E_2 \hat{+} (y-1)$ are measurable

$\Rightarrow m(E_1 \hat{+} y) = m(E_1 + y) = m(E_1)$

and $m(E_2 \hat{+} y) = m(E_2 + (y-1)) = m(E_2)$ $\longrightarrow (a)$

$\because E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$

$\Rightarrow E \hat{+} y = (E_1 \hat{+} y) \cup (E_2 \hat{+} y)$ and $(E_1 \hat{+} y) \cap (E_2 \hat{+} y) = \phi$

$\Rightarrow m(E \hat{+} y) = m(E_1 \hat{+} y) + m(E_2 \hat{+} y)$ (union of two disjoint)

$= m(E_1) + m(E_2)$ by (a)

$m(E \hat{+} y) = m(E)$ by (1) (proved)

(9) A subset $\phi \neq E \subseteq \mathbb{R}$ is measurable $\Leftrightarrow$ for given $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$ with $m^*(U-E) < \varepsilon$ (for finite)

Sol: Suppose E is measurable.

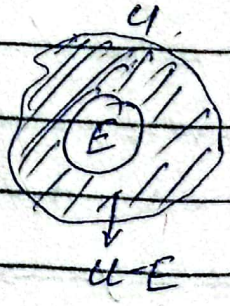
If $m^*(E) < \infty$, $\forall\, \phi \neq E \subseteq \mathbb{R}$, for $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$, $\Rightarrow m^*(U) < m^*(E) + \varepsilon \longrightarrow (a)$

$E \subseteq U \Rightarrow U = E \cup (U-E)$

$\Rightarrow m^*(U) = m^*(E) + m^*(U-E)$

$\Rightarrow m^*(U-E) = m^*(U) - m^*(E) < \varepsilon$ (from eq (a))

$\Rightarrow m^*(U-E) < \varepsilon$ (proved)



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δ Delta σ sigma

(10) Prove that every Borel set is measurable.

Sol: Every open and closed set is measurable.
⇒ countable union or intersection of closed or open set is also measurable.
Therefore Borel set is also measurable being union or intersection of closed or open sets.

(11) Give examples of Borel set.

Sol: Examples of Borel set:

(i) $G_\delta = \bigcap_{n=1}^{\infty} u_n$ (ii) $F_\sigma = \bigcup_{n=1}^{\infty} F_n$

(12) Let $\{f_n\}$ be a sequence of extended real valued measurable function on "D" then prove that $\sup_{n\in N}\left(\inf_{k\geq n} f_k\right)$ is also measurable.

Sol: let $p = \sup_{n\in N}\left(\inf_{k\geq n} f_k\right)$ and

$G_n = \inf\{f_n, f_{n+1}, f_{n+2}, \ldots f_{n+k}, \ldots\}, \forall n\in N$

Each G_n is measurable by the measurability of f_n for each n and $\inf_{n\in N} f_n$ is measurable.
Consequently $p = \sup_{n\in N} G_n$ is a measurable function because $\sup_{n\in N} f_n$ is measurable.

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(13) Let $f: E \to R \cup \{\pm\infty\}$, $\phi \neq E \in M$, where M be measurable space. Then show that:

$$\{x \in E : f(x) < a\} \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$$

Sol:

Let $y \in \{x \in E : f(x) \leq a\} \to (a)$

$\Rightarrow f(y) \leq a, \forall\ a \in R.$

$\Rightarrow f(y) < a + \frac{1}{n}, \forall\ n \in R$

$\Rightarrow y \in \{x \in E : f(x) < a + \frac{1}{n}\}, \forall\ n \in R.$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \to (b)$

From eq. (a) and (b)

$$\{x \in E : f(x) \leq a\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$$

(proved)

(14) Evaluate $m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$ where interval I_{nm} are pairwise disjoint and $m(I_{nm}) = \ell(I_{nm}) = \frac{1}{3^n}$

Sol:

Let $m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{nm}) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} \ell(I_{nm})$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} \left(\frac{1}{3^n}\right)$$

$$= \sum_{m=1}^{2^{1-1}} \left(\frac{1}{3^1}\right) + \sum_{m=1}^{2^{2-1}} \left(\frac{1}{3^2}\right) + \sum_{m=1}^{2^{3-1}} \left(\frac{1}{3^3}\right) + \cdots$$

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$$m(G_1) = \frac{1}{3} + \left(\frac{1}{3^2} + \frac{1}{3^2}\right) + \left(\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}\right) + \cdots\cdots$$

$$m(G_1) = \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^3} + \cdots\cdots$$

$$m(G_1) = 1$$

$a = \frac{1}{3}$, $r = \frac{2}{3}$

$$S_\infty = \frac{a}{1-r} = \frac{1/3}{1-2/3}$$

$$S_\infty = \frac{1/3}{1/3} = 1$$

$$m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right) = 1 \quad \text{Ans}$$

(15) Check the measurability of the cantor set in nth steps.

Sol: In general, at nth step, 2^{n-1} interval I_{nm}, $m = 1, 2, 3, \ldots 2^{n-1}$ are removed. Each I_{nm} has obviously length equal $\frac{1}{3^n} = 3^{-n}$. We have a sequence $\{I_{nm}\}$ of pairwise disjoint open sets. Thus the set $G_1 = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$ being the union of a sequence of open sets is open and measurable. The compliment of G_1 with respect to $[0, 1]$ is called cantor set and is denoted by C. The set C is closed and so it must be a measurable.

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(16) If "f" and "g" are two extended real valued and measurable functions defined on same domain ($\phi \neq I$) then prove that $f \wedge g$ is a measurable.

Sol: $\because f \wedge g = \min(f, g) = \frac{1}{2}[(f+g) - |f-g|]$

So $f \wedge g$ is measurable because $(f+g)$ and $|f-g|$ are measurable and "kf" is measurable $\forall\, k \in \mathbb{R}$.

OR

$(f \wedge g)(x) = \min(f(x), g(x))$

Observe that any $a \in \mathbb{R}$, we have

$\{x : (f \wedge g)(x) > a\} = \{x : f(x) > a\} \cap \{x : g(x) > a\}$

$f(x)$ and $g(x)$ are measurable functions and the intersection of two measurable function is also measurable $\Rightarrow (f \wedge g)(x)$ is measurable.

(17) Determine the measure 1st step construction of contor set from $[0, 1]$

Sol: We know that

$$m(G_1) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$$

For 1st step

put $n = 1 \Rightarrow m = 2^{n-1} = 2^{1-1} = 2^0 = 1$

$$m(G_1) = m(I_{11}) = \ell(I_{11})$$

$$m(G_1) = m(I_{11}) = \ell(I_{11}) = \frac{1}{3^1} = \frac{1}{3} \qquad \because \ell(I_{nm}) = \frac{1}{3^n}$$

Ans

(18) Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$ and $\phi \neq E \in M$, be the measurable space then prove that $\{x \in E : f(x) > 0\}$ is also measurable

Sol: By the definition of measurable function $f: E \to \mathbb{R}$ is lebesgue measurable function on $\phi \neq E \in M$ if $\{x \in E : f(x) > a\} \in M$ (measurable) for every $a \in \mathbb{R}$.

So $\{x \in E : f(x) > 0\}$ is also measurable because $0 \in \mathbb{R}$.

(19) Let $\phi \neq E \subset \mathbb{R}$, show that there exist F_σ set $F \subset E$.

Sol: For each n, let F_n be a closed set with $F_n \subseteq E$ and $m^*(E - F_n) < \frac{1}{n}$. Then $E = \bigcup_{n=1}^{\infty} F_n$ is an F_σ set $F_n \subseteq \bigcup_{n=1}^{\infty}$

$= F_n \subseteq E$ proved

(20) If $\phi = A, B \subset X$ and same member x are in $A \cup B$ Then verify that $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$

Sol: Given that $A, B = \phi \Rightarrow \chi_A = \chi_B = 0$

$\Rightarrow A \cup B = \phi$ and $A \cap B = \phi \Rightarrow \chi_{A \cup B} = 0$ & $\chi_{A \cap B} = 0$

Hence prove that

$\chi_{A \cup B} = 0 = \chi_A + \chi_B = \chi_{A \cap B}$

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(20) Let $\phi \neq E \subseteq \mathbb{R}$, If given that ① G_δ set "G" with $E \subseteq G$ and $m^*(G-E)=0$ then prove that ② F_σ set "F" with $F \subseteq E$ and $m^*(E-F)=0$

Sol: ① ⇒ ② ∵ $\phi \neq E \subseteq \mathbb{R} \Rightarrow \phi = E' \subseteq \mathbb{R}$

∴ For $\phi \neq E'$ given $E' \subseteq G$, $m^*(G-E')=0$

$\Rightarrow m^*(G \cap (E')') = 0$

$\Rightarrow m^*(G \cap E) = 0 \Rightarrow m^*(E \cap G) = 0$

$\Rightarrow m^*(E \cap (G')') = 0 \Rightarrow m^*(E - G') = 0$

$\Rightarrow m^*(E-F)$ where F is an F_σ set being the complement of G_δ set G. i.e. $G_\delta' = F_\sigma$

(21) Let $\phi \neq E$ be a measurable set, $f: E \to \mathbb{R} \cup \{\pm\infty\}$ then prove that $\{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

Sol: Let $z \in \{x \in E : f(x) = -\infty\}$

$\Rightarrow z \in \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

$\Rightarrow z \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$, "

$\Rightarrow \{x \in E : f(x) = -\infty\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\} \to (a)$

Now let $s \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

$\Rightarrow s \in \{x \in E : f(x) < -n\}$, "

$\Rightarrow f(s) < -n \Rightarrow s \in \{x \in E : f(x) = -\infty\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\} \subseteq \{x \in E : f(x) = -\infty\} \to (b)$

From (a) and (b)

$\therefore \{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$ (proved)

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(22) If f and g are extended real valued (and measurable) functions. Then prove $\bigcup_{p\in Q}\left[\{x: f(x)>p\}\cap\{x: a-g(x)<p\}\right]\subseteq\{x:(f+g)(x)>a\}$

Sol: Let $t\in\bigcup_{p\in Q}\left[\{x: f(x)>p\}\cap\{x: a-g(x)<p\}\right]$

$\Rightarrow t\in\{x: f(x)>p\}\cap\{x: a-g(x)<p\}$ for some $p\in Q$

$\Rightarrow t\in\{x: f(x)>p\}$ and $t\in\{x: a-g(x)<p\}$

$\Rightarrow f(t)>p$ and $a-g(t)<p \Rightarrow p>a-g(t)$

$\Rightarrow f(t)>p>a-g(t) \Rightarrow f(t)>a-g(t)$

$\Rightarrow f(t)+g(t)>a \Rightarrow (f+g)(t)>a$

$\Rightarrow t\in\{x:(f+g)(x)>a\}$

So $\bigcup_{p\in Q}\left[\{x: f(x)>p\}\cap\{x: a-g(x)<p\}\right]\subseteq\{x:(f+g)(x)>a\}$ (proved)

(23) If "f" is real valued and measurable function. Then f^2 is also measurable.

Sol: We need to prove $\{x: f^2(x)>a\}\in M$, $\forall a\in R$

$$\{x: f^2(x)>a\}=\begin{cases}\{x: f(x)>\sqrt{a}\}\cup\{x: f(x)<-\sqrt{a}\}; & a>0\\ \text{Domain}; & a<0\end{cases}$$

$\because f$ is measurable $\Rightarrow \{x: f(x)>\sqrt{a}\}\in M$ and $\{x: f(x)<-\sqrt{a}\}\in M$

$\Rightarrow \{x: f(x)>\sqrt{a}\}\cup\{x: f(x)<-\sqrt{a}\}\in M$ and Domain is given to be measurable. i.e. Domain $\in M$

$\therefore \{x: f^2(x)>a\} \Rightarrow$ "f^2" is also measurable.

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(24) If $\phi \neq A, B \subseteq X$ Then prove that $\chi_{A-B} = \chi_A - \chi_{A\cap B}$

Sol: Case I: If $x \in A-B$

$\Rightarrow x \in A$ but $x \in B$, $x \notin A\cap B \Rightarrow \chi_{A-B} = 1$

$\chi_A - \chi_{A\cap B} = 1 - 0 = 1 = \chi_{A-B}$

Case II: If $x \notin A-B$

$$x \notin A-B = \begin{cases} x \notin A, x \notin B, x \notin A\cap B \longrightarrow (a) \\ x \notin A, x \in B, x \notin A\cap B \longrightarrow (b) \\ x \in A, x \in B, x \in A\cap B \longrightarrow (c) \end{cases}$$

$\Rightarrow \chi_{A-B} = 0$

$$\chi_A - \chi_{A\cap B} = \begin{cases} (a) \Rightarrow 0-0 = 0 \\ (b) \Rightarrow 0-0 = 0 \\ (c) \Rightarrow 1-1 = 0 \end{cases} = \chi_{A-B}$$

$\therefore$ For all cases $\chi_{A-B} = \chi_A - \chi_{A\cap B}$ (proved)

(25) Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$ and $\phi \neq E \in \mathcal{M}$, where $\mathcal{M}$ be the measurable space. Then show that

$$\{x \in E : f(x) \geq a\} \subseteq \bigcap_{n=1}^{\infty} \left\{x \in E : f(x) > a - \frac{1}{n}\right\}, \forall\, n \in \mathbb{N}$$

Sol: Let $y \in \{x \in E : f(x) \geq a\} \Rightarrow f(y) \geq a, \forall\, a \in \mathbb{R}$

$\Rightarrow f(y) > a - \frac{1}{n}, \forall\, n \in \mathbb{N}$

$\Rightarrow y \in \left\{x \in E : f(x) > a - \frac{1}{n}\right\}, \forall\, n \in \mathbb{N}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \left\{x \in E : f(x) > a - \frac{1}{n}\right\}$

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$\Rightarrow \{x \in E : f(x) \geq a\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$ (proved)

(26) For a given $\phi \neq E \subset R$ and $\varepsilon > 0$, there is a a closed set $F \subset E$ such that $m^*(E-F) < \varepsilon$, then show that $\exists$ an open set satisfying $m^*(O-E) < \varepsilon$.

Sol: For $\phi \neq E \subset R \Rightarrow E' \subset R$ .. $\exists$ a closed set "F" and $\varepsilon > 0$ such that $F \subset E'$ and $m^*(E'-F) < \varepsilon \rightarrow (1)$

$F \subset E' \Rightarrow E = E'' \subseteq E' = O$ (open set) say

$\Rightarrow m^*(O-E) = m^*(O \cap E') = m^*(E' \cap O)$

$\Rightarrow m^*(O-E) = m^*(E'-O') = m^*(E'-F) < \varepsilon$ from (1)

$\Rightarrow m^*(O-F) < \varepsilon$ (proved)

(27) If $m^*(G-E) = 0$, then show that $m^*(G-E^c) = 0$ (2)

Sol: $\because m^*(G-E) = 0$, $\phi \neq E \subset R \Rightarrow \phi \neq E' \subseteq R$

$\therefore$ for $\phi \neq E'$ gives $E' \subseteq G$ $\quad \therefore E \subseteq G$

Hence $m^*(G-E^c) = 0$ (proved)

(28) Let f be an extended real valued measurable function on D and $\phi \neq A \subset D$. Then show that $\{x \in A : f(x) > a\}$ is measurable. $\forall a \in R$ (3)

Sol: Refer ... Measurable Function ...

For Solution See Question No. 36

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(29) Let "f" be an extended real valued and measurable functions defined on some domain ($\phi \neq D$) the prove that $|f|$ is measurable function.

Sol: ∵ "f" is measurable $\Rightarrow \{x : f(x) > a\} \in M, \forall a \in \mathbb{R}$

To prove $|f|$ is measurable, we need to show that

$$\{x : |f|(x) > a\} \in M, \forall a \in \mathbb{R}$$

$$\{x : |f|(x) > a\} = \{x : |f(x)| > a\}$$

$$\{x : |f|(x) > a\} = \begin{cases} \{x : f(x) > a\} \cup \{x : f(x) < -a\} ; & a > 0 \\ D ; & a \leq 0 \end{cases}$$

In all cases on R.H.S, we get a measurable sets.

$\Rightarrow \{x : |f|(x) > a\}$ is measurable $\forall a \in \mathbb{R}$

Therefore $|f|$ is also measurable.

(30) Countable set are Borel sets with zero measure.

Sol: Case-I Singleton

$$\forall a \in \mathbb{R} ; \{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a + \frac{1}{n}\right) ; n \in \mathbb{N}$$

∵ $\left\{\left\{a - \frac{1}{n}, a + \frac{1}{n}\right\}, \downarrow\right\}$ is decreasing sequence.

with $m(E_n) = \frac{2}{n} < \infty$ ∴ where $E_n = \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$

$$\Rightarrow m\{a\} = m\left\{\bigcap_{n=1}^{\infty} E_n\right\} = \lim_{n \to \infty} m(E_n) = \lim_{n \to \infty} m(E_n)$$

$$\Rightarrow m\{a\} = \lim_{n \to \infty} \left(\frac{2}{n}\right) = 0$$

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Case II : Finite set

let $\{x_1, x_2, x_3, \ldots x_n\} = \bigcup_{i=1}^{n} \{x_i\}$

$\Rightarrow m\left(\bigcup_{i=1}^{n} \{x_i\}\right) = \sum_{i=1}^{n} m\{x_i\} = 0$

Case III : Infinite set

Let $\{x_1, x_2, x_3, \ldots\} = \bigcup_{i=1}^{\infty} \{x_i\}$

$\Rightarrow m\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) = \sum_{i=1}^{\infty} m\{x_i\} = 0$

(proved)

(31) Open interval $(a,b) \subset \mathbb{R}$ is measurable.

Sol: (a,∞) is measurable.

$\Rightarrow (a,\infty)^c = (-\infty, a]$ is also measurable.

and $(-\infty, b-\frac{1}{n}]$ is also measurable $\forall n \in \mathbb{N}$, as $n \to \infty$

and $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b-\frac{1}{n}]$ is also measurable.

because being countable union of measurable set

$\Rightarrow (a,\infty) \cap (-\infty, b) = (a,b)$ is also measurable.

because being intersection of two measurable set.

(32) If $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of X. Then shows that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Sol: Case I: If $x \in A \Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow x \in A_k$ exactly for $n=k$

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$x \notin \bigcup_{n=1}^{\infty} A_n = \{A_k\}$

$\Rightarrow \chi_{A_k}(x) = 1 \;,\; \chi_{A_m}(x) = 0, \quad \forall\ n \neq k$

$x \in A \Rightarrow \chi_A(x) = 1$

$\therefore \chi_A(x) = 1 = 0+0+\cdots+\underset{n=k}{1}+0+0+\cdots$

$\chi_A(x) = 1 = \sum_{n=1}^{\infty} \chi_{A_n}(x) \Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Case II: $x \notin A = \bigcup_{n=1}^{\infty} A_n \Rightarrow x \notin A_n, \forall n \in \mathbb{N}$

$\Rightarrow \chi_A(x) = 0 \quad | \quad \Rightarrow \chi_{A_n}(x) = 0, \forall n \in \mathbb{N}$

$\chi_A(x) = 0+0+0+\cdots+0+\cdots = \sum_{n=1}^{\infty} \chi_{A_n}(x)$

$\Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$ (proved)

Ex: (33) Let $\{E_n, \uparrow\}_{n=1}^{\infty}$ be an increasing sequence of measurable sets. Then show that

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

Sol: Given that $(E_n, \uparrow)$, such that

$\Rightarrow E_1 \subset E_2 \subset E_3 \subset \cdots$ of measurable sets.

Defining: $F_1 = E_1,\ F_2 = E_2 - E_1,\ F_3 = E_3 - F_2, \ldots$

$F_n = E_n - E_{n-1}, \ldots$ where $\{F_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable sets

$\Rightarrow \bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} E_k \Rightarrow \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k$

$\therefore m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} F_k\right) = m\left(\bigcup_{k=1}^{\infty} E_k\right)$

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$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k) = \lim_{n\to\infty} \sum_{k=1}^{\infty} m(E_k)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$= \lim_{n\to\infty} m^*(E_n) \qquad \text{(proved)}$$

(34)

Let $\{E_n, \downarrow\}_{n=1}^{\infty}$ be a decreasing sequence of measurable sets such that $m(E_k) < \infty$ for some $k \in \mathbb{N}$. Then shows that $m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$

Sol: Let $m(E_1) < \infty$, say

$\because \langle E_n \downarrow \rangle \Rightarrow E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots\cdots$

$\Rightarrow \phi = E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \cdots\cdots E_1 - E_n \subseteq \cdots\cdots$

$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} (E_1 - E_n)\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$

$$\bigcup_{n=1}^{\infty} (E_1 - E_n) = \bigcup_{n=1}^{\infty} (E_1 \cap E_n')$$

$$= (E_1 \cap E_1') \cup (E_2 \cap E_2') \cup \cdots\cdots$$

$$= E_1 \cap [E_1' \cup E_2' \cup E_3' \cup \cdots]$$

$$= E_1 \cap [E_1 \cap E_2 \cap E_3 \cap \cdots]'$$

$$= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)'$$

$$\bigcup_{n=1}^{\infty} (E_1 - E_n) = E_1 - \left(\bigcap_{n=1}^{\infty} E_n\right)$$

$$\lim_{n\to\infty} m^*(E_1 - E_n) = m^*\left(E_1 - \bigcap_{n=1}^{\infty} E_n\right)$$

$$\lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n) = m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right)$$

$$m^*(E_1) - \lim_{n\to\infty} m^*(E_n) = m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right)$$

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$\Rightarrow m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$ (proved)

(3a) If "f" and "g" are two extended real valued and measurable functions defined on same domain ($\phi \neq D$) then prove that $f \vee g = \max(f, g)$ is measurable.

Sol: $\because f \vee g = \max(f, g) = \frac{1}{2}[(f+g) + |f-g|]$

So $f \vee g$ is measurable because $(f+g)$ and $|f-g|$ are measurable and kf is measurable for all $k \in R$.

OR

$(f \vee g)(x) = \max(f(x), g(x))$

For any $a \in R$, we have

$\{x : (f \vee g)(x) > a\} = \{x : f(x) > a\} \cup \{x : g(x) > a\}$

$f(x)$ and $g(x)$ are measurable functions and the Union of two measurable functions is also measurable.

$\Rightarrow (f \vee g)(x)$ is measurable.

(3b) Let f be an extended real valued measurable function on D and $\phi \neq A \subset D$. Then show that restriction of "f" to A is also measurable and $A \in M$. (OR). $\{x \in A : f_A(x) > a\}$ is measurable. $\forall a \in R$.

Sol: $\because$ "f" is measurable

$\rightarrow \{x \in D : f(x) > a\} \in M$, $\forall a \in R$.

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To prove "f_A" is measurable we have
$\{x \in A : f_A(x) > a\} \in \mathcal{M}$.

Note: $\{x \in A : f_A(x) > a\} = \{x \in D : f(x) > a\} \cap A$

On R.H.S both sets are measurable.

∴ Intersection of two measurable sets is measurable

⇒ "f_A" is measurable; $\{x \in A : f_A(x) > a\} \in \mathcal{M}$

(proved)

(37) Show that $f = f^+ - f^-$

Sol: If $f(x) \geq 0 \Rightarrow f^+(x) = f(x)$, $f^-(x) = 0$

$\therefore (f^+ - f^-)(x) = f^+(x) - f^-(x) = f(x) - 0 = f$

If $f(x) < 0 \Rightarrow f^+(x) = 0$, $f^-(x) = -f(x)$

$\therefore (f^+ - f^-)(x) = f^+(x) - f^-(x) = 0 - (-f(x)) = f$

$\Rightarrow f^+ - f^- = f$ (proved)

(38) Show that $f^+ = \frac{f + |f|}{2}$

Sol: If $f(x) \geq 0 \Rightarrow f^+(x) = f(x) \Rightarrow f^+ = f$

$|f(x)| = |f|(x) = f(x)$

$\therefore \frac{f + |f|}{2}(x) = \frac{f(x) + |f|(x)}{2} = \frac{f(x) + f(x)}{2} = f(x) = f^+(x)$

If $f(x) < 0 \Rightarrow f^+(x) = 0$, $\therefore |f|(x) = |f(x)| = -f(x)$

$\therefore \frac{f + |f|}{2}(x) = \frac{f(x) + |f|(x)}{2} = \frac{f(x) - f(x)}{2} = 0 = f^+(x)$

$\Rightarrow \frac{f + |f|}{2} = f^+$ (proved)

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Solved by Mahar Afaq Safdar Muhammadi

دعاؤں میں یاد رکھیے گا۔ جزاک اللہ

(39) Define Borel set.

Sol: A set "E" is said to be a Borel set if it can be expressed as countable union or intersection of open or closed sets.

e.g. $\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ is Borel set.

↳ G_δ

(40) Define Sum Modulo 1 and Translation Modulo 1.

Sol: Sum Modulo 1:

Let $x, y \in A = [0, 1)$ and we define sum modulo 1 of x and y as

$$x \hat{+} y = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{if } x + y \geq 1 \end{cases}$$

Translation Modula 1.

Let $E \subseteq A = [0, 1)$ and $y \in A$, we define a translation modula 1 of "E" as

$$E \hat{+} y = \{x \hat{+} y, x \in E\}$$

$\overline{E}$:

Set of all Sum Modula 1.

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(41) Show that $|f| = f^+ + f^-$.

Sol. If $f(x) \geq 0 \Rightarrow f^+(x) = f(x)$, $f^-(x) = 0$

$\therefore (f^+ + f^-)(x) = f^+(x) + f^-(x) = f(x) + 0 = f(x) = |f(x)|$

If $f(x) < 0 \Rightarrow f^+(x) = 0$, $f^-(x) = f(x)$

$\therefore (f^+ + f^-)(x) = f^+(x) + f^-(x) = 0 + f(x) = f(x) = |f(x)|$

$\Rightarrow |f| = f^+ + f^-$ (proved)

(42) Show that $\min(f,g) = \dfrac{(f+g) - |f-g|}{2}$

Sol:

$$|f-g|(x) = |f(x) - g(x)| = \begin{cases} f(x) - g(x) & ; f(x) \geq g(x) \\ g(x) - f(x) & ; f(x) < g(x) \end{cases}$$

Case I:

If $f(x) \geq g(x) \Rightarrow f(x) - g(x) \geq 0$

$\Rightarrow |f(x) - g(x)| = f(x) - g(x)$ and

If $f(x) \geq g(x) \Rightarrow \min(f,g) = g(x)$

$$\therefore \frac{(f+g) - |f-g|}{2}(x) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

$$= \frac{f(x) + g(x) - (f(x) - g(x))}{2}$$

$$= \frac{f(x) + g(x) - f(x) + g(x)}{2} = \frac{2g(x)}{2} = g(x)$$

$$\Rightarrow \frac{(f+g) - |f-g|}{2} = \min(f,g)$$

Case II: If $f(x) < g(x) \Rightarrow f(x) - g(x) < 0$

$\Rightarrow |f(x) - g(x)| = g(x) - f(x)$ and

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If $f(x) < g(x) \Rightarrow \min(f,g) = f(x)$

$\therefore \frac{(f+g)-|f-g|}{2}(x) = \frac{f(x)+g(x)-|f(x)-g(x)|}{2}$

$= \frac{f(x)+g(x)-(g(x)-f(x))}{2}$

$= \frac{f(x)+g(x)-g(x)+f(x)}{2} = \frac{2f(x)}{2} = f(x)$

$\Rightarrow \frac{(f+g)-|f-g|}{2} = \min(f,g)$ (proved)

(43) Show that $\max(f,g) = \frac{(f+g)-|f-g|}{2}$

Sol:

$$|f-g|(x) = |f(x)-g(x)| = \begin{cases} f(x)-g(x); & f(x) \geq g(x) \\ g(x)-f(x); & f(x) < g(x) \end{cases}$$

Case I: If $f(x) \geq g(x) \Rightarrow f(x)-g(x) \geq 0$

$\Rightarrow |f(x)-g(x)| = f(x)-g(x)$ and

If $f(x) \geq g(x) \Rightarrow \max(f,g) = f(x)$

$\therefore \frac{(f+g)+|f-g|}{2}(x) = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$

$= \frac{f(x)+g(x)+f(x)-g(x)}{2} = \frac{2f(x)}{2} = f(x)$

$\Rightarrow \frac{(f+g)+|f-g|}{2} = \max(f,g)$

Case II: If $f(x) < g(x) \Rightarrow f(x)-g(x) < 0$

$\Rightarrow |f(x)-g(x)| = g(x)-f(x)$ and

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If $f(x) \le g(x) \Rightarrow \max(f,g) = g(x)$

$$\therefore \frac{(f+g)+|f-g|}{2}(x) = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$$

$$= \frac{f(x)+g(x)+g(x)-f(x)}{2} = \frac{2g(x)}{2} = g(x)$$

$$\Rightarrow \frac{(f+g)+|f-g|}{2} = \max(f,g) \qquad \text{(proved)}$$

(44) Show that $\chi_{A\cap B} = \chi_A \cdot \chi_B$

Sol: If $x \in A\cap B \Rightarrow x \in A$ and $x \in B$

$$\Rightarrow \chi_{A\cap B} = 1 = \chi_A = \chi_B = 1.1 = \chi_A \cdot \chi_B$$

If $x \notin A\cap B \Rightarrow$
- $x \in A$ but $x \notin B \Rightarrow \chi_A = 1, \chi_B = 0$
- $x \in B$ but $x \notin A \Rightarrow \chi_B = 1, \chi_A = 0$
- $x \notin A$ and $x \notin B \Rightarrow \chi_A = 0, \chi_B = 0$

$$x \notin A\cap B \Rightarrow \chi_{A\cap B} = 0 \quad \{\because \chi_A \cdot \chi_B = 0$$

For all, $x \notin A\cap B$; $\chi_{A\cap B} = 0 = \chi_A \cdot \chi_B$

$$\Rightarrow \chi_{A\cap B} = \chi_A \cdot \chi_B \qquad \text{(proved)}$$

(45) Define open set in $\mathbb{R}$.

Sol: Any open set in $\mathbb{R}$ say "u" can be written as countable union of disjoint open intervals i.e. $u = \bigcup_{n=1}^{\infty} \{I_n\}$ and each open intervals is measurable and countable union of measurable set is measurable.

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(46) If $f(x) = \cos x$, Find f^+, $-\pi \le x \le \pi$

Sol: We know that

$$f^+(x) = \frac{f + |f|}{2}(x) = \frac{f(x) + |f|(x)}{2}$$

$\because f(x) = \cos x$

$$|f|(x) = |f(x)| = \begin{cases} \cos(x) & ; \ -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ -\cos(x) & ; \ \frac{\pi}{2} < x \le \pi, \ -\pi \le x < \frac{\pi}{2} \end{cases}$$

$$f^+(x) = \begin{cases} \frac{\cos x + \cos x}{2} = \cos x & ; \ -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \frac{\cos x - \cos x}{2} = 0 & ; \ \frac{\pi}{2} < x \le \pi, \ -\pi \le x < -\frac{\pi}{2} \end{cases}$$

(47) Let $\{E_i\}_{i=1}^n$ be a sequence in a measurable space $\mathcal{M}$, the m is countably additive.

Sol: $\forall \ \phi \neq A \subseteq \mathbb{R}$ and for the sequence $\{E_i\}_{i=1}^n$ of pairwise disjoint measurable sets:

$$m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

put $A = \mathbb{R}$ $\Rightarrow$ $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

Hence m is countably additive.

Note: If the question asked for "finitly additive" then in this solution replaced "∞" by "n".

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(48) Let $\{E_i\}_{i=1}^{n}$ be a sequence in a measurable space $\mathcal{M}$, then m is monotonic.

Sol: Let $\phi \neq A, B \in \mathcal{M}$, such that $B \subseteq A$

$\Rightarrow A = B \cup (A-B)$

$\because m^*$ is finitely additive.

$\Rightarrow m^*(A) = m^*(B) + m^*(A-B)$

$\Rightarrow m^*(A) \geq m^*(B) \quad \because m^*(A-B) \geq 0$

$\Rightarrow m^*(B) \leq m^*(A)$

(Proved)

(49) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of extended real valued measurable function on D. $f = \inf_{n \in \mathbb{N}} \{f_n\}$, further if $\{x : f(x) > a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\}$, $\forall a \in \mathbb{R}$, then show that f is measurable. (2)

Sol: Given that $f = \inf_{n \in \mathbb{N}} \{f_n\}$ and

$\{x : f(x) > a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\}$; $\forall a \in \mathbb{R}$

The intersection of a countable number of measurable sets is measurable. So f must be measurable.

(50) In the construction of cantor set $[0,1]$, evaluate the measure of dropped interval in the first step (case $n=1$) (2)

Sol: We know that $m = 2^{n-1} = 2^{1-1} = 2^0 = 1$, $\because n=1$

$m(G_1) = m(I_{mm}) = m(I_{11}) = l(I_{11}) = \frac{1}{3}$

$\because m[0,1] = G \cup C \Rightarrow m[0,1] = m(G_1) + m(C)$

$\Rightarrow m(C) = m[0,1] - m(G_1) = (1-0) - \frac{1}{3} = 1 - \frac{1}{3} = \frac{2}{3}$ Ans

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(51) Show that $f^- = \frac{|f| - f}{2}$

Sol: If $f(x) \geq 0 \Rightarrow f^-(x) = 0$, $|f(x)| = |f|(x) = f(x)$

$\therefore \frac{|f| - f}{2}(x) = \frac{|f|(x) - f(x)}{2} = \frac{f(x) - f(x)}{2} = 0 = f^-(x)$

If $f(x) < 0 \Rightarrow f^-(x) = -f(x)$, $|f(x)| = |f|(x) = -f(x)$

$\therefore \frac{|f| - f}{2}(x) = \frac{|f|(x) - f(x)}{2} = \frac{-f(x) - f(x)}{2} = -f(x) = f^-(x)$

$\Rightarrow \frac{|f| - f}{2} = f^-$ (proved)

(52) If $f(x) = \sin x$ Find f^+ and f^-, $-\pi \leq x \leq \pi$

Sol: We know that

$$f^+(x) = \frac{f + |f|}{2}(x) = \frac{f(x) + |f|(x)}{2}$$

$\because f(x) = \sin x$

$$|f|(x) = |f(x)| = \begin{cases} \sin x & ; \ 0 \leq x \leq \pi \\ -\sin x & -\pi < x \leq -\frac{\pi}{2}, \ -\frac{\pi}{2} < x < 0 \end{cases}$$

$$f^+(x) = \frac{f(x) + |f|(x)}{2} = \begin{cases} \frac{\sin x + \sin x}{2} = \sin x & ; \ 0 \leq x \leq \pi \\ \frac{\sin x - \sin x}{2} = 0 & ; \ -\pi < x \leq -\frac{\pi}{2}, \ -\frac{\pi}{2} < x < 0 \end{cases}$$

We know that $f^-(x) = \frac{|f| - f}{2}(x) = \frac{|f|(x) - f(x)}{2}$

$$f^-(x) = \frac{|f|(x) - f(x)}{2} = \begin{cases} \frac{\sin x - \sin x}{2} = 0 ; & 0 \leq x \leq \pi \\ \frac{-\sin x - \sin x}{2} = -\sin x ; & -\pi < x \leq \frac{-\pi}{2}, \ -\frac{\pi}{2} < x < 0 \end{cases}$$

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(53) If $f(x)=\cos x$, Find f^-, $-\pi \le x \le \pi$

Sol: We know that

$$f^-(x)=\frac{|f|-f(x)}{2}=\frac{|f|(x)-f(x)}{2}$$

$\because f(x)=\cos x$

$$|f|(x)=|f(x)|=\begin{cases}\cos x & ; \frac{-\pi}{2}\le x\le\frac{\pi}{2}\\ -\cos x & ; \frac{\pi}{2}\le x\le\pi,\ -\pi\le x<\frac{-\pi}{2}\end{cases}$$

$$f^-(x)=\frac{|f|(x)-f(x)}{2}=\begin{cases}\frac{\cos x-\cos x}{2}=0 & ; \frac{-\pi}{2}\le x\le\frac{\pi}{2}\\ \frac{-\cos x-\cos x}{2}=-\cos x & ; \frac{\pi}{2}<x\le\pi,\ -\pi\le x<\frac{-\pi}{2}\end{cases}$$

(54) Evaluate $E\hat{+}\frac{1}{5}$, the translation modulo 1 of the set $E=\left\{0,\frac{1}{2},\frac{1}{3},\frac{2}{3}\right\}\subset[0,1[$ by $\frac{1}{5}$.

Sol. We know that $E\hat{+}y=\{x\hat{+}y\ ,\ x\in E\}$

$y=\frac{1}{5}$

$$E\hat{+}\frac{1}{5}=\left\{0\hat{+}\frac{1}{5},\frac{1}{2}\hat{+}\frac{1}{5},\frac{1}{3}\hat{+}\frac{1}{5},\frac{2}{3}\hat{+}\frac{1}{5}\right\}$$

$$E\hat{+}\frac{1}{5}=\left\{0+\frac{1}{5},\frac{1}{2}+\frac{1}{5},\frac{1}{3}+\frac{1}{5},\frac{2}{3}+\frac{1}{5}\right\}$$

$\because x\hat{+}y=\begin{cases}x+y\ ; & x+y<1\\ x+y-1\ ; & x+y\ge1\end{cases}$

$$E\hat{+}\frac{1}{5}=\left\{\frac{1}{5},\frac{7}{10},\frac{8}{15},\frac{13}{15}\right\}$$ Ans

(55) In $[0,1[$, the sum modulo 1, $x\hat{+}y$ of $\frac{5}{2}$ and $\frac{1}{2}$ = ... (MCQs)

Sol: $x\hat{+}y=\begin{cases}x+y\ ; & x+y<1\\ x+y-1\ ; & x+y\ge1\end{cases}$ ✓

$\left\{\frac{5}{2}+\frac{1}{2}=\frac{6}{2}=3>1\right.$

$$x\hat{+}y=\frac{5}{2}+\frac{1}{2}-1=3-1=2$$ Ans

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(56) An extended real valued function defined on a set of measure zero is measurable.

Sol: Let $\phi \neq E \subseteq \mathbb{R}$, such that $m^*(E) = 0$ and $f: E \to \mathbb{R} \cup \{+\infty\}$, To prove f is measurable, we need to show that $\{x \in E : f(x) > a\} \in M$, $\forall\, a \in \mathbb{R}$.

$\because \{x \in E : f(x) > a\} \subseteq E$

$\Rightarrow m^*\{x \in E : f(x) > a\} \leq m^*(E)$ $\because$ By monotocity of "m^*"

$\Rightarrow m^*\{x \in E : f(x) > a\} \leq 0 \rightarrow$ (a) (given)

But by non-negativity of "m^*"

$\Rightarrow m^*\{x \in E : f(x) > a\} \geq 0 \rightarrow$ (b)

From (a) and (b) $\Rightarrow m^*\{x \in E : f(x) > a\} = 0$, $\forall\, a \in \mathbb{R}$.

$\Rightarrow \{x \in E : f(x) > a\} \in M$, $\forall\, a \in \mathbb{R}$.

So f is measurable. (proved)

(57) If "f" is measurable and $k < 0$, then show that kf is measurable.

Sol: $\because f$ is measurable.

$\Rightarrow \{x : f(x) > a\} \in M$, $\forall\, a \in \mathbb{R}$

If $k < 0$, $f(x) > a$, $\forall\, a \in \mathbb{R}$ (given)

$\Rightarrow f(x) < \frac{a}{k} \Rightarrow kf(x) < a$ $\because (k < 0)$

$\Rightarrow \{x : kf(x) < a\} \in M$, $\forall\, a \in \mathbb{R}$

$\Rightarrow$ "kf" is measurable.

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(58) Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$, $\phi \neq E \in M$, If $\{x \in E : f(x) \geq a\}$ is measurable, then $\{x \in E : f(x) \leq a\}$ is measurable for all $a \in \mathbb{R}$.

Sol: Given that $\{x \in E : f(x) \geq a\} \in M$, $\forall a \in \mathbb{R}$.

We know that

$$\{x \in E : f(x) < a\} = \{x \in E : f(x) \geq a\}^c, \forall a \in \mathbb{R}$$

$\because \{x \in E : f(x) \geq a\} \in M \Rightarrow \{x \in E : f(x) \geq a\}^c \in M$

Hence $\Rightarrow \{x \in E : f(x) < a\} \in M, \forall a \in \mathbb{R}$

(proved)

(59) Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$, $\phi \neq E \in M$, If $\{x \in E : f(x) \leq a\}$ is measurable, the $\{x \in E : f(x) > a\}$ is measurable for all $a \in \mathbb{R}$.

Sol:

Given that $\{x \in E : f(x) \leq a\} \in M$, $\forall a \in \mathbb{R}$.

We know that

$$\{x \in E : f(x) > a\} = \{x \in E : f(x) \leq a\}^c, \forall a \in \mathbb{R}.$$

$\because \{x \in E : f(x) \leq a\} \in M \Rightarrow \{x \in E : f(x) \leq a\}^c \in M$.

$\Rightarrow \{x \in E : f(x) > a\}$ is measurable, $\forall a \in \mathbb{R}$.

(proved)

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(60) Check the measurability of ~~Cantor~~ contor set in nth steps.

Sol: "G" is open being countable union of disjoint open Intervals $\Rightarrow$ "G" is measurable.

$\because C = G' = [0,1] - G$ closed and measurable.

$\therefore G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$

$$m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$$

$\because$ m is countably additive.

$$m(G) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{nm})$$

$$m(G) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} \ell(I_{nm})$$

expanding firstly by "n"

$$m(G) = \sum_{m=1}^{2^{1-1}} \ell(I_{1m}) + \sum_{m=1}^{2^{2-1}} \ell(I_{2m}) + \sum_{m=1}^{2^{3-1}} \ell(I_{3m}) + \cdots$$

$$m(G) = \underbrace{\ell(I_{11})} + \underbrace{\ell(I_{21}) + \ell(I_{22})} + \underbrace{\ell(I_{31}) + \ell(I_{32}) + \ell(I_{33}) + \ell(I_{34})} + \cdots$$

$$m(G) = \frac{1}{3} + \left(\frac{1}{3^2} + \frac{1}{3^2}\right) + \left(\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}\right) + \cdots \quad \because \ell(I_{nm}) = \frac{1}{3^n}$$

$$m(G) = \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^3} + \cdots$$

$$\boxed{m(G) = 1}$$

$a = \frac{1}{3}, \; r = \frac{2}{3}$

$$S_\infty = \frac{a}{1-r} = \frac{1/3}{1-2/3} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

$\because [0,1] = G \cup C$

$\Rightarrow m[0,1] = m(G) + m(C)$

$\Rightarrow 1 = 1 + m(C)$

$\Rightarrow \boxed{m(C) = 0}$

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Current Paper of Math644

Question No 1

If $K>0$ then show that Kf is Measurable?

Solution:-

Since f is measurable

$\Rightarrow \{x: f(x)>a\} \in M \quad \forall a \in \mathbb{R}$

if $K>0$

$f(x)>a \quad \forall a \in R$

$\Rightarrow f(x) > a/K$

$\Rightarrow Kf(x) > a \quad \forall a \in R$

So

$\therefore \{x: Kf(x)>a\} \in M \quad \forall a \in R$

$\Rightarrow$ "Kf" is measurable.

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Date:

Q.2

If f, g and $f+g$ are measurable then $f-g$ is measurable?

Solution:-

~~If f+f=f~~

Given that f, g and $f+g$ are measurable

So

$\Rightarrow (-1)g = -g$ is also measurable

$\Rightarrow f+(-1)g = f-g$ is also measurable.

So

$f \pm g$ is also measurable

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Date:

Q-3

Countable sets are Borel sets with zero measure.

Proof:-

Case(I) Sigleton

$\forall\ a \in \mathbb{R}$

$$\{a\} \subseteq \bigcap_{n=1}^{\infty} (a-1/n,\ a+1/n);\ n \in \mathbb{N}$$

Since

$\{\{a-1/n,\ a+1/n\},\ \forall\}$ is a decreasing sequence with

$$m(E_n) = \frac{2}{n} < \infty$$

where

$$E_n = (a-1/n,\ a+1/n)$$

$$\Rightarrow m\{a\} = m\left\{\bigcap_{n=1}^{\infty} E_n\right\}$$

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Date:

$$= \lim_{n \to \infty} m(E_n)$$

$$= \lim_{n \to \infty} l(E_n)$$

$$= \lim_{n \to \infty} \frac{2}{n}$$

$$= \frac{2}{\infty}$$

$$m\{a_1\} = 0$$

case(II) Finite set

$$\{x_1, x_2, \cdots x_n\} = \bigcup_{i=1}^{n} \{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{n} \{x_i\}\right) = \sum_{i=1}^{n} m\{x_i\}$$

B/c $\{x\}$ is singleton.

case (III) Infinite set:-

$$\{x_1, x_2, \cdots\} = \bigcup_{i=1}^{\infty} \{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) = \sum_{i=1}^{\infty} m\{x_i\}$$

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$$m\{\overset{\infty}{\underset{i=1}{\cup}}\{x_i\}\} = 0$$

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Question No 4

$f(x) = |x+2|$

find f^+ $f(x) \geq 0$

Solution:-

Given that

$f(x) = |x+2|$

As if $f(x) \geq 0$

$$f^+ = \frac{f + |f|}{2}$$

$$= \frac{|x+2| + ||x+2||}{2}$$

$$= \frac{x+2+x+2}{2}$$

$$= \frac{2x+4}{2}$$

$$= x+2$$

$f^+ = |x+2|$ ABS.

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Q=5

Define Borel Measure Space?

Solution:-

Restriction of Lebsegue measure "m" to the class of Boral Set 'B' is said to be Borel space i,e (R, B, m).

Question:6

Let f be an extanded real valued measurable function on 'D'. let $\phi \neq A \subseteq D$ Then restriction of 'f' to 'A' is also measurable.

Proof:-

since

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'f' is measureable.

$\Rightarrow \{x \in D : f(x) > a\} \in M, \forall a \in \mathbb{R}$

for any $a \in \mathbb{R}$ To Prove f_A is meaurable we show that

$\{x \in A : f_A(x) > a\} \in M$,

~~(for any $a \in \mathbb{R}$ / To prove)~~

Note that

$\{x \in A : f_A(x) > a\} = \{x \in D : f(x) > a\} \cap A$

on R.H.S both sets are meaurable.

$\Rightarrow \{x \in A : f_A(x) > 0\} \in M$

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Intersection of two measurable sets is measurable

$\Rightarrow$ "f_A" is measureable

Q:7

In R if the interval (a,∞) is measurable than Show That bounded interval a,b is also measureable?

Sol:-

As we know that (a,∞) is measurable

$\Rightarrow (a,\infty)^c = (-\infty, a)$ is also measurable

and

$(-\infty, b-1/n)$ is also measurable

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$\forall\ n \in \mathbb{N}$

and if $n \to \infty$

$$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - 1/n)$$

is also measurable being countable union also measurable being countable union of two measurable sets.

Q=8

Lebesegue measure 'm' is translational invarient.

Sol:-

Give that

$\phi \neq E \in M$ is measurable

Since we know that

'E' is measurable iff for given $\varepsilon > 0$ $\exists$ an open set '$u \supseteq E$' such

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Date:

that $m^*(u-E) < \varepsilon$.

As

'u' is open set in $\mathbb{R}$.

$\Rightarrow u+y$ is also open in $\mathbb{R}$, $\forall\, y \in \mathbb{R}$.

If $E \subseteq u$

$\Rightarrow E+y \subseteq u+y$

$\Rightarrow (u+y) - (E+y) = (u-E)+y$

$\Rightarrow m^*[(u+y)-(E+y)] = m^*[(u-E)+y]$

Since

Lebsegue outer measure is translational invarient so

$m^*[(u+y)-(E+y)] = m^*(u-E) < \varepsilon$

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But $m^* = m$

$\Rightarrow m[(A+y)-(E+y)] = m[A-E] < \varepsilon$.

So

$E+y$ is also measurable

so m is also translational invariant.

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Q=9

let $E \subseteq A = [0,1)$ be a measurable set then for each $y \in A$ the set $(E \hat{+} y)$ is measurable.

$m(E) = m^*(E \hat{+} y)$.

Sol:-

Suppose

$E_1 = E \cap [0, 1-y)$ and

$E_2 = E \cap [1-y, 1)$

since

$E, [0, 1-y), [1-y, 1)$ are

measurable.

=> E_1 and E_2 are also measurable

Also

$E_1 \cap E_2 = \{E \cap [0, 1-y)\} \cap \{E \cap [1-y, 1)\}$

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$= E \cap \{[0, 1-y) \cap [1-y, 1)\}$

$= E \cap \phi$

$= \phi$

and

$E_1 \cup E_2 = E \cup \phi - E$

$\Rightarrow m(E) = m(E_1) + m(E_2) \rightarrow (i)$

$\because$ m is finitely additive.

Let

$y_1 \in A$

if $y_1 \in E_1$

$\Rightarrow y_1 < 1 - y$

$\Rightarrow y_1 + y < 1$

$\Rightarrow y_1 \hat{+} y = y_1 + y$

So

$\Rightarrow E_1 \hat{+} y = E_1 + y \rightarrow 2$

if $y_2 \in E_2$

$\Rightarrow y_2 \geq 1 - y$

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$\Rightarrow y_2 + y \geq 1$

So

$y_2 \hat{+} y = y_2 + y - 1$

$\Rightarrow E_2 \hat{+} y = E_2 + (y-1)$

Since we know that 'm' is translational invariant -

$\Rightarrow E_1 \hat{+} y$ and $E_2 \hat{+} y$ are measurable.

So

$E_1 \hat{+} y$ and $E_2 \hat{+} (y-1)$ are also measurable

Therefore

$m(E_1 \hat{+} y) = m(E_2 + (y-1)) = m(E_2)$

Since

$E_1 \cup E_2 = E_1$

and

$E_1 \cap E_2 = \phi$

$\Rightarrow E \hat{+} y = (E_1 + y) \cup (E_2 + y)$

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and

$(E_1 + y) \cap (E_2 + y) = \emptyset$

$E \hat{+} y$ being union of two disjoint measurable sets is measurable.

So

$$m(E \hat{+} y) = m(E_1 \hat{+} y) + m(E_2 \hat{+} y)$$
$$= m(E_1 + y) + m(E_2 + y)$$
$$= m(E_1) + m(E_2)$$
$$= m(E)$$

$$m(E \hat{+} y) = m(E)$$

So

m is also invariant w.r.t translation.

Date:

Q=10

A subset $\phi \neq E \subseteq R$ is measurable for given $\varepsilon > 0$ ∃ an open set $U \supseteq E$

$$m^*(U - E) < \varepsilon$$

for first case.

Sol:-

suppose 'E' is measurable

Case1:-

If $m^*(E) < \infty$ (finite)

$\forall\ \phi \neq E \subseteq R$

∃ an open set 'U' and for any $\varepsilon > 0$

$$m^*(U) < m^*(E) + \varepsilon \longrightarrow (1)$$

Since

$$E \subseteq U$$

$$\Rightarrow U = E \cup (U - E)$$

$$\Rightarrow m^*(U) = m^*(E) + m^*(U - E)$$

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Date:

$\Rightarrow m^*(u-E) \neq m^*(u) - m^*(E)$

from (1) we get

$m^*(u-E) = m^*(u) - m^*(E) < \varepsilon$

so

$m^*(u-E) < \varepsilon$

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Topic # 49

Theorem:

A subset $\phi \neq E \subseteq \mathbb{R}$ is measurable for given $\varepsilon > 0$ there exist an open set $U \supseteq E$ with

$$m^*(U - E) < \varepsilon.$$

Proof:-

Suppose 'E' is measurable.

Case I:-

If $m^*(E) < \infty$ (finite)

$\forall \phi \neq E \subseteq \mathbb{R}$

$\exists$ an open set 'U'

such that $E \subseteq U$

and for any $\varepsilon > 0$

$$m^*(U) < m^*(E) + \varepsilon \longrightarrow (1)$$

Since open set ka measurable

$E \subseteq U$

$\Rightarrow U = E \cup (U - E)$

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$\Rightarrow m^*(U) = m^*(E) + m^*(U-E)$

$\Rightarrow m^*(U-E) = m^*(U) - m^*(E)$

from (1) we get

$m^*(U-E) = m^*(U) - m^*(E) < \varepsilon$ ✓

Case I:

If $m^*(E) = \infty$ (infinite)

Since $E = \bigcup_{n=1}^{\infty} I_n$, where $l(I_n) < \infty \ \forall\, n \in \mathbb{N}$

and $I_j \cap I_k = \phi$, $j \neq k$

Put $E_n = E \cap I_n$

$\Rightarrow E_n \subseteq I_n \quad \forall\ 'n'$

$\Rightarrow m^*(E_n) \leq m^*(I_n) = l(I_n) < \infty$

So by Case I

$\forall\ E_n \ \exists$ an open set say V_n such that $E_n \subseteq V_n$ and for $\varepsilon > 0$

$$m^*(V_n - E_n) < \frac{\varepsilon}{2^n} \rightarrow (2)$$

Also

$V = \bigcup_{n=1}^{\infty} V_n$, an open set.

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n)$$

$$= E \cap \left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$= E \cap \mathbb{R}$$

$$\bigcup_{n=1}^{\infty} E_n = E$$ ✓

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$\bigcup_{n=1}^{\infty} V_n = E$ $\quad \because E_n \subseteq V_n$

$E = V$ $\quad \because V = \bigcup_{n=1}^{\infty} V_n$

Also

$$V - E = \bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{\infty} E_n$$

$$\Rightarrow m^*(V-E) \leq \sum_{n=1}^{\infty} m^*(V_n - E_n)$$

$$= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$

$$= \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \Rightarrow \text{sum} = 1$$

$$= \varepsilon (1)$$

$$\Rightarrow m^*(V-E) < \varepsilon$$

Conversely:-

Let $\varepsilon = \frac{1}{n}$ (say)

and $\exists$ an open set

$O_n \supseteq E$ such that $E \subseteq O_n$ $\forall n$

and

$$m^*(O_n - E) < \varepsilon = \frac{1}{n}$$

(Next topic)

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Topic # 50

Theorem:-

For a subset E' of $\mathbb{R}$ and given $\varepsilon > 0$, $\exists$ an open set $O \supseteq E$ and $m^*(O - E) < \varepsilon$ if and only if $\exists$ a closed set $F \subseteq E$ and $m^*(E - F) < \varepsilon$.

Proof:-

For $\phi \neq E \subseteq \mathbb{R}$

$\Rightarrow E' \subseteq \mathbb{R}$ $\exists$ (for given $\varepsilon > 0$) an open set $O \supseteq E$ and $m^*(O - E) < \varepsilon \rightarrow (1)$

$O \supseteq E' \Rightarrow E = E'' \supseteq O' = F$ (say)

i.e closed set.

$O' \subseteq E'' = E$

$E \supseteq O' = F$

$\Rightarrow E \supseteq F$

$m^*(E - F) = m^*(E \cap F')$

$= m^*(F' \cap E)$

$= m^*(O - E') < \varepsilon$ by (1)

Converse:-

For the set $E' \subseteq \mathbb{R}$ $\exists$ a closed set 'F' and $\varepsilon > 0$ such that $F \subseteq E'$ and $m^*(E' - F) < \varepsilon \rightarrow (2)$

$F \subseteq E'$

$E = E'' \subseteq F' = O$ (open set)

$\Rightarrow m^*(O - E) = m^*(O \cap E')$

$= m^*(E' \cap O)$

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$= m^*(E' - O^-)$ ①

$= m^*(E' - I^-) < \varepsilon$ by ②

Topic # 51

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Topic # 51

Theorem:-

Let $\phi \neq E \subseteq R$, then the following statements are equivalent.

i) There is a G_δ set "G" with $E \subseteq G$ (countable union of open set)

$$m^*(G-E)=0 \rightarrow \text{given}$$

ii) There is a F_σ set F with $F \subseteq E$ ($\subseteq \cap F$)

$$m^*(E-F)=0 \rightarrow \text{Proof:}$$

Proof:- First we show that

(i) $\Rightarrow$ (ii)

Since we have

$$\phi \neq E \subseteq R \quad \checkmark$$

$$\Rightarrow \phi \neq E' \subseteq R$$

For

$$\phi \neq E' \text{ given that } E' \subseteq G$$

$$m^*(G-E') = 0$$

$$\Rightarrow \quad 0 = m^*(G-E')$$

$$= m^*(G \cap (E')')$$

$$= m^*(G \cap E)$$

$$= m^*(E \cap G)$$

$$= m^*(E \cap (G')')$$

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$$= m^*(E - G')$$
$$= m^*(E - F)$$

where $G_\delta = G = \bigcup_{n=1}^{\infty} G_n$

$$\Rightarrow G'_\delta = G' = \left(\bigcup_{n=1}^{\infty} G_n\right)' = \bigcap_{n=1}^{\infty} G_n' = \bigcap_{n=1}^{\infty} F_n = F = F_\delta$$

Conversely:- (ii) $\Rightarrow$ (i)

again for

$E' \subseteq \mathbb{R}$ with $F \subseteq E'$

$$m^*(E' - F) = 0 \quad \text{(given)}$$
$$\geq m^*(G - E)$$

$$\Rightarrow 0 = m^*(E' - F)$$
$$= m^*(E' \cap F')$$
$$= m^*(E' \cap E')$$
$$= m^*(F' - E)$$
$$= m^*(G - E)$$

where

$$F'_\delta = F' = \left(\bigcap_{k=1}^{\infty} F_k\right)' = \bigcup_{k=1}^{\infty} F_k' = \bigcup_{k=1}^{\infty} G_k = G = G_\delta$$

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$m^*[(U+y)-(E+y)] = m^*(U-E) < \varepsilon$

But

$m^* = m$

$\Rightarrow m[(U+y)-(E+y)] = m[U-E] < \varepsilon$

So

$E+y$ is also measurable. So

m is Translation Invariant.

Topic # 55

Theorem:-

Suppose E_1 and E_2 are two measurable sets, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

Proof:-

i) If $m(E_1) = m(E_2) = \infty$ (infinite)

then the result is obvious.

ii) if $m(E_1) < \infty$, $m(E_2) < \infty$ (finite).

Since

if "E_1" is measurable then

for any $\phi \neq A \subseteq R$ Carlaria

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2^c) \rightarrow (1)$$

therefore

for particularly if $A = E_2$

(1) be comes

$$m(E_2) = m^*(E_2 \cap E) + m^*(E_2 \cap E_1') \rightarrow (2)$$

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Proof: if ~~$E_1 \to E_2$~~ ② belongs

$E_2 \to E_1$ in ②

$m^*(E_1) = m^*(E_1 \cap E_2) + m^*(E_1 \cap E_2') \to ③$

adding ② and ③

$m^*(E_1) + m^*(E_2) = m^*(E_1 \cap E_2) + m^*(E_1 \cap E_2') + m^*(E_1 \cap E_1') + m^*(E_1 \cap E_2)$

As $(m^* \equiv m)$ so,

$m(E_1) + m(E_2) = m(E_1 \cap E_2) + m(E_1 \cap E_2') + m(E_1 \cap E_1') + m(E_1 \cap E_2) \to (A)$

As we know that

$E_1 \cup E_2 = (E_1 - E_2) \cup (E_2 - E_1) \cup (E_1 \cap E_2)$

$= (E_1 \cap E_1') \cup (E_2 \cap E_1') \cup (E_1 \cap E_2)$

$m(E_1 \cup E_2) = m[(E_1 \cap E_2') \cup (E_2 \cap E_1') \cup (E_1 \cap E_2)]$

Since

m is finitely additive

$\Rightarrow m(E_1 \cup E_2) = m(E_1 \cap E_2') + m(E_2 \cap E_1') + m(E_1 \cap E_2)$

So (A) belongs.

$m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2)$

(Proved).

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Topic # 54

Theorem:
Lebesgue measure "m" is translational invariant.

Proof:
Give that
$\phi \neq E \subseteq M$ is measurable

Since we know that
"E" is measurable iff for given $E>0$, $\exists$ an open set "$U \supseteq E$" such that $m^*(U-E) < E$.

As
"U" is open set in $\mathbb{R}$
$\Rightarrow U+y$ is also open in $\mathbb{R}$, $\forall y \in \mathbb{R}$.

If $E \subseteq U$
$\Rightarrow E+y \subseteq U+y$
$\Rightarrow (U+y)-(E+y) = (U-E)+y$
$\Rightarrow m^*[(U+y)-(E+y)] = m^*[(U-E)+y]$
$=$

Since
Lebesgue outer measure is translation invariant.

So,

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Topic # 59

Theorem:-

Let $\{E_n, \uparrow\}_{n=1}^{\infty}$ be an increasing sequence of measurable sets. Then

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

Proof:-

Given that

$(E_n, \uparrow)$ such that

$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ of measurable set

Defining

$F_1 = E_1$, $F_2 = E_2 - E_1$, $F_3 = E_3 - E_2$, ...

$F_n = E_n - E_{n-1}$

where $\{F_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable sets.

$$\Rightarrow \bigcup_{K=1}^{n} F_K = \bigcup_{K=1}^{n} E_K \quad \text{— finite}$$

$$\Rightarrow \bigcup_{K=1}^{\infty} F_K = \bigcup_{K=1}^{\infty} E_K \quad \text{infinite}$$

So

$$m^*\left(\bigcup_{K=1}^{\infty} F_K\right) = m\left(\bigcup_{K=1}^{\infty} E_K\right)$$

$$= m\left(\bigcup_{K=1}^{\infty} F_K\right)$$

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$$= \sum_{k=1}^{\infty} m(E_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} m(E_k)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{n} E_k\right)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$= \lim_{n\to\infty} m^*(E_n)$$

Topic # 60

Theorem:-

Let $\{E_n, \downarrow\}$ be a decreasing sequence of measurable sets such that $m(E_k) < \infty$, for some $k \in \mathbb{N}$, then

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

Proof:-

Let $m(E_1) < \infty$, say (without loss of generality)

Since $\{E_n, \downarrow\}$ is decreasing sequence

so

$E_n \downarrow \Rightarrow E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots\cdots$

$\Rightarrow \phi = E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \cdots\cdots$

$\cdots\cdots E_1 - E_n \subseteq \cdots$

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$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty}(E_1 - E_n)\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$$

Since

$$\bigcup_{n=1}^{\infty}(E_1 - E_n) = \bigcup_{n=1}^{\infty}(E_1 \cap E_n')$$

$$= (E_1 \cap E_1') \cup (E_1 \cap E_2') \cup \cdots$$

$$= E_1 \cap [E_1' \cup E_2' \cup \cdots]$$

$$= E_1 \cap [E_1 \cap E_2 \cap \cdots]'$$

$$= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)'$$

$$= E_1 - \left(\bigcap_{n=1}^{\infty} E_n\right)$$

So

$$m^*\left(E_1 - \bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} (E_1 - E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$$

$$= \lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

We get

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

$$= m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \cdots = \lim_{n\to\infty} m^*(E_n)$$

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n) \quad \text{Ag}$$

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Topic # 62

Borel Measure Space:-

Theorem

Every Borel set is measurable.

Proof:-

As we know that

Every open or closed set is measurable.

$\Rightarrow$ countable union or intersection of closed or open sets is also measurable.

So,

Borel set is also measurable being union or intersection of closed or open sets.

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Topic # 65

Theorem:-

Let $E \subseteq A = [0,1)$ be a measurable set. then for each $y \in A$, the set $E \hat{+} y$ is measurable and

$$m(E) = m(E \hat{+} y)$$

Proof:-

Suppose

$$E_1 = E \cap [0, 1-y) \quad \text{and}$$

$$E_2 = E \cap [1-y, 1)$$

Since $E_1 \subseteq [0, 1-y)$, $[1-y, 0)$ are measurable.

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$\Rightarrow E_1$ and E_2 are also measurable

Also

$E_1 \cap E_2 = \{\xi \in \cap [0, 1/y)\} \cap \{E \cap [1/y, 1)\}$

$= E \cap \{[0, 1/y) \cap [1/y, 1)\}$

$= E \cap \phi$

$= \phi$

and

$E_1 \cup E_2 = E \cup \phi = E$

$\Rightarrow m(E) = m(E_1) + m(E_2) \rightarrow (1)$

$\therefore$ m is finitely additive.

Let

$y_1 \in A$

if $y_1 \in E_1$

$\Rightarrow y_1 < 1 - y$

$\Rightarrow y_1 + y < 1$

$\Rightarrow y_1 \hat{+} y = y_1 + y$

So

$\Rightarrow E_1 \hat{+} y = E_1 + y \rightarrow (2)$

if $y_2 \in E_2$

$\Rightarrow y_2 \geq 1 - y$

$\Rightarrow y_2 + y \geq 1$

So

$y_2 \hat{+} y = y_2 + y - 1$

$\Rightarrow E_2 \hat{+} y = E_2 + (y - 1)$

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Since we know that "m" is translation invariant.

$\Rightarrow E_1 + y$ and $E_2 + (y-1)$ are measurable.

So

$E_1 \hat{+} y$ and $E_2 \hat{+} (y-1)$ are also measurable.

Therefore.

$$m(E_1 \hat{+} y) = m(E_1 + y) = m(E_1)$$

Similarly

$$m(E_2 \hat{+} y) = m(E_2 + (y-1)) = m(E_2)$$

Since

$$E_1 \cup E_2 = E$$

and

$$E_1 \cap E_2 = \emptyset$$

$$\Rightarrow E \hat{+} y = (E_1 + y) \cup (E_2 + y)$$

and

$$(E_1 + y) \cap (E_2 + y) = \emptyset$$

"$E \hat{+} y$" being union of two disjoint measurable sets is measurable.

So

$$\begin{aligned} m(E \hat{+} y) &= m(E_1 \hat{+} y) + m(E_2 \hat{+} y) \\ &= m(E_1 + y) + m(E_2 + y) \\ &= m(E_1) + m(E_2) \\ &= m(E) \end{aligned}$$

So

"m" is also invariant w.r.t translation modulo "1".

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which is also measurable.

Topic # 67

Theorem:

Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$, $\emptyset \neq E \in M$

then

i) f is measurable

$\Updownarrow$

ii) $\{x \in E : f(x) \geq a\}$ is measurable $\forall\, a \in \mathbb{R}$

$\Updownarrow$ MSQC

iii) $\{x \in E : f(x) > a\}$ is measurable $\forall\, a \in \mathbb{R}$

$\Updownarrow$

iv) $\{x \in E : f(x) < a\}$ is measurable $\forall\, a \in \mathbb{R}$

Proof:

First we show that

(i) $\Rightarrow$ (ii)

Means that

if f is measurable then

$\{x \in E : f(x) \geq a\}$ is also measurable.

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$$\{x \in E : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \tfrac{1}{n}\}$$

Let $y \in \{x : f(x) \geq a\}$

$\Rightarrow f(y) \geq a \quad \forall\, a \in \mathbb{R}$

$\Rightarrow f(y) > a - \frac{1}{n} \quad \forall\, n \in \mathbb{N}$

$\Rightarrow y \in \{x : f(x) > a - \frac{1}{n}\}, \quad \forall\, n \in \mathbb{N}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$

So

$$\{x : f(x) \geq a\} \subset \bigcap_{n=1}^{\infty} \{x : f(x) > a - \tfrac{1}{n}\}$$

Conversely

$$t \in \bigcap_{n=1}^{\infty} \{x : f(x) > a - \tfrac{1}{n}\}$$

$\Rightarrow f(t) > a - \frac{1}{n}, \quad \forall\, n \in \mathbb{N}$

$\Rightarrow f(t) \geq a, \quad \forall\, a \in \mathbb{R}$

$\Rightarrow t \in \{x : f(x) \geq a\}$

$$\Rightarrow \bigcap_{n=1}^{\infty} \{x : f(x) > a - \tfrac{1}{n}\} \subseteq \{x : f(x) \geq a\}$$

$$\Rightarrow \{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x : f(x) > a - \tfrac{1}{n}\}$$

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Since $\{x : f(x) > q - \frac{1}{n}\}$ is measurable.

$\forall\ n \in \mathbb{N}$

$\bigcap_{n=1}^{\infty} \{x : f(x) > q - \frac{1}{n}\}$ is also measurable.

$\Rightarrow \{x \in E : f(x) \geq q\}$ is also measurable.

Topic # 68

(ii) $\Rightarrow$ (iii)

Means

$\{x \in E : f(x) \geq q\} \in M$ (Measurable)

$\Downarrow$

$\{x \in E : f(x) < q\} \in M$ (Measurable).

we know that if

$\{x : f(x) < q\} = \{x \in E : f(x) \geq q\}^c$

$\forall\ a \in R$

Since

$\{x : f(x) \geq 0\}$ is measurable.

and $\{x : f(x) \geq 0\}^c$ is also measurable.

So

$\{x : f(x) < q\}$ is also measurable.

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Topic # 73

Theorem:-

Show that

i) $f^+ = \frac{f + |f|}{2}$ WSVC ii) $f^- = \frac{|f| - f}{2}$

iii) $f = f^+ - f^-$ WSVC iv) $|f| = f^+ + f^-$

v) $\max(f, g) = \frac{(f+g) + |f-g|}{2}$

Proof:-

(i)

if $f(x) \geq 0$

$\Rightarrow f^+(x) = f(x)$

$\Rightarrow f^+ = f$

$|f(x)| = |f|(x)$

$= f(x)$

$\frac{f + |f|}{2}(x) = \frac{f(x) + |f|(x)}{2}$

$= \frac{f(x) + f(x)}{2}$

$= f(x)$

$= f^+(x)$

if $f(x) < 0$

$\Rightarrow f^+(x) = 0$

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$|f(x)| = |f(x)| = -f(x)$

$\Rightarrow \frac{f + |f|}{2}(x) = \frac{f(x) + |f|(x)}{2}$

$= \frac{f(x) - f(x)}{2}$

$= 0$

$\rightarrow \frac{f + |f|}{2} = f^+$

(iii)

$f = f^+ - f^-$

if $f(x) \neq 0$

$f^+(x) = f(x)$, $f^-(x) = 0$

$\Rightarrow (f^+ - f^-)(x) = f^+(x) - f^-(x)$

$= f(x) - 0$

$= f(x)$

$f^+ - f^- = f$

(v)

$\max(f, g) = \frac{(f+g) + (f-g)}{2}$

Since

$|f-g|(x) = |f(x) - g(x)|$

$= \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x) \\ g(x) - f(x) & \text{if } f(x) \leq g(x) \end{cases}$

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case (a)

If $f(x) \geq g(x)$

$\Rightarrow f(x) - g(x) \geq 0$

$\Rightarrow |f(x) - g(x)| = f(x) - g(x)$

So,

$$\frac{(f+g) + (f-g)}{2}(x) = \frac{f(x) + g(x) + |f-g|(x)}{2}$$

$$= \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

$$= \frac{f(x) + g(x) + f(x) - g(x)}{2}$$

$$= f(x)$$

$$= \max(f, g)$$

case (b)

If $f(x) < g(x)$

$\Rightarrow f(x) - g(x) < 0$

$\Rightarrow |f(x) - g(x)| = -(f(x) - g(x))$

$= g(x) - f(x)$

and

$\max(f, g) = g(x)$

So

$$\frac{(f+g) + (f-g)}{2}(x) = \frac{f(x) + g(x) + |f-g|(x)}{2}$$

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$= g(x)$

$= \max(f, g)$

(iv)

$|f| = f^+ + f^-$

If

$f(x) \geq 0$

$f^+(x) = f(x)$ and $f^-(x) = 0$

So

$(f^+ + f^-)(x) = f^+(x) + f^-(x)$

$= f(x) + 0$

$= f(x) \rightarrow (1)$

If $f(x) < 0$

$f^+(x) = 0$, $f^-(x) = -f(x)$

So

$(f^+ + f^-)(x) = f^+(x) + f^-(x)$

$= 0 + (-f(x))$

$= -f(x) \rightarrow (2)$

By (1) and (2)

$f^+ + f^- = |f|$

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$A \times B = \{(x,y) : x \in A, y \in B\}$.

Topic # 78

Theorem:- Cartesian Product of characteristic function:

prove that

$$\chi_{A \times B} = \chi_A \cdot \chi_B$$

Proof:-

Let

(x,y) be a ordered pair.

and

$(x,y) \in A \times B$

case (i)

if $x \in A$ and $x \in B$

$\Rightarrow \chi_A(x) = 1$, $\chi_B(x) = 1$

$\Rightarrow \chi_{A \times B}(x,y) = 1$

$\chi_{A \times B} = \chi_A \cdot \chi_B$

case (ii)

if $(x,y) \notin A \times B$

- $x \in A$ but $y \notin B$ → (a)
- $x \notin A$ but $y \in B$ → (b)
- $x \notin A$ but $y \notin B$ → (c)

$\Rightarrow \chi_{A \times B}(x,y) = 0$

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$\lambda_A \cdot \lambda_B$ = { a ⇒ $1 \cdot 0 = 0$; b ⇒ $0 \cdot 1 = 0$; c ⇒ $0 \cdot 0 = 0$

In all cases

$\lambda'_{A \times B} = \lambda_A \cdot \lambda_B$

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Topic # 79

Theorem:-

If $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of X then show that

$$\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$$

Proof:-

If $x \in A$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow x \in A_k$ (exactly for $n=k$)

$\chi_A(x) = 1$

$x \notin \bigcup_{n=1}^{\infty} A_n := \{A_k\}$

$\Rightarrow \chi_{A_k}(x) = 1 \;,\; \chi_{A_n}(x) = 0 \;,\; \forall\, n \neq k$

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$\chi_A(x) = 1 = 0 + 0 + \cdots + \underbrace{1}_{n=k} + 0 + \cdots$

$\chi_A(x) = 1 = \sum_{n=1}^{\infty} \chi_{A_n}(x)$

$\Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Case ii)

$x \notin A = \bigcup_{n=1}^{\infty} A_n$

$\chi_A(x) = 0$

$\Rightarrow x \notin A_n, \quad \forall n \in \mathbb{N}$

$\Rightarrow \chi_{A_n}(x) = 0, \quad \forall n \in \mathbb{N}$

So

$\chi_A(x) = 0 + 0 + \cdots + 0 + \cdots$

$= \sum_{n=1}^{\infty} \chi_{A_n}(x)$

$\Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Hence we prove that

$\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

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Topic # 80

Theorem: Let $\phi \neq X \supseteq A$ and for characteristic function $\chi_A : X \to \mathbb{R}$, χ_A is measurable if and only if "A" is measurable.

Proof:

Suppose χ_A is measurable (function).

$\Rightarrow \forall a \in \mathbb{R}$

The set $\{x : \chi_A(x) > a\}$ is measurable.

In particular for $a = 0$

$\{x : \chi_A(x) > 0\}$ (by def χ_A).

So A is measurable.

Converse:

Let A is measurable and we prove "χ_A" is measurable.

In another words we want to prove that

$$\{x : \chi_A(x) > a\} \in M \quad \forall\, a \in \mathbb{R}.$$

$$\{x : \chi_A(x) > a\} = \begin{cases} \phi & , a \geq 1 \\ A & 0 \leq a < 1 \\ A \cup A^c & a < 0 \end{cases}$$

If $A \in M$

$\Rightarrow A^c \in M$

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$A \cup A^c$ is also measurable.

and

f is also measurable.

$\Rightarrow \{x : \chi_A(x) > a\} \in M$

So

χ_A is measurable

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Topic # 87

(vi)

If $|f|$ is measurable.

Proof:-

Since

f is measurable.

$\Rightarrow \{x : f(x) > a\} \in M, \forall a \in \mathbb{R}$

To prove

If $|f|$ is measurable, we need to show

$\{x : |f|(x) > a\} \in M, \forall a \in \mathbb{R}$

$$\{x : |f|(x) > a\} = \{x : |f(x)| > a\}$$

$$= \begin{cases} \{x : f(x) > a\} \cup \{x : f(x) < -a\} & \forall a > 0 \\ D \in M & a \leq 0 \end{cases}$$

In all case in R.H.S, we get a measurable set.

$\Rightarrow \{x : |f|(x) > a\} \in M \quad \forall a \in \mathbb{R}$

$\Rightarrow |f|$ is also measurable

Corollaries:-

1)

$$f \vee g = \max(f, g)$$

$$= \frac{1}{2}\{(f+g) + (f-g)\}$$

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Since

$f+g$ is measurable. and kf is measurable.

$\rightarrow$ $f \vee g$ is measurable.

ii)

$f \wedge g = \min(f, g)$

$= \frac{1}{2}\{(f+g) - (|f-g|)\}$

$f \wedge g$ is measurable.

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Topic #48

Theorem: Open interval $(a,b) \subseteq \mathbb{R}$ is measurable.

proof: $\because (a,\infty)$ is measurable

$\Rightarrow (a,\infty)^c = (-\infty, a]$ is also measurable.

And $(-\infty, b-\frac{1}{n}]$ is also measurable $\forall\ n \in \mathbb{N}$ as

and $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b-\frac{1}{n}]$ is also measurable

because being countable union of meriurable

$\Rightarrow (a,\infty) \cap (-\infty, b) = (a,b)$ i.e also measurable being intersection of two measurable set

Open set in $\mathbb{R}$:

Any open set in $\mathbb{R}$ say U can be written countable union of disjoint open intervals

i.e $U = \bigcup_{n=1}^{\infty} \{I_n\}$ and each open intervals is

MCQ's Topic # 63

Theorem: Countable sets are Borel sets with zero measure.

Proof: (Case I) Singleton:

$\forall\ a \in \mathbb{R}: \{a\} = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n})$; $n \in \mathbb{N}$

Countable
↓
finite ↔ infinite
$\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{Q}$ ---

$\because \{(a - \frac{1}{n}, a + \frac{1}{n})_{n \in \mathbb{N}} \downarrow\}$ is decreasing sequence

with $m(E_n) = \frac{2}{n} < \infty$

where $E_n = (a - \frac{1}{n}, a + \frac{1}{n})$

$\Rightarrow m\{a\} = m\left\{\bigcap_{n=1}^{\infty} E_n\right\} = \lim_{n \to \infty} m(E_n)$

$= \lim_{n \to \infty} \ell(E_n)$

$= \lim_{n \to \infty} \left(\frac{2}{n}\right) = \frac{2}{\infty}$

$\infty = 0$

Case - II

Finite Set

$\{x_1, x_2, x_3, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\}$

$\Rightarrow m\left(\bigcup_{i=1}^{n} \{x_i\}\right) = \sum_{i=1}^{n} m\{x_i\} = 0$

Topic #65

Long Q

Theorem: Let $E \subset A = [0,1)$ be a measurable set. Then for each $y \in A$. The set $E \hat{+} y$ is measurable and $m(E) = m(E \hat{+} y)$.

proof: Suppose $E_1 = E \cap [0, 1-y)$ and

$E_2 = E \cap [1-y, 1)$

$\therefore E_1 [0, 1-y), [1-y, 1)$ are measurable

$\Rightarrow E_1$ and E_2 are also measurable being the intersection of two measurable sets.

Also $E_1 \cap E_2 = \{E \cap [1-y, 1)\} \cap \{E \cap [0, 1-y]\}$

$= E \cap \{[0, 1-y) \cap [1-y, 1)\}$

$= E \cap \phi = E$

Topic #79

Theorem: If $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of X. Then show that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Topic # 87

(vii) $|f|$ is measurable:

proof: $\because$ "f" is measurable $\Rightarrow \{x : f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

To prove $|f|$ is measurable, we need to show:

$\{x : |f|(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

$$\{x : |f|(x) > a\} = \{x : |f(x)| > a\}$$

$$= \begin{cases} \{x : f(x) > a\} \cup \{x : f(x) < -a\}, & \forall a > 0 \\ \in \mathcal{M} \quad \in \mathcal{M} \\ \{x : |f|(x) > a\} = \\ \emptyset \in \mathcal{M}, & a < 0 \end{cases}$$

In all cases on R.H.S, we get a measurable set. $\Rightarrow \{x : |f|(x) > a\} \in \mathcal{M}$ (measurable), $\forall a \in \mathbb{R}$

$\Rightarrow |f|$ is also measurable.