

MT644

Measure Theory

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Topic #01

Introduction:

Mathematics is a place where you can do things which you can't do in the real world.

(Marcus Du Santoy)

Why the measure theory?

Do the measurements really need Abstraction and Concrete vigorous grounds?

yes!! ... Due to ... Infinity ∞ and Infinitesimals ϵ .

Srinivasa Ramanujan: the man who knew infinity (∞)!

There are different ways to look at the size of a set.

$\{0, 3, 6, 9\}$ has order 4, but on real line, these are "Negligible".

For $\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup \dots, 1\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup \dots$
 $= 1 + 2 + 3 + \dots = ?$

But still on real line these are "Negligible".

So a single point takes up no real width on the real line.

The mathematical concept of measure is the generalization of lengths (size) in $\mathbb{R}$, area in $\mathbb{R}^2$, volume in $\mathbb{R}^3$ and hence in $\mathbb{R}^n$ and other spaces.

Starting from the lengths of intervals on $\mathbb{R}$.

Intuitively:

I) $|[0,1]| = 1$, and all $(0,1), (0,1], [0,1) \subset [0,1]$

$$\Rightarrow |(0,1)|, |(0,1]|, |[0,1)| \leq |[0,1]|$$

(Monotonicity preserves with respect to measure in $\mathbb{R}$)

And, $|(2,\infty)| = \infty$

Informally, the measure is a monotonic mapping from a subsets of $\mathbb{R}$ to non-negative extended real numbers.

II) $[-1/2, 1/2]$ is $[0,1]$ i.e. translated " $1/2$ " unit to the left. And both have length (measure) "1".

(Translation Invariant)

$$\text{III) } |[0,2]| = |[0,1] \cup (1,2]|$$

$$\Rightarrow 2 = 1 + 1$$

Similarly, $(0,1] = (0,1) \cup \{1\} \Rightarrow 1 = 1 + 0$, means:

"the whole is the sum of its parts" (additivity).

But for, $|[0,1]| = \left| \bigcup_{x \in [0,1]} \{x\} \right| \Rightarrow 1 = 0 + 0 + 0$

Therefore, we take the "whole is the sum of its parts", if a set is decomposed into a countable number of disjoint sets. (Countable Subadditivity)

Topic #02

Ring of Sets:

Let $X \neq \phi$, and $\mathcal{A}$ be a class or collection of subsets of X . Then $\mathcal{A}$ is said to be a ring if:

- i) $A \cup B \in \mathcal{A}$
 - ii) $A - B \in \mathcal{A}$
- } $\forall A, B \in \mathcal{A}$

For Examples:

(i) $X = \{a, b, c\}$ and $\mathcal{A} = \{\phi, X, \{b\}, \{a, c\}\}$
all are subset of X .

(ii) $X \neq \phi$, $\mathcal{P}(X) = \{A : A \subseteq X\}$

$\Rightarrow \mathcal{P}(X)$ is a ring of X .

(iii) N , $\mathcal{A} = \{A : A \subseteq N \text{ such that } |A| < \infty\}$

$\Rightarrow A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$, $A - B \in \mathcal{A}$,
order of A

Topic #3

Sigma Ring:

Let $X \neq \phi$, and $\mathcal{A}$ be collection of

subsets of "X". Then "A" is said to a σ -ring if:

- i) $\forall A, B \in \mathcal{A} \Rightarrow A - B \in \mathcal{A}$
- ii) For any sequence: $\{A_i\}_{i=1}^{\infty}$ of subsets of "X"
 $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

Examples:

$X \neq \emptyset$, $\mathcal{A} = \{\emptyset, X\}$ is σ -ring.
and $\mathcal{P}(X) = \{A : A \subseteq X\}$ is σ -ring.

Topic #04

Ring vs Sigma Ring:

σ -ring is also a ring because σ -ring is satisfied the both properties of ring.

For N , $\mathcal{A} = \{A \subseteq N : |A| < \infty\}$ is ring.
 $\downarrow$ finite

$$\Rightarrow A - B \in \mathcal{A}$$

Let $\{n_1\}, \{n_2\}, \{n_3\} \dots$ be a sequence finite subsets of N .

But $\left| \bigcup_{i=1}^{\infty} \{n_i\} \right| \neq \infty \Rightarrow \mathcal{A}$ is not σ -ring.

Topic #05

Boolean Algebra (Algebra of Sets):

Let $X \neq \emptyset$, and "A" is collection of "X". Then "A" is said to a boolean algebra if

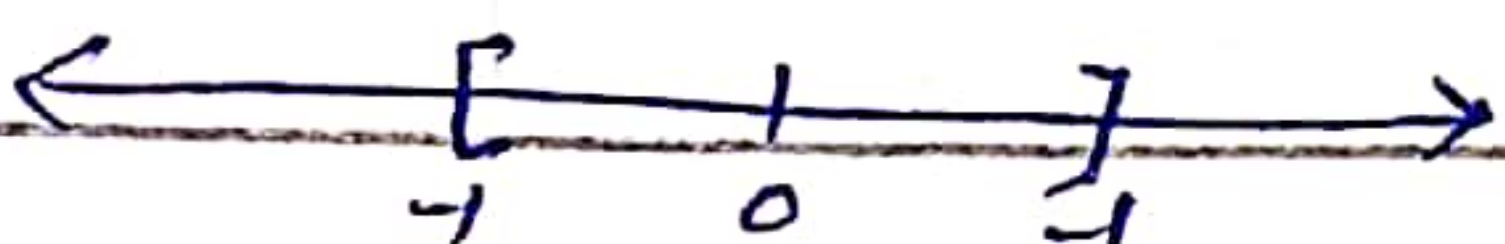
$$i) \forall A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \quad ii) \forall A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$$

Examples:

(i) $X = \{a, b, c\}$, $\mathcal{A} = \{\phi, X, \{a\}, \{b, c\}\}$

$\mathcal{A}$ is boolean algebra.

(ii) $X \neq \phi$, $\mathcal{P}(X) = \{A \subseteq X\}$ is boolean algebra.

(iii) let $X = [-1, 1] \subseteq \mathbb{R}$  and $\mathcal{A} = \{\phi, X, [-1, 0), [0, 1]\}$ is boolean algebra.

Topic #06

Theorem: let $\mathcal{A}$ be a ring of subsets of $X \neq \phi$, then (i) $\phi \in \mathcal{A}$, (ii) $A \Delta B \in \mathcal{A}$, (iii) $A \cap B \in \mathcal{A}$
Symmetric relation

Proof: (i) $\forall A, B \in \mathcal{A} \Rightarrow A - B \in \mathcal{A}$

In particular if $A = B$.

$A - B = A - A = \phi \in \mathcal{A} \because$ by definition of ring.

(ii) $\forall A, B \in \mathcal{A}$

$\Rightarrow A - B, B - A \in \mathcal{A}$

$\Rightarrow (A - B) \cup (B - A) \in \mathcal{A}$

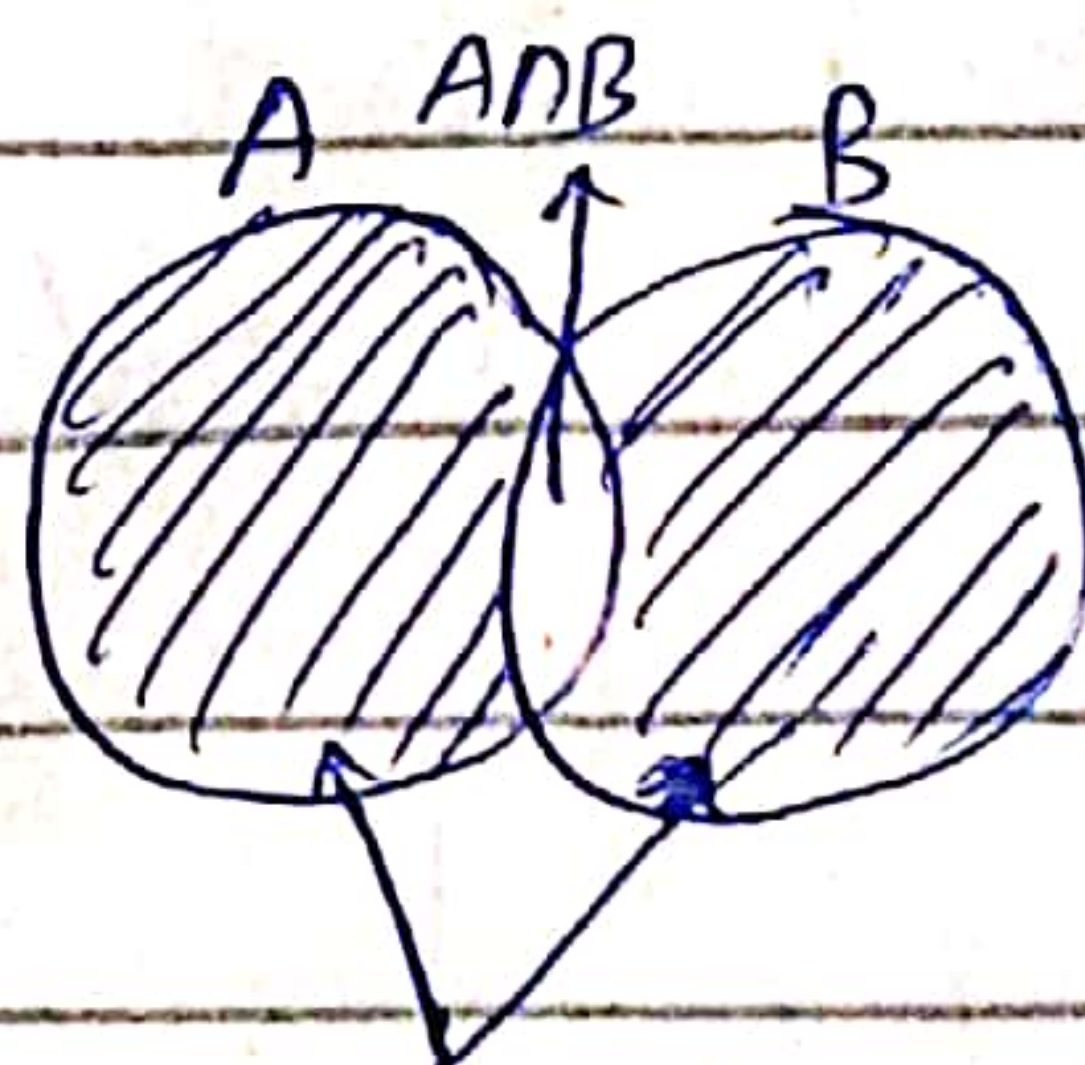
$\Rightarrow A \Delta B \in \mathcal{A}$

(iii) $\forall A, B \in \mathcal{A}$

$\Rightarrow A \Delta B \in \mathcal{A}, A \cup B \in \mathcal{A}$

by definition of ring.

$(A - B) \cup (B - A) = A \Delta B$
 $\downarrow$
 A symmetric B



$(A - B) \cup (B - A) = A \Delta B$

From figure

$A \cup B = [(A - B) \cup (B - A)] \cup (A \cap B) = (A \Delta B) \cup (A \cap B)$

and $A \cap B = [(A \cup B) - (A \Delta B)] \in \mathcal{A}$

$\Rightarrow A \cap B \in \mathcal{A}$

TOPIC # 07

Theorem 1: Every Algebra is a Ring.

proof: By definition of Algebra " $\mathcal{A}$ " on set $X \neq \emptyset$:

$$\forall A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \text{ and } A^c \in \mathcal{A}$$

$$\therefore A' \in \mathcal{A} \quad \forall A \in \mathcal{A} \Rightarrow (A' \cup B) \in \mathcal{A}$$

$\Rightarrow$ $\therefore$ By definition of Algebra

$$\Rightarrow (A' \cup B)' \in \mathcal{A}$$

$$\Rightarrow A'' \cap B' \in \mathcal{A} \quad \therefore \text{By De-Morgan's law}$$

$$\Rightarrow A \cap B' \in \mathcal{A} \quad \therefore A'' = A \quad \forall \text{ set } "A"$$

$$\Rightarrow A - B \in \mathcal{A} \quad \therefore A \cap B' = A - B \quad \forall A, B \in X$$

$$\Rightarrow \mathcal{A} \text{ is also a Ring.}$$

TOPIC # 08

Theorem 2: For a $X \neq \emptyset$, Then ring " $\mathcal{A}$ " if subsets of " X " is also an algebra iff $X \in \mathcal{A}$.

proof: p : ring " $\mathcal{A}$ " is an algebra.

$$q: X \in \mathcal{A}$$

Given: " p " and To prove: " q ";

$$\therefore \text{For } X \neq \emptyset, \mathcal{A} \text{ is a ring} \Rightarrow \forall A \in \mathcal{A}$$

$$\Rightarrow \emptyset \in \mathcal{A} \quad \therefore A - A = \emptyset \Rightarrow (A - A) \in \mathcal{A}$$

$$\Rightarrow \emptyset' \in \mathcal{A} \Rightarrow \emptyset' = X \in \mathcal{A}$$

Now Given: " q " and To prove: " p "

$$(i) \quad \therefore X \in \mathcal{A} \text{ and } \mathcal{A} \text{ is a ring}$$

$$\Rightarrow \forall A \in \mathcal{A} \Rightarrow (X - A) \in \mathcal{A} \quad \therefore \text{by definition of ring}$$

$$\Rightarrow A' \in \mathcal{A}$$

$$(ii) \forall A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$$

$\therefore$ (i) and (ii)

$\Rightarrow$ any $\mathcal{A}$ of subset of X is also an algebra.

Topic # 09

σ -Algebra of sets:

let $X \neq \emptyset$, and $\mathcal{A}$ be a class of subset of X , Then $\mathcal{A}$ is said to be a σ -algebra if

$$(i) \forall A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$$

$$(ii) \text{ whenever } \{A_1, A_2, A_3, \dots\} \in \mathcal{A}$$

$$\text{or } \{A_i\}_{i=1}^{\infty} \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

Example: (i) For any $X \neq \emptyset$, $\mathcal{A} = \{\emptyset, X\}$ is a σ -algebra.

(ii) For any $X \neq \emptyset$; $\mathcal{P}(X) = \{A : A \subseteq X\}$ is also a σ -algebra.

(iii) For $X = [-1, 1] \subseteq \mathbb{R}$

$\Rightarrow \mathcal{A} = \{\emptyset, X, [-1, 0), [0, 1]\}$ is also a σ -algebra.

(iv) For $X = \mathbb{N}$, $\mathcal{A} = \{A \subseteq \mathbb{N} : |A| < \infty \vee A \text{ is countable}\}$

$\vee A' \text{ is countable?}$

$$A = E = \{2n : n \in \mathbb{N}\}, E' = \emptyset = \{2n-1, n \in \mathbb{N}\}$$

if $X = \mathbb{R}$ (uncountable)

$$\mathcal{A} = \{A \subseteq \mathbb{R} : |A| < \infty \vee A' \text{ countable} \dots\}$$

$Z \subseteq \mathbb{R}$ but Z' is not countable.

(Not Holds)

Topic #10

Theorem: Let $X \neq \emptyset$, and $\mathcal{A}$ is a class of subsets of X ; then σ -ring $\mathcal{A}$ is also a σ -algebra iff $X \in \mathcal{A}$.

proof: See Topic #08 for proof.

Theorem: Arbitrary intersection of σ -algebra on $X \neq \emptyset$ is also a σ -algebra.

proof: Let $\mathcal{Q} = \{\mathcal{A}_\alpha : \alpha \in I\}$ be a class of σ -algebra on $X \neq \emptyset$; clearly $\mathcal{Q} \neq \emptyset$.

To prove $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is also a σ -algebra.

(i) For any $A \subseteq X$, such that $A \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$

$$\Rightarrow A \in \mathcal{A}_\alpha \quad \forall \alpha \in I$$

But each $\mathcal{A}_\alpha$ is a σ -algebra.

$$\Rightarrow A' \in \bigcup_{\alpha \in I} \mathcal{A}_\alpha; \quad \forall \alpha \in I$$

$$\Rightarrow A' \in \bigcap_{\alpha \in I} \bigcup_{\alpha \in I} \mathcal{A}_\alpha \rightarrow \textcircled{1}$$

If $x \in A \cap B$

Then $x \in A$ and $x \in B$

(ii) Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets of $X \neq \emptyset$, such that $\{A_i\}_{i=1}^{\infty} \subseteq \bigcap_{\alpha \in I} \mathcal{A}_\alpha \Rightarrow \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}_\alpha \quad \forall \alpha \in I$

$\therefore$ each $\mathcal{A}_\alpha$ is a σ -algebra

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_\alpha \quad \forall \alpha \in I$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bigcap_{\alpha \in I} \bigcup_{\alpha \in I} \mathcal{A}_\alpha \rightarrow \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$

$$\Rightarrow \bigcap_{\alpha \in I} \mathcal{A}_\alpha \text{ is also a } \sigma\text{-algebra.}$$

Topic # 11

problem: Give a counter example to show that union of two σ -algebra is not necessarily a σ -algebra.

Solution: Let $X = \{a, b, c\}$, and defining two σ -algebra $\mathcal{A}_1 = \{\phi, X, \{b\}, \{a, c\}\}$ and

$$\mathcal{A}_2 = \{\phi, X, \{c\}, \{a, b\}\}$$

Now taking

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \{\phi, X, \{b\}, \{c\}, \{a, c\}, \{a, b\}\}$$

Taking finite (countable) sequence of subsets of $\mathcal{A}_1 \cup \mathcal{A}_2$ i.e. $\{b\}, \{c\}$, Then

$$\{b\} \cup \{c\} = \{b, c\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$$

$\Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ -algebra.

Topic # 12

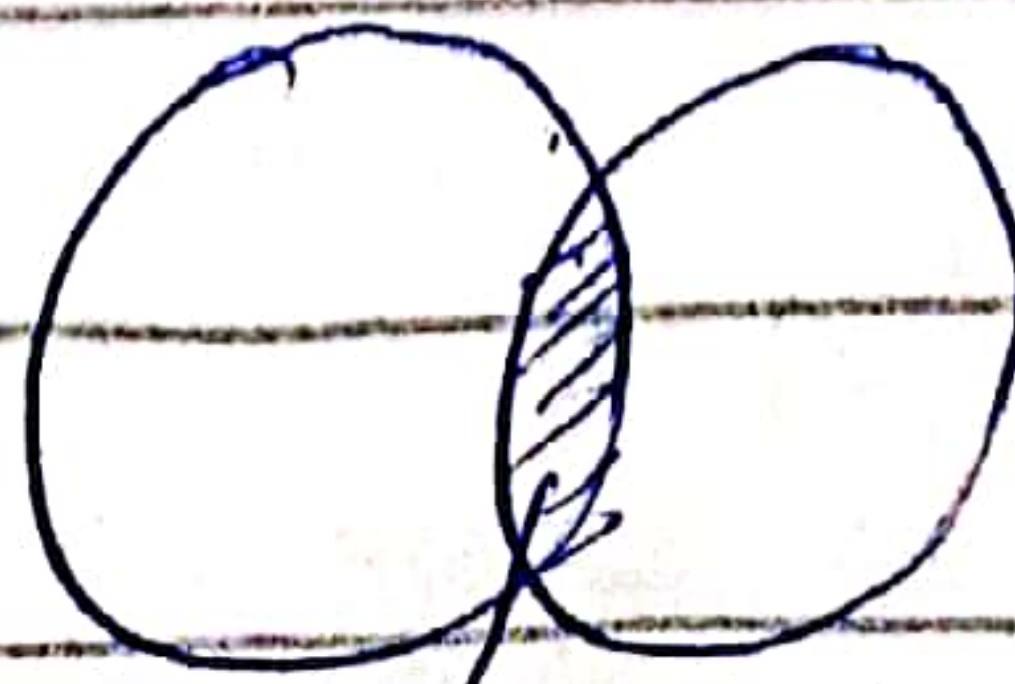
Sigma(σ)-Algebra Generated by a Class:

For $X \neq \phi$, $A \subseteq X$ and $B = \{A\}$, Then $\{\phi, X, A, A'\}$ is a smallest σ -algebra generated by "B".

Definition: Let $X \neq \phi$ and B be a family of subsets of X , Then σ -algebra generated by "B" defined as the intersection of all σ -algebra containing "B"

$$\bigcap_{i \in I} \{\mathcal{A}_i : \mathcal{A}_i \text{ is a } \sigma\text{-algebra, } B \subseteq \mathcal{A}_i, \forall i \in I\}$$

X



Smallest / Intersection

Topic #13

Theorem: Let "B" be a family of subsets of $X \neq \emptyset$, Then there exist a smallest sigma algebra containing "B".

proof: let $\mathcal{Q} = \{A_i : B \subseteq A_i \forall i \in I\}$
 $\because P(X)$ is a σ -algebra containing "B"
 $\Rightarrow \mathcal{Q} \neq \emptyset$.

We will prove:

(i) Existence (ii) Minimality

$$(i) \mathcal{G} = \bigcap_{i \in I} \{A_i : B \subseteq A_i, \forall i \in I\} \rightarrow \textcircled{1}$$

$\because$ Arbitrary intersection of σ -algebra is a σ -algebra
 and $B \subseteq A_i, \forall i \in I \Rightarrow B \subseteq \bigcap A_i = \mathcal{G}$.

$\Rightarrow$ "G" is a σ -algebra containing "B".

(ii) let "J" is a σ -algebra containing "B".

$$\Rightarrow J \in \mathcal{Q} \Rightarrow \bigcap_{i \in I} A_i \subseteq J \quad \because A \cap B \subseteq A$$

$\Rightarrow \mathcal{G} \subseteq J \Rightarrow$ "G" is a smallest σ -algebra.

Topic #14

Theorem: Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of elements of σ -algebra "A" of subsets of $X \neq \emptyset$, Then there exists also a sequence $\{F_n\}_{n=1}^{\infty}$ of ^{Subsets} ~~elements~~ of "A" such that

$$(i) F_n \in \mathcal{A} \forall n \in \mathbb{N}, \quad (ii) F_n \subseteq E_n \forall n \in \mathbb{N},$$

$$(iii) F_n \cap F_m = \emptyset, \forall m \neq n, \quad (iv) \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

proof: (i) Let we construct:

$$F_1 = E_1, \quad F_2 = E_2 - E_1, \quad F_3 = E_3 - (E_1 \cup E_2)$$

$$\text{by induction: } F_n = E_n - (E_1 \cup E_2 \cup E_3 \dots \cup E_{n-1})$$

$$\begin{aligned}
 \Rightarrow F_n &= E_n \cap (E_1 \cup E_2 \cup \dots \cup E_{n-1})' \because A-B = A \cap B' \\
 &= E_n \cap (E_1' \cap E_2' \cap \dots \cap E_{n-1}') \quad \text{By De Morgan's Law} \\
 \because E_n \in \mathcal{A} \text{ and } \because E_1, E_2, \dots, E_{n-1} \in \mathcal{A} \\
 \Rightarrow E_1', E_2', \dots, E_{n-1}' &\in \mathcal{A} \\
 \Rightarrow F_n &= E_n \cap (E_1' \cap E_2' \cap \dots \cap E_{n-1}') \in \mathcal{A} \rightarrow (1)
 \end{aligned}$$

(ii) From eq. (1)

$$\Rightarrow F_n \subseteq E_n \quad \forall n \in \mathbb{N} \rightarrow (2)$$

(iii) $\forall m \neq n \Rightarrow m \not\subseteq n$.



$$A \cap B \subseteq A, B$$

Say without loss of generality $m < n$.

$$\text{From (2)} \quad F_m \subseteq E_m \Rightarrow F_m \cap F_n \subseteq E_m \cap F_n \rightarrow (3)$$

$$F_m \cap F_n = E_m \cap [E_n \cap (E_1' \cap E_2' \cap \dots \cap E_{n-1}')] \quad \downarrow \text{From eq. (1)}$$

$$F_m \cap F_n = (E_m \cap E_m') \cap [E_n \cap (E_1' \cap E_2' \cap \dots \cap E_{n-1}')] \quad \downarrow$$

$$F_m \cap F_n = \phi$$

$$\Rightarrow F_m \cap F_n \subseteq \phi \Rightarrow F_m \cap F_n = \phi \quad \forall m \neq n$$

(iv) $\because F_n \subseteq E_n \quad \forall n \in \mathbb{N}$

$$\Rightarrow \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n \rightarrow (3)$$

Let $y \in \bigcup_{n=1}^{\infty} E_n \Rightarrow y \in E_n$ for some $n \in \mathbb{N}$

~~And say~~ Suppose "m" be a least positive integer

$$y \in E_m \text{ but } y \notin E_1, E_2, \dots, E_{m-1}$$

$$y \in A \text{ by } y \notin B$$

$$\Rightarrow y \notin E_1 \cup E_2 \cup \dots \cup E_{m-1}$$

$$\Rightarrow y \in A - B$$

$$\Rightarrow y \in E_m - (E_1 \cup E_2 \cup \dots \cup E_{m-1})$$

$$\Rightarrow y \in F_m \Rightarrow y \in \bigcup_{m=1}^{\infty} F_m = \bigcup_{n=1}^{\infty} F_n$$

$\sim q = \text{negation } q$

$$\therefore \Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n$$
$$\Rightarrow \boxed{\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n}$$

Topic #15

An Important Result of Analysis:

Theorem: Suppose that for any $\epsilon > 0$, if $a < b + \epsilon$, then $a \leq b$.

Proof: $p: a < b + \epsilon$ and $q: a \leq b$

proof scheme: $\begin{cases} p \rightarrow \text{true} \text{ and } \sim q \rightarrow \text{true} \\ \text{and seek for contradiction.} \end{cases}$

$$p \Rightarrow a < b + \epsilon \text{ and } \sim q \Rightarrow a > b$$
$$\Rightarrow (a - b) > 0$$

$$\begin{cases} p \rightarrow q \text{ if } p \rightarrow \text{true} \\ q \rightarrow \text{false, } p \rightarrow \text{false} \end{cases}$$

But we have given from $p: a < b + \epsilon$

$\therefore$ In particular taking $\epsilon = (a - b) > 0$

$$\therefore p \Rightarrow a < b + (a - b) \Rightarrow a < a$$

i.e. a contradiction

$$\therefore a < b + \epsilon \Rightarrow a \leq b \quad ; \quad \epsilon \rightarrow 0 \text{ arbitrarily} \\ (\epsilon > 0)$$

Topic #16

Set Function:

Let $X \neq \phi$ and $\mathcal{A} \subseteq \mathcal{P}(X)$, be a class of subsets of X , then a function defined on " $\mathcal{A}$ " as: $\mu: \mathcal{A} \rightarrow [0, \infty) \cup \{\infty\}$, i.e. every element (set) is mapped to a non-negative

real extended real number.

For examples: (i) $[0, \infty) \subseteq \mathbb{R} \Rightarrow \mu([0, \infty)) = \infty$
(0, 1) $\subseteq \mathbb{R} \Rightarrow \mu(0, 1) = 1 - 0 = 1$

(ii) $\mu: P(\mathbb{N}) \rightarrow [0, \infty) \cup \{\infty\} \quad \forall E \subseteq P(\mathbb{N})$
if $E = \{1, 2, 3\} \Rightarrow \mu(E) = O(E) = 3$
 $E = \{1, 2, 3, \dots\} \Rightarrow \mu(E) = O(E) = \infty$

Topic #17

Measure on a σ -Algebra:

Let $X \neq \emptyset$ and $\mathcal{A}$ be a σ -algebra on X , then a set function $\mu: \mathcal{A} \rightarrow [0, \infty) \cup \{\infty\}$ is said to be (i) Finitely subadditive if

$$\mu\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \mu(E_i)$$

(ii) Countably subadditive if

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

(iii) Finitely additive if

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i), \quad \forall E_i \cap E_j = \emptyset \quad \forall i \neq j$$

(iv) Countably additive if

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i), \quad \forall E_i \cap E_j = \emptyset \quad \forall i \neq j$$

(v) Monotone if

$$\mu(A) \leq \mu(B) \text{ whenever } A \subseteq B$$

$$\mu: A \rightarrow [0, \infty) \cup \{\infty\}$$

measure on "A" if

$$(i) \mu(\emptyset) = 0$$

$$(ii) \mu(A) \geq 0 \quad \forall A \in A$$

(iii) μ is ~~count~~ countably additive

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \quad \forall E_i \cap E_j = \emptyset \quad \forall i \neq j$$

Topic #18

Monotonicity of Measure:

(i) If $\emptyset \neq A, B \in A$ such that $B \subseteq A$, then

$$\mu(B) \leq \mu(A)$$

proof: $\because B \subseteq A \Rightarrow A = B \cup (A-B)$

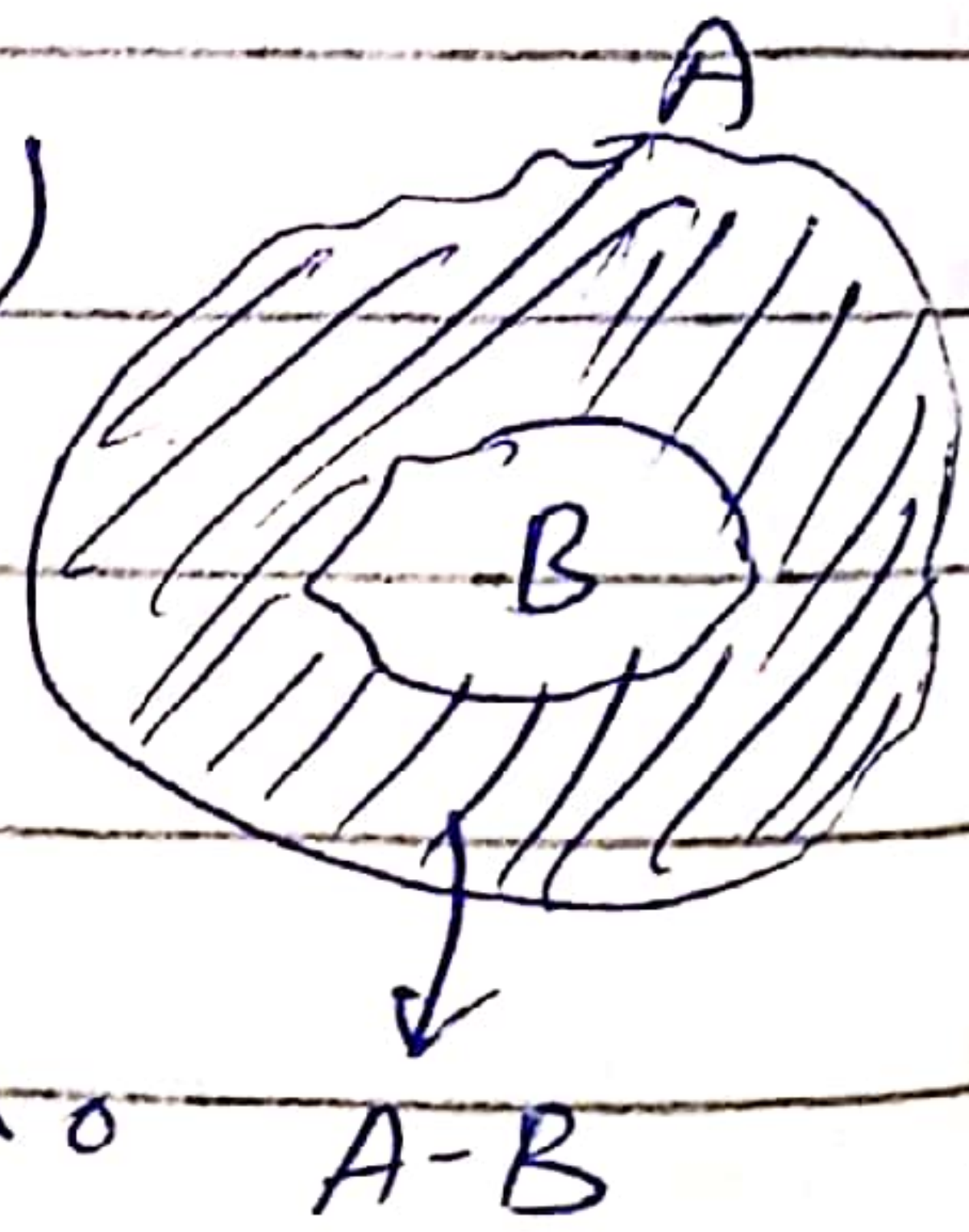
$$\Rightarrow \mu(A) = \mu[B \cup (A-B)]$$

$$\mu(A) = \mu(B) + \mu(A-B) \quad \text{--- (1)}$$

$$\Rightarrow \mu(A) \geq \mu(B)$$

$$\Rightarrow \mu(B) \leq \mu(A)$$

$$\begin{cases} a \neq b+c \\ a \geq b, c \geq 0 \\ \Rightarrow a \geq b \end{cases}$$



Topic #19

(ii) If $\mu(B) < \infty$, then $\mu(A-B) = \mu(A) - \mu(B)$

proof: From eq. (1) $\Rightarrow \mu(A) - \mu(B) = \mu(A-B)$

$[0, \infty]$

subtraction

Difference

operation on sets

Topic #20

Measurable Space:

For $X \neq \emptyset$, let $\mathcal{A}$ be the σ -algebra on X . Then the triple $(X, \mathcal{A}, \mu)$ is called measurable space and members of $\mathcal{A}$ are called measurable sets.

$\mathcal{A} \rightarrow$ class of subsets of X .

Topic #21

Theorem:- Let $(X, \mathcal{A}, \mu)$ be a measure space. Then if $\exists$ a $\phi \neq A \in \mathcal{A}$ such that $\mu(A) < \infty$ then $\mu(\phi) = 0$.

proof: $\because A = A \cup \phi$

$$\Rightarrow \mu(A) = \mu(A \cup \phi) = \mu(A) + \mu(\phi)$$

By countable additivity of μ

$$\Rightarrow \mu(\phi) = 0$$

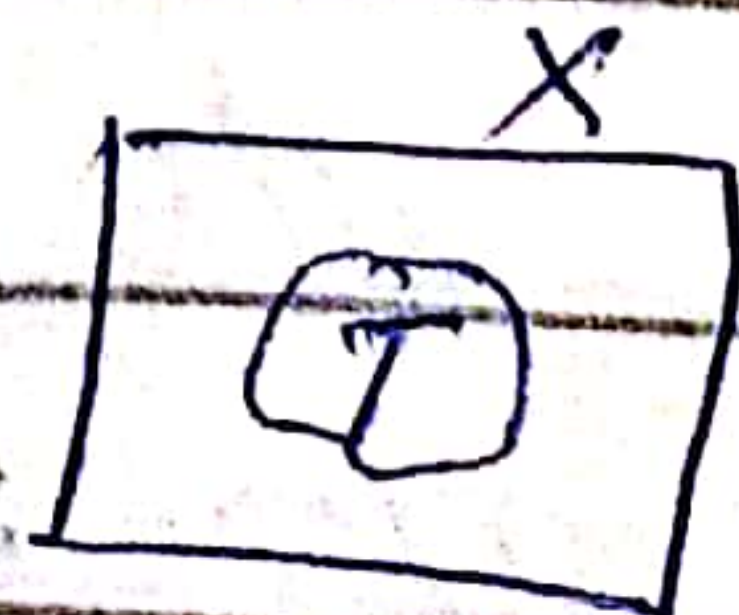
Topic #22

Theorem: Let $\{A_n\}$ is a sequence of pair wise disjoint subsets of X in σ -algebra $\mathcal{A}$ then for any $T \in \mathcal{A}$,

$$\mu(T) = \sum_{n=1}^{\infty} \mu(A_n \cap T) + \mu\left(T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right).$$

proof: $\because$ For any $T \in \mathcal{A} \Rightarrow T = T \cap X$

$$\Rightarrow T = T \cap \left[\left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right]$$



$$\Rightarrow T = \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)\right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right]$$

$$\begin{aligned} A \cup A^c &= X \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

$$\begin{cases} T \cap \left(\bigcup_{n=1}^{\infty} A_n \right) = T \cap (A_1 \cup A_2 \cup A_3 \cup \dots) \\ = (T \cap A_1) \cup (T \cap A_2) \cup \dots \\ = \bigcup_{n=1}^{\infty} (T \cap A_n) \end{cases}$$

$$\Rightarrow T = \left[\bigcup_{n=1}^{\infty} (T \cap A_n) \right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right]$$

$$\Rightarrow \mu(T) = \mu \left\{ \left[\bigcup_{n=1}^{\infty} (T \cap A_n) \right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right] \right\}$$

By additive property of " μ "

$$\Rightarrow \mu(T) = \mu \left[\bigcup_{n=1}^{\infty} (T \cap A_n) \right] + \mu \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right]$$

$\because$ All $T \cap A_n$'s are disjoint

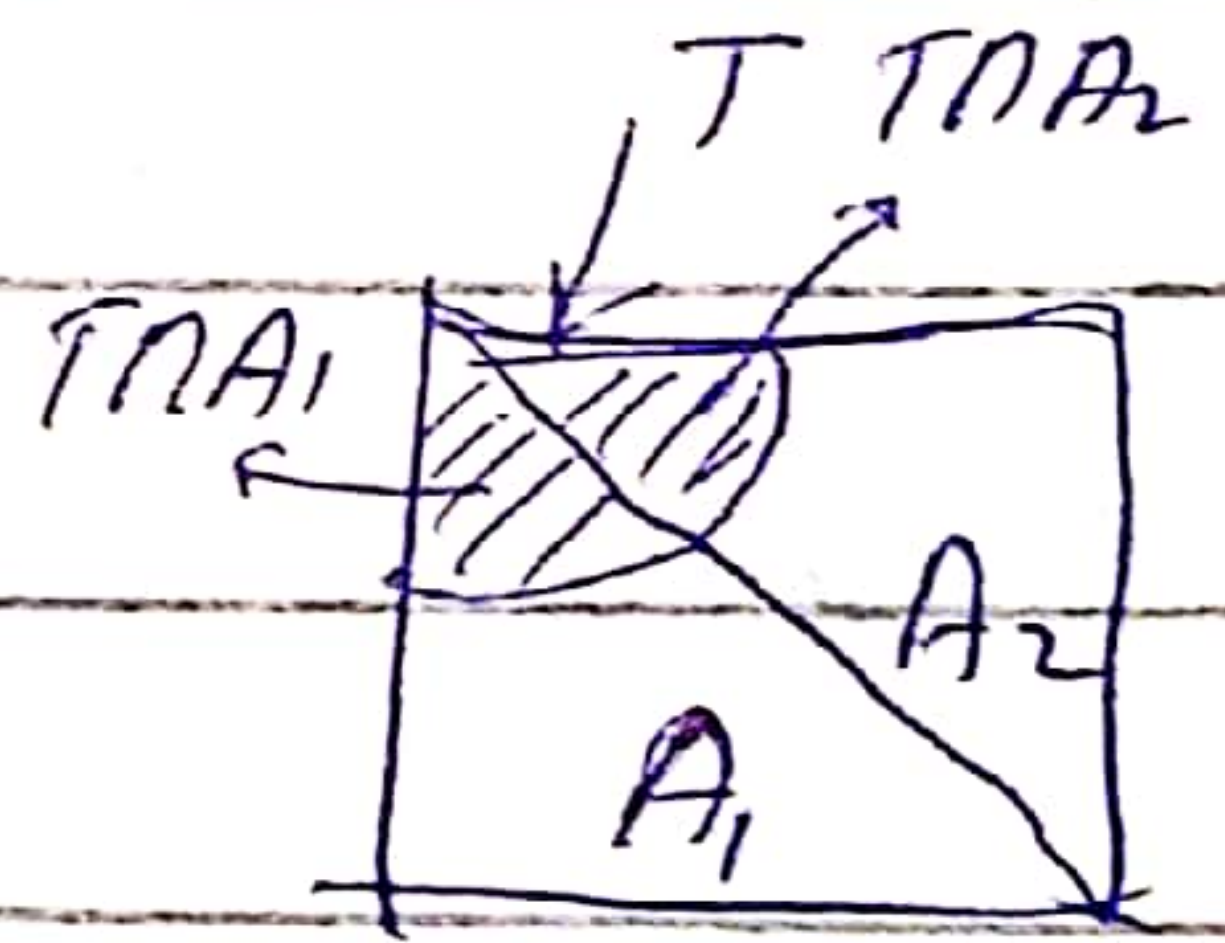
$\therefore$ by additive property of " μ "

$$\mu \left(\bigcup_{n=1}^{\infty} B_n \right) = \mu(B_1 \cup B_2 \cup \dots)$$

$$= \mu(B_1) + \mu(B_2) + \dots$$

$$= \sum_{n=1}^{\infty} \mu(B_n)$$

$$\Rightarrow \mu(T) = \sum_{n=1}^{\infty} \mu(T \cap A_n) + \mu \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right]$$



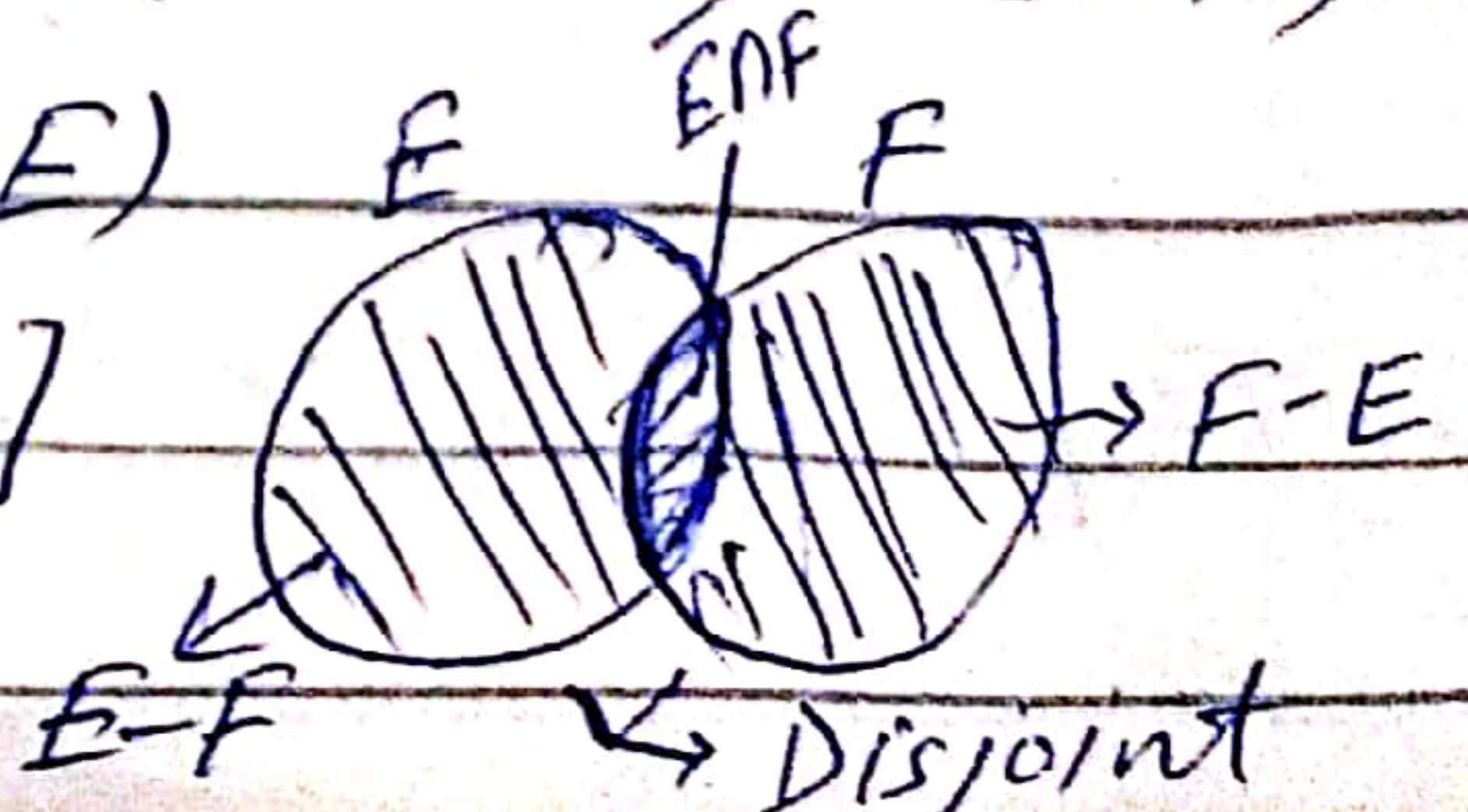
Topic #23

Theorem: In a measure space $(X, \mathcal{A}, \mu)$, if

$\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F) = \mu(E \cup F) = \mu(E \cap F)$

Proof: $\because E \Delta F = (E - F) \cup (F - E)$

$$\Rightarrow \mu(E \Delta F) = \mu[(E - F) \cup (F - E)]$$



E symmetric F



$$\Rightarrow \mu(E \Delta F) = \mu(E-F) + \mu(F-E)$$

Given that $\mu(E \Delta F) = 0$

$$\Rightarrow 0 = \mu(E-F) + \mu(F-E)$$

$$\Rightarrow \mu(E-F) = 0 \text{ and } \mu(F-E) = 0$$

Note that $F = (F-E) \cup (F \cap E)$

$\therefore F-E$ and $F \cap E$ are disjoint

$$a+b=0$$

$$a, b \geq 0$$

$$\Rightarrow a=0$$

$$\text{and } b=0$$

$$\Rightarrow \mu(F) = \mu[(F-E) \cup (F \cap E)]$$

$$\Rightarrow \mu(F) = \mu(F-E) + \mu(F \cap E)$$

$$\Rightarrow \mu(F) = 0 + \mu(F \cap E)$$

$$\Rightarrow \mu(F) = \mu(F \cap E) \quad \text{EDF} = (E-F) \cup (F-E)$$

①

Also $E \cup F = (E \Delta F) \cup (E \cap F)$

$$\Rightarrow \mu(E \cup F) = \mu[(E \Delta F) \cup (E \cap F)]$$

$$\Rightarrow \mu(E \cup F) = \mu(E \Delta F) + \mu(E \cap F)$$

$$\Rightarrow \mu(E \cup F) = 0 + \mu(E \cap F)$$

$$\Rightarrow \mu(E \cup F) = \mu(E \cap F) \rightarrow \text{②}$$

Now $E = (E-F) \cup (E \cap F)$

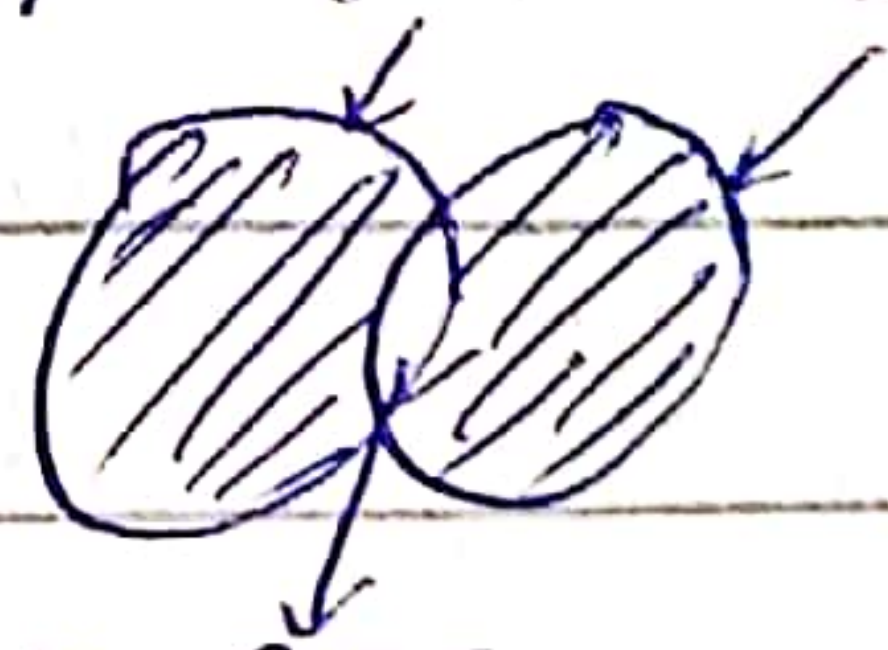
$$\Rightarrow \mu(E) = \mu[(E-F) \cup (E \cap F)]$$

$$\Rightarrow \mu(E) = \mu(E-F) + \mu(E \cap F)$$

$$\Rightarrow \mu(E) = 0 + \mu(E \cap F)$$

$$\Rightarrow \mu(E) = \mu(E \cap F) \rightarrow \text{③}$$

From ①, ② and ③ $\Rightarrow \mu(E) = \mu(F) = \mu(E \cup F) = \mu(E \cap F)$



$E-F$ and $E \cap F$ are disjoint

Topic # 24

Finite Measure:

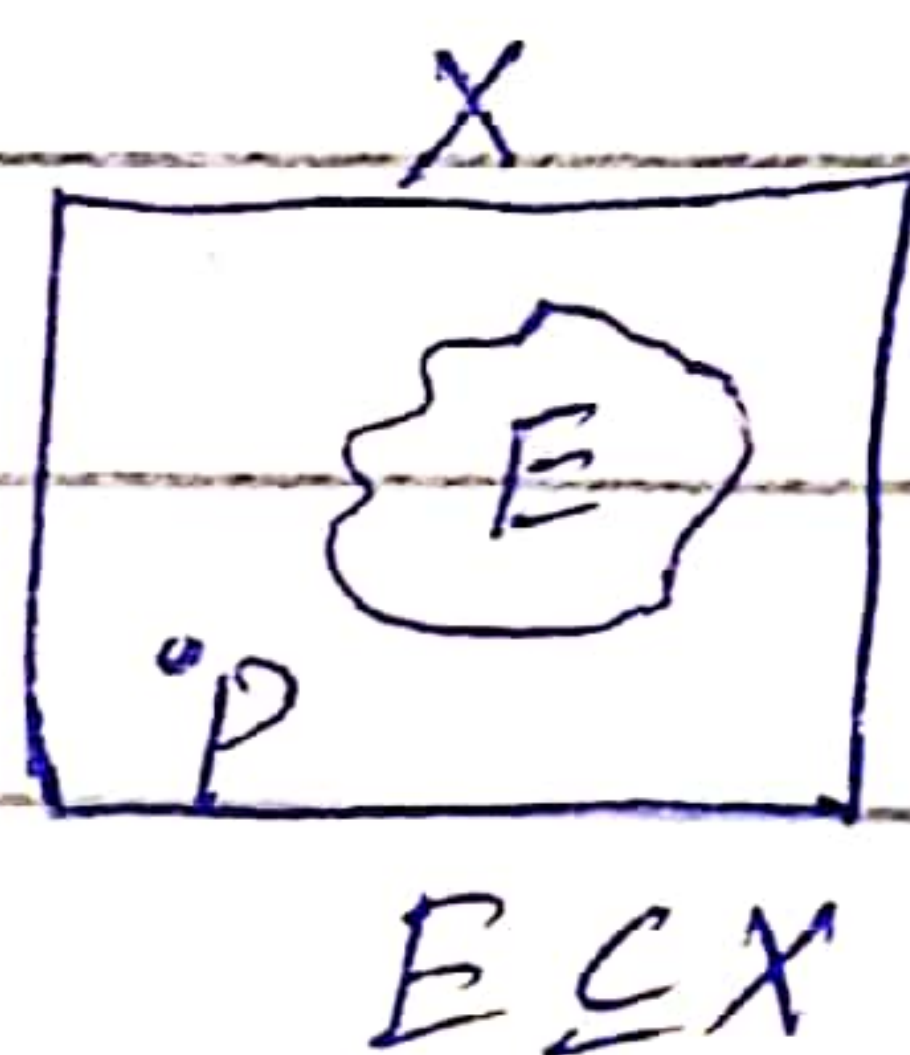
For $\phi \neq X$ the triplet $(X, \mathcal{A}, \mu)$ is said to be finite measure space if $\mu(X) < \infty$ and " μ " is called finite measure.

Sigma(σ)-Finite Measure:

For $\phi \neq X$, the triple $(X, \mathcal{A}, \mu)$ is σ -finite measure if and only if $\mu(X) < \infty$ and $\exists$ a sequence $\{A_n\}$ of subsets of X , such that $\bigcup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty \quad \forall A_n \in \mathcal{A}$

For example:

$$(i) \quad X \neq \phi, \mu: \mathcal{P}(X) \rightarrow [0, \infty]$$
$$\mu(E) = \begin{cases} 0 & \text{if } p \notin E \\ 1 & \text{if } p \in E \end{cases}$$



for any fixed point " p " in " X ".

$$(ii) \quad \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$
$$\mu: \mathcal{P}(\mathbb{Z}) \rightarrow [0, \infty]$$

$$\mu(A) = \begin{cases} |A| = \alpha(A) = \#(A) & \text{if } "A" \text{ is finite} \\ \infty & \text{if } "A" \text{ is infinite.} \end{cases}$$

Counting measure.

Topic #25

For $\phi \neq X$, defining; $\mu: P(X) \rightarrow [0, \infty]$, as

$$\mu(A) = \begin{cases} 0 & ; p \notin A \\ 1 & ; p \in A \end{cases}$$

for fixed point $p \in X$, Then show that " μ " is a finite measure on " X ."

proof: (i): $p \in \phi \Rightarrow \mu(\phi) = 0$

by given definition of " μ "

(ii) for any $A \subseteq X, \forall A \in \mathcal{A}$; by definition

$$\mu(A) \geq 0$$

(iii) let $\{A_n\}$ be the sequence of pairwise disjoint subset of " X " in " $\mathcal{A}$ ".

μ on σ -algebra on " X "

i) $\mu(\phi) = 0$

ii) $\mu(A) \geq 0$

iii) $\{A_n\} \rightarrow \text{disjoint}$
 $\in \mathcal{A}$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

- Finite μ

$$\mu(X) < \infty$$

Case I: let $p \in \left(\bigcup_{n=1}^{\infty} A_n\right) \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 1$

$\Rightarrow p \in A_k$ for some fixed $k = n$.

$\Rightarrow \mu(A_k) = 1$ and $\mu(A_n) = 0 \quad n \neq k$

$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = 1 = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

Case II: $p \notin \left(\bigcup_{n=1}^{\infty} A_n\right) \Rightarrow p \notin A_n \quad \forall n$

$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$ and $\mu(A_n) = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = 0$

So $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = 0$

Now, Trivially $p \in X \Rightarrow \mu(X) = 1$

$\Rightarrow \mu(X) < \infty$

$\eta \rightarrow \text{Eta}$

Topic # 26

Closed Operations of Measure:-

(i) Sum of Measures:

Let " μ " and " η " are measure on $\mathcal{B}$ -algebra of $X \neq \emptyset$, Then

$$(i) \quad \mu(\emptyset) = 0, \quad \eta(\emptyset) = 0$$

$$(ii) \quad \mu(A) \geq 0, \quad \eta(A) \geq 0, \quad \forall A \in \mathcal{A}$$

(iii) For any sequence $\{A_n\}$ in $\mathcal{A}$ of disjoint pairs

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

and

$$\eta\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \eta(A_n)$$

$$(1) \quad (\mu + \eta)(\emptyset) = \mu(\emptyset) + \eta(\emptyset) = 0 + 0 = 0$$

$$(2) \quad (\mu + \eta)(A) = \mu(A) + \eta(A) \geq 0 \quad \because \mu(A), \eta(A) \geq 0$$

$$(3) \quad (\mu + \eta)\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) + \eta\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= \sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \eta(A_n)$$

$$= \sum_{n=1}^{\infty} (\mu(A_n) + \eta(A_n))$$

$$= \sum_{n=1}^{\infty} (\mu + \eta)(A_n)$$

For any say $\alpha \in \mathbb{R}$

$$\alpha \mu(A) = \alpha(\mu A)$$

scalar composition.

Counter case: For $A \in \mathcal{A}$

$$\mu(A) \geq 0, \quad \eta(A) \geq 0$$

But $\mu(A) - \eta(A) < 0$

if $\eta(A) > \mu(A)$

$\Rightarrow$ " $\mu - \eta$ " is not defined

Topic #27

Outer Measure:

For $X \neq \emptyset$, a function $\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$ is said to be outer measure if:

(i) $\mu^*(\emptyset) = 0$

(ii) whenever $A \subseteq B: \mu^*(A) \leq \mu^*(B)$

(iii) If $\{A_n\}$ be a sequence of subsets of X
i.e. $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

For Example $X \neq \emptyset$: define $\mu^*: \mathcal{P}(X) \rightarrow [0, \infty]$

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset \end{cases}$$

Sol: (i) If $\emptyset = A \Rightarrow \mu^*(A) = 0$ trivially true.

(ii) If $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(a) $A = B = \emptyset \Rightarrow \mu^*(A) = 0 = \mu^*(B)$

(b) $A = \emptyset, B \neq \emptyset \Rightarrow \mu^*(A) = 0 < \mu^*(B) = 1$

(c) $A \neq B \neq \emptyset \Rightarrow \mu^*(A) = 1 = \mu^*(B)$

In all three cases: $\mu^*(A) \leq \mu^*(B)$

(iii) Let $\{A_n\}$ be the sequence of subsets in " X ".

(a) If all $A_n = \phi \quad \forall n \in \mathbb{N} \Rightarrow \left(\bigcup_{n=1}^{\infty} A_n\right) = \phi$
 $\Rightarrow \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) = 0 = \mu^*(A_n) = \sum_n \mu^*(A_n)$

(b) If $A_n \neq \phi$ for some $n \in \mathbb{N}$

$\Rightarrow \mu^*(A_n) = 1$ for some $n \in \mathbb{N}$

$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) \geq 1 \rightarrow \textcircled{1}$

and If $A_n \neq \phi \Rightarrow \bigcup_{n=1}^{\infty} A_n \neq \phi$

$\Rightarrow \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 \rightarrow \textcircled{2}$

From $\textcircled{1}$ and $\textcircled{2}$

$\Rightarrow \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Topic # 28

Lebesgue Outer Measure:

For $\phi \neq E \subseteq \mathbb{R}$, the Lebesgue measure of " E " defined on $\mathcal{P}(\mathbb{R})$ as $m^*: \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty) \cup \{\infty\}$ and given by:

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ where } \{I_n\} \text{ is a sequence of open intervals covering } E \right\}$$

Here $l(I_n) \rightarrow$ length of n th interval $\{I_n\}$.

Topic #29

Non-negativity and Monotonicity of Lebesgue Outer Measure:

Since $l(I_n) \geq 0, \forall n \in \mathbb{N}$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \geq 0 \quad \because E \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\Rightarrow \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n, \{I_n\} \text{ be the sequence of interval covering } E \right\} \geq 0$$

$$\Rightarrow m^*(E) \geq 0$$

Let $\emptyset \neq A, B \in \mathcal{P}(\mathbb{R})$ such that $A \subseteq B$ and $\{I_n\}$ be a sequence of intervals covering "B" in $\mathbb{R}$.

$$\Rightarrow B \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow A \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\Rightarrow m^*(A) \leq \sum_{n=1}^{\infty} l(I_n), \quad B \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\Rightarrow m^*(A) \text{ is lower bound of } \left\{ \sum_{n=1}^{\infty} l(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

$$\Rightarrow m^*(A) \leq \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

$$\Rightarrow m^*(A) \leq m^*(B)$$

$$\{x\} \subseteq \mathbb{R}, \quad \forall \varepsilon > 0$$

$$\{x\} \subseteq (x - \varepsilon, x + \varepsilon)$$

$$\Rightarrow m^*\{x\} \leq \frac{2\varepsilon}{2} = \varepsilon$$

$$\left| \begin{array}{l} A \subseteq B \\ m^*(A) \leq m^*(B) \end{array} \right.$$

$$\Rightarrow m^*\{x\} = 0 \quad \text{where } \varepsilon \rightarrow 0 \text{ arbitrary}$$

Topic #30

Countable Subadditivity of Lebesgue Outer Measure:

Theorem: For any arbitrary seq $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathbb{R})$
 $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$ i.e. m^* is countably subadditive.

Proof: (i) For fixed $n \in \mathbb{N}$ if $m^*(A_n) = \infty$
 $\Rightarrow m^*(\bigcup_{n=1}^{\infty} A_n) = \infty = m^*(A_n) = \sum_{n=1}^{\infty} m^*(A_n)$

(ii) For $m^*(A_n) < \infty, \forall n \in \mathbb{N}$

But $m^*(A_n) = \inf \left\{ \sum_{m=1}^{\infty} l(I_{nm}) : A_n \subseteq \bigcup_{m=1}^{\infty} I_{nm} \right\}$

Where $\{I_{nm}\}_{m=1}^{\infty}$ is open covering of A_n

For $\epsilon > 0, \Rightarrow \frac{\epsilon}{2n} > 0$

$\Rightarrow$ there exists a sequence $\{I_{nm}\}_{m=1}^{\infty}$

$$\sum_{m=1}^{\infty} l(I_{nm}) < m^*(A_n) + \frac{\epsilon}{2n} \rightarrow \textcircled{1}$$

$$\because A_n \subseteq \bigcup_{m=1}^{\infty} I_{nm} \Rightarrow \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} I_{nm} \right)$$

Here, countable union $\bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} I_{nm} \right)$ is countable sets: $\bigcup_{m=1}^{\infty} I_{nm}$ is countable.

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \inf \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} l(I_{nm}) : \right.$$

$$\left. \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} I_{nm} \right) \right\} \rightarrow$$

where $\{I_{nm}\}$ is sequence of open intervals covering $\bigcup_{n=1}^{\infty} A_n$

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} l(I_{nm})$$

$$\text{From (1)} \Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \left(m^*(A_n) + \frac{\epsilon}{2^n}\right)$$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow \text{Geometric Series.}$$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \epsilon \cdot 1$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

$$S = \frac{a}{1-r}$$

$$S = \frac{1}{1 - \frac{1}{2}} = 1$$

Topic #31

Lebesgue Outer Measure of Countable set:

Since every countable set can be expressed in the form of sequence i.e. $1, 2, 3, \dots, n, \dots$

$$\Rightarrow A = \{x_1, x_2, x_3, \dots, x_n, \dots\} = \bigcup_{n=1}^{\infty} \{x_n\}_{n=1}^{\infty}$$

$$0 \leq m^*(A) = m^*\left(\bigcup_{n=1}^{\infty} \{x_n\}_{n=1}^{\infty}\right) \leq \sum_{n=1}^{\infty} m^*\{x_n\} = 0$$

$\therefore$ Lebesgue outer Measure of $\{x\} = 0$

$$\Rightarrow m^*(A) = 0$$

Topic #32

Theorem: For any $\phi \neq A \subseteq \mathbb{R}$ and any $\epsilon > 0$, there is an open set " U " such that " $A \subseteq U$ " and $m^*(U) < m^*(A) + \epsilon$

Proof: By definition

$m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : A \subseteq \bigcup I_n, \text{ where } \{I_n\}_{n=1}^{\infty} \text{ is a sequence of open intervals covering "A" in } \mathbb{R} \text{ and } l(I_n) \text{ is the length of } n\text{th interval} \right\}$

For any $\epsilon > 0$, $\exists \{I_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that $m^*(A) + \epsilon > \sum_{n=1}^{\infty} l(I_n)$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) < m^*(A) + \epsilon \rightarrow \textcircled{1}$$

Union of open interval in " $\mathbb{R}$ " is an open set $\Rightarrow$ there exists U an open set.

$$\bigcup_{n=1}^{\infty} I_n = U (\text{open set})$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} I_n\right) = m^*(U)$$

$$\Rightarrow m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \rightarrow \textcircled{2}$$

$$\therefore m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} m^*(I_n)$$

$$\text{From } \textcircled{2} \Rightarrow m^*(U) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} l(I_n)$$

$$\text{From } \textcircled{1} \Rightarrow m^*(U) < m^*(A) + \epsilon$$

Topic #33

The Lebesgue Outer Measure of an Interval:

Theorem: Lebesgue outer measure of an interval is its length. i.e. $m^*(a, b) = m^*[a, b] = m^*(a, b] = m^*[a, b) = b - a$.

proof: Case I: Let $I = [a, b]$ in $\mathbb{R}$. By definition:
 $m^*(I) = \inf \left\{ \sum_{i=1}^{\infty} l(I_i) : I \subseteq \bigcup_{i=1}^{\infty} I_i, \text{ where } \{I_i\}_{i=1}^{\infty} \text{ are open intervals covering } I \text{ in } \mathbb{R} \text{ and } l(I_i) \text{ is length of } i\text{th interval} \right\} \longrightarrow \textcircled{1}$

In particular for $\varepsilon > 0$, $(a - \varepsilon, b + \varepsilon)$ will also serve as covering $I = [a, b]$.

i.e. $[a, b] \subset \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right)$

By definition $\textcircled{1} \Rightarrow m^*(I) < l\left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right) = b - a + \varepsilon$
 $\Rightarrow m^*(I) \leq b - a$, $\varepsilon > 0$ and $\varepsilon \rightarrow 0$ arbitrary.
 $\hookrightarrow \textcircled{2}$

For proving: $m^*(I) \geq b - a$, let $\{I_k\}_{k=1}^{\infty}$ be a countable family of open intervals covering $I = [a, b]$

$\because [a, b]$ is bounded and closed $\downarrow$ i.e. $[a, b] \subseteq \bigcup_{k=1}^{\infty} I_k$

$\Rightarrow$ By Heine Borel; there exist a finite sub-cover (collection) $\{I_j\}_{j=1}^n \Rightarrow [a, b] \subseteq \bigcup_{j=1}^n I_j$

$$\Rightarrow l[a, b] \leq \sum_{j=1}^n l(I_j)$$

$$\Rightarrow \sum_{j=1}^n l(I_j) \geq b - a \longrightarrow \textcircled{3}$$

i.e. true for all families of sequence of open intervals covering $I = [a, b]$.

$$\Rightarrow \sum_{j=1}^n l(I_j) \leq \sum_{k=1}^{\infty} l(I_k)$$

Taking Infimum in (3)

$$\inf \left\{ \sum_{j=1}^n l(I_j) : I \subseteq \bigcup_{j=1}^n I_j, \text{ whenever } \{I_j\}_{j=1}^n \text{ is a covering of } I \right\} \geq b-a$$

$$m^*(I) \geq b-a \quad (4)$$

$$\text{From (2) and (4)} \Rightarrow \boxed{m^*([a, b]) = b-a} \rightarrow (5)$$

$$\text{Case II: If } I = (a, b) \Rightarrow (a, b) \subset [a, b]$$

$$\Rightarrow m^*(a, b) < m^*[a, b] \text{ by monotonicity of } m^* \text{ for } \varepsilon > 0; \left[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2} \right] \subset (a, b)$$

$$\Rightarrow m^*\left[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}\right] < m^*(a, b)$$

$$\Rightarrow (b-a) - \varepsilon \leq m^*(a, b)$$

$$\Rightarrow b-a \leq m^*(a, b) \rightarrow (7)$$

$$\text{From (5), (6) and (7)} \Rightarrow \boxed{m^*(a, b) = b-a} \rightarrow (8)$$

$$\text{Case III: If } I = (a, b] \text{ or } [a, b)$$

$$\Rightarrow (a, b) \subset I \subset [a, b]$$

$$\Rightarrow m^*(a, b) \leq m^*(I) \leq m^*[a, b]$$

$$\Rightarrow b-a \leq m^*(I) \leq b-a \quad \text{From (5) and (8)}$$

$$\Rightarrow m^*(I) = b-a$$

$$\Rightarrow \boxed{m^*(a, b] = m^*[a, b) = b-a}$$

Topic #34

Determining Uncountability by Lebesgue Outer Measure:

Theorem: Intervals in " $\mathbb{R}$ " are uncountable.

proof: Let $[a, b]$ be a interval and it is countable.

$$\therefore m^*[a, b] = 0 \quad \because \text{it is countable}$$

But $m^*[a, b] = l[a, b] = b - a \neq 0$, if $a \neq b$
i.e. contradiction.

$\Rightarrow [a, b]$ in $\mathbb{R}$ is uncountable.

Topic #35

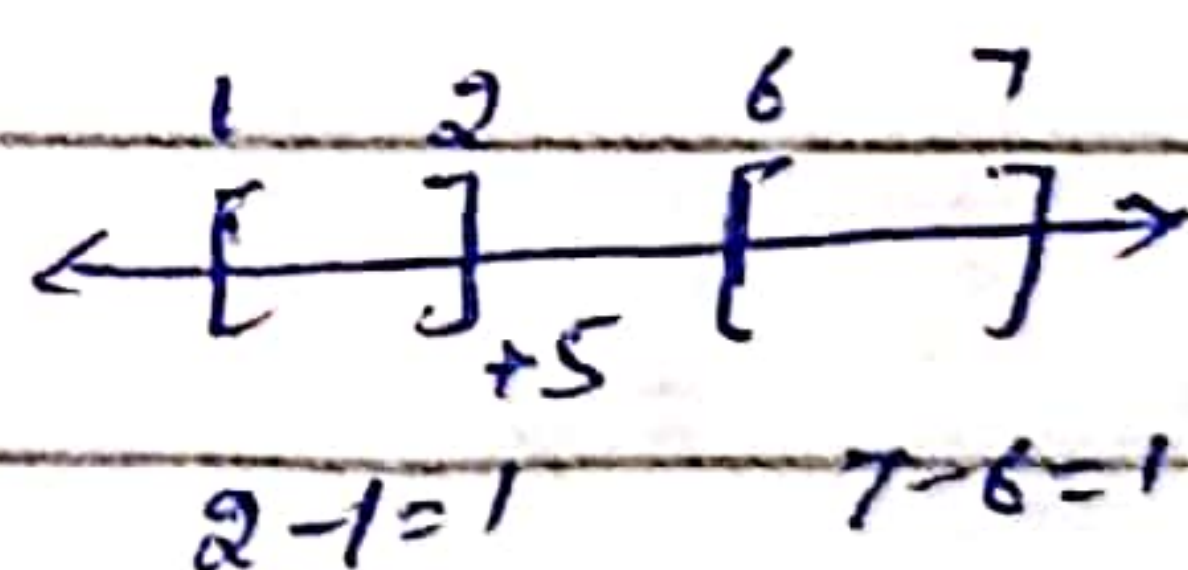
Translation Invariance of Lebesgue Outer Measure:

Theorem: Show that m^* is translation invariant.

proof: Let $\phi \neq E \subseteq \mathbb{R}$ and

$$x \in E \Rightarrow p + E = \{p + x, x \in E\},$$

i.e. in translation of " E " by " p " units.



By definition:

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n, \text{ where } \{I_n\}_{n=1}^{\infty} \text{ is sequence of intervals covering } E \right\}$$

$$\text{let } I_n = (a_n, b_n) \Rightarrow l(I_n) = b_n - a_n \quad \text{--- (1)}$$

$$\text{if } E \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow p + E \subseteq \bigcup_{n=1}^{\infty} J_n, \text{ such that}$$

$$J_n = (p + a_n, p + b_n) \Rightarrow l(J_n) = b_n - a_n = l(I_n)$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} l(J_n) \quad \text{--- (2)}$$

From ① and ②

$$\Rightarrow m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(J_n) : p+E \subseteq \bigcup_{n=1}^{\infty} J_n, \text{ where } \{J_n\}_{n=1}^{\infty} \text{ sequence of open intervals covering } p+E \right\} = m^*(E+p).$$

Topic #36

Least Length of Open Covering of All Rationals in an Interval:

Theorem: If $E = \{p \in \mathbb{Q} : 0 < p < 1\}$ and $\{I_n\}_{n=1}^k$ is the finite sequence of open intervals such that $A \subseteq \bigcup_{n=1}^k I_n$, then $\sum_{n=1}^k l(I_n) \geq 1$

proof:

$$\begin{aligned} \because A \subseteq \bigcup_{n=1}^k I_n &\Rightarrow (0,1) \subseteq \bigcup_{n=1}^k I_n \\ \Rightarrow m^*(0,1) &\leq m^*\left(\bigcup_{n=1}^k I_n\right) \quad \text{by monotonicity of } m^* \\ &\leq \sum_{n=1}^k m^*(I_n) \quad \because m^* \text{ is sub-additive} \\ m^*(0,1) &\leq \sum_{n=1}^k l(I_n) \rightarrow \textcircled{1} \end{aligned}$$

$$\text{But } m^*(0,1) = l(0,1) = 1 - 0 = 1$$

$$\text{From } \textcircled{1} \Rightarrow 1 \leq \sum_{n=1}^k l(I_n)$$

$$\Rightarrow \sum_{n=1}^k l(I_n) \geq 1$$

Topic # 37

Lebesgue Measurable Sets:

A $\phi \neq E \subseteq \mathbb{R}$ is said to be Lebesgue measurable or simply measurable if $\forall A \subseteq \mathbb{R}$, we have, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \rightarrow (1)$

Note: $A = A \cap \mathbb{R} = A \cap (E \cup E^c)$

$$A = (A \cap E) \cup (A \cap E^c)$$

$$\Rightarrow m^*(A) = m^*[(A \cap E) \cup (A \cap E^c)]$$

$$\Rightarrow m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c) \rightarrow (2)$$

In order to prove (1); we will go for:

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \rightarrow (3)$$

Topic # 38

Lebesgue Measurability of Complement of a set:

Theorem: A set "E" is measurable if and only if its complement "E^c" is measurable.

Proof: $\because E \subseteq \mathbb{R}$ is measurable if $\forall A \subseteq \mathbb{R}$

$$\Leftrightarrow m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

$$\Leftrightarrow m^*(A) = m^*(A \cap E^c) + \cancel{m^*(A \cap E)} m^*(A \cap E^c)$$

$$\Leftrightarrow \cancel{m^*(A \cap E)} E^c \text{ is measurable.}$$

Topic # 39

Theorem: " ϕ " and $\mathbb{R}$ are measurable.

$$\text{Proof: } m^*(A) = m^*(A) + m^*(\phi) \quad \left| \quad \phi^c = \mathbb{R} \right.$$

$$\because m^*(\phi) = 0 \quad \left| \quad \mathbb{R}^c = \phi \right.$$

$$A = A \cap R$$

$$\phi = A \cap \phi$$

$$\Rightarrow m^*(A) = m^*(A \cap R) + m^*(A \cap \phi)$$

$$\Rightarrow m^*(A) = m^*(A \cap R) + m^*(A \cap R^c)$$

$\Rightarrow R$ is measurable.

$\Rightarrow R^c = \phi$ is also measurable.

$\therefore$ {From previous theorem

Topic #40

Theorem: If $m^*(E) = 0$, then $E \subseteq R$ is measurable.

proof: For any $\phi \neq A \subseteq R$; $E \cap A \subseteq E$

$$\Rightarrow m^*(A \cap E) \leq m^*(E) \quad \because m^* \text{ is monotonic.}$$

$$\Rightarrow m^*(A \cap E) \leq 0 \rightarrow \textcircled{1} \quad \because \text{given}$$

$$\text{But } m^*(A \cap E) \geq 0 \rightarrow \textcircled{2} \quad \text{By definition.}$$

$$\therefore \textcircled{1} \text{ and } \textcircled{2} \Rightarrow m^*(A \cap E) = 0 \text{ or } 0 = m^*(A \cap E)$$

Similarly: $A \cap E^c \subseteq A$

$$\Rightarrow m^*(A \cap E^c) \leq m^*(A)$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E^c) \rightarrow \textcircled{4}$$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

$$m^*(A) \geq \quad \quad \quad$$

$\therefore$ From $\textcircled{3}$ and $\textcircled{4} \Rightarrow$

$$m^*(A) + 0 \geq m^*(A \cap E) + m^*(A \cap E^c)$$

$\Rightarrow$ "E" is measurable.

Topic #41

Theorem: Union and Intersection of two measurable sets is measurable.

proof: $\forall A \subseteq R, E \rightarrow \text{measurable}$

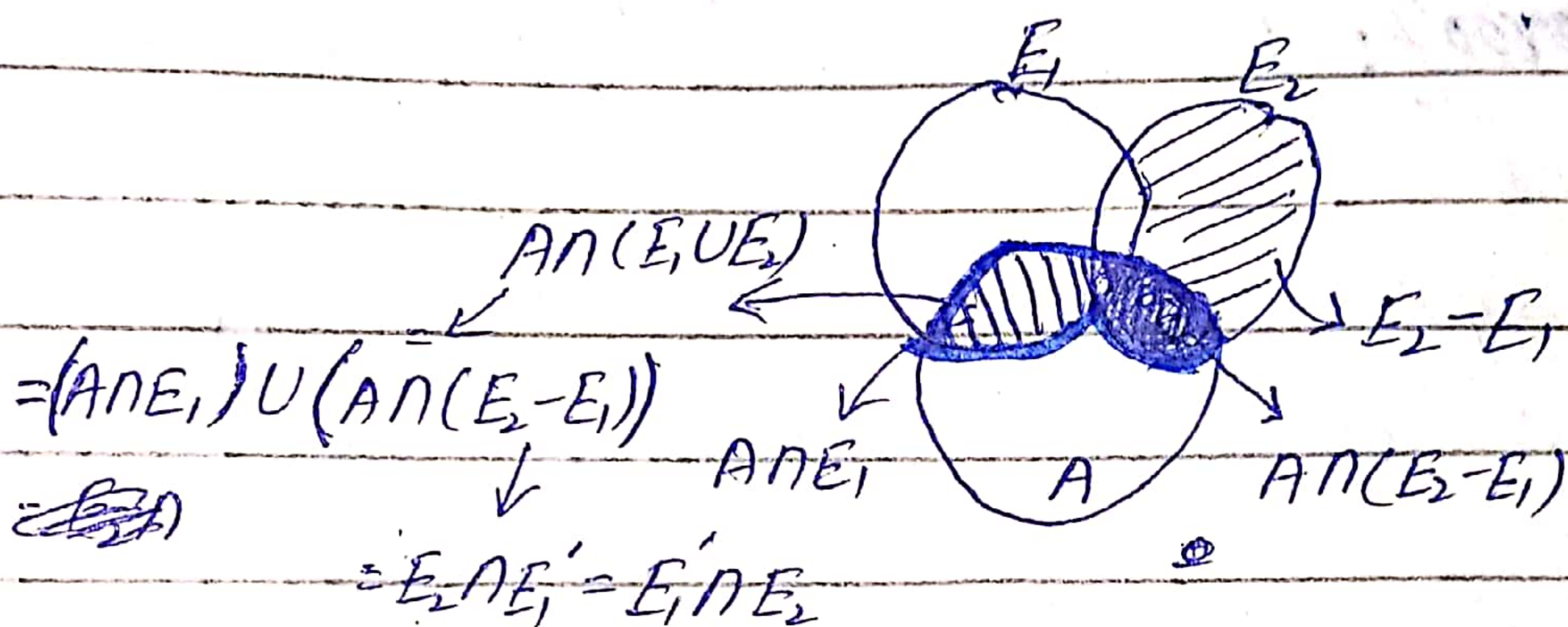
$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

let E_1, E_2 are two measurable sets.

To prove $E_1 \cup E_2$ is measurable:

$$m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c] \quad \rightarrow (1)$$

$$R.H.S = m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$



$$\Rightarrow R.H.S = m^*[(A \cap E_1) \cup (A \cap (E_2 - E_1))] + m^*[A \cap (E_1 \cup E_2)^c]$$

By sub-additivity of m^* .

$$\Rightarrow R.H.S \leq m^*(A \cap E_1) + m^*(A \cap (E_2 - E_1)) + m^*(A \cap (E_1 \cup E_2)^c)$$

$\therefore$ " E_2 " is measurable and using test set $(A \cap E_1')$

$$\Rightarrow m^*(A \cap E_1') = m^*(A \cap E_1' \cap E_2) + m^*(A \cap E_1' \cap E_2')$$

$$\therefore R.H.S \leq m^*(A \cap E_1) + m^*(A \cap E_1')$$

Since again " E_1 " is measurable.

$$R.H.S \leq m^*(A)$$

$$\Rightarrow m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$

$\therefore$ E_1 and E_2 are measurable.

$\Rightarrow$ E_1' and E_2' are measurable.

$\Rightarrow$ $(E_1' \cup E_2')$ is also measurable.

$\Rightarrow (E_1 \cup E_2)' = E_1 \cap E_2$ is measurable.

Topic # 42

Theorem: For any $\phi \neq A \subseteq \mathbb{R}$ and $\{E_i\}_{i=1}^n$ be a family of pairwise disjoint measurable sets, then

$$m^*[A \cap (\bigcup_{i=1}^n E_i)] = \sum_{i=1}^n m^*(A \cap E_i) \quad \text{--- (A)}$$

proof: We will prove this by mathematical induction. For $n=1$, result is self-evident.

Suppose (A) is true for $n=k-1$;

$$\therefore \text{(A)} \Rightarrow m^*[A \cap (\bigcup_{i=1}^{k-1} E_i)] = \sum_{i=1}^{k-1} m^*(A \cap E_i) \quad \text{--- (B)}$$

$$\text{Consider: } A \cap (\bigcup_{i=1}^k E_i) \cap E_k = A \cap [(\bigcup_{i=1}^k E_i) \cap E_k]$$

$$= A \cap [(E_1 \cup E_2 \cup \dots \cup E_k) \cap E_k]$$

$$A \cap (\bigcup_{i=1}^k E_i) \cap E_k = A \cap \underbrace{(E_1 \cap E_k)}_{=\phi} \cup \underbrace{(E_2 \cap E_k)}_{=\phi} \cup \dots \cup \underbrace{(E_k \cap E_k)}_{=E_k}$$

$$\because E_i \cap E_j = \phi \quad \forall i \neq j$$

$$\Rightarrow A \cap (\bigcup_{i=1}^k E_i) \cap E_k = A \cap E_k \rightarrow \text{--- (1)}$$

Now

$$A \cap (\bigcup_{i=1}^k E_i) \cap E_k' = A \cap [(\bigcup_{i=1}^k E_i) \cap E_k']$$

$$= A \cap [(E_1 \cup E_2 \cup \dots \cup E_k) \cap E_k']$$

$$= A \cap [(\underbrace{E_1 \cap E_k'}_{=\phi}) \cup (\underbrace{E_2 \cap E_k'}_{=\phi}) \cup \dots \cup (\underbrace{E_k \cap E_k'}_{=\phi})]$$

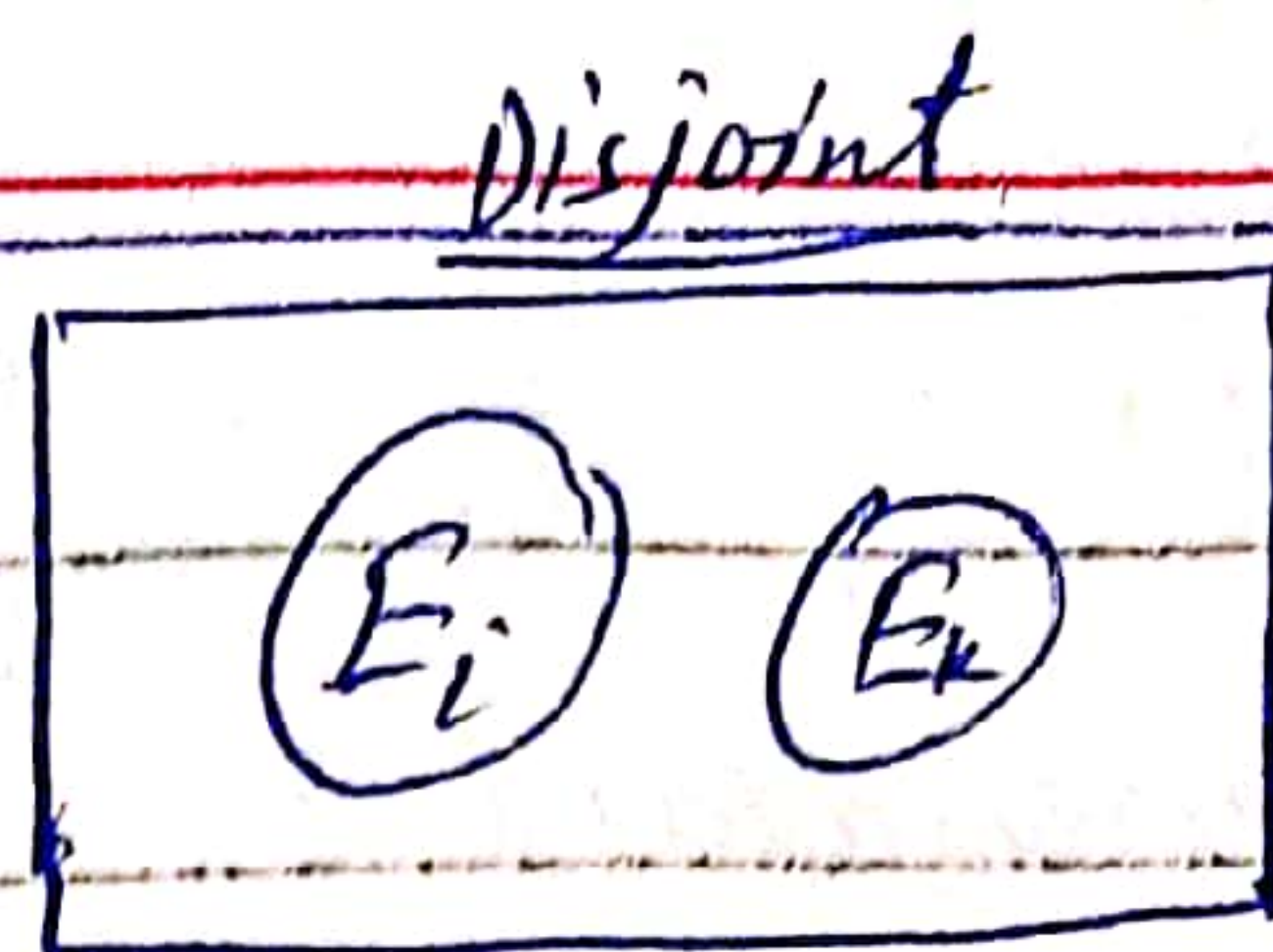
$$\downarrow$$

$$\phi$$

$$\Rightarrow A \cap \left(\bigcup_{i=1}^k E_i \right) \cap E'_k$$

$$= A \cap (E_1 \cup E_2 \cup \dots \cup E_{k-1})$$

$$\Rightarrow A \cap \left(\bigcup_{i=1}^k E_i \right) \cap E'_k = A \cap \left(\bigcup_{i=1}^{k-1} E_i \right) \rightarrow (2)$$



$$E_i \subseteq E'_k$$

$$\Rightarrow E_i \cap E'_k = E_i$$

Taking " $A \cap \left(\bigcup_{i=1}^k E_i \right)$ " as test set and using Lebesgue measurability of " E_k ";

$$m^* \left[A \cap \left(\bigcup_{i=1}^k E_i \right) \right] = m^* \left[A \cap \left(\bigcup_{i=1}^k E_i \right) \cap E_k \right] + m^* \left[A \cap \left(\bigcup_{i=1}^k E_i \right) \cap E'_k \right] \text{ By def.}$$

$$m^* \left[A \cap \left(\bigcup_{i=1}^k E_i \right) \right] = m^* (A \cap E_k) + m^* \left[A \cap \left(\bigcup_{i=1}^{k-1} E_i \right) \right]$$

$$= m^* (A \cap E_k) + \sum_{i=1}^{k-1} m^* (A \cap E_i)$$

$$m^* \left[A \cap \left(\bigcup_{i=1}^k E_i \right) \right] = \sum_{i=1}^k m^* (A \cap E_i)$$

$\therefore$ Truth of $n \leq k-1$ also implies its truth for $n \leq k$. This proves the given theorem.

Topic #43

Theorem: The class " M " of all measurable sets is a σ -algebra.

Proof: (i) If " E " is measurable
 $\Rightarrow E'$ is also measurable
 $\Rightarrow E \in "M"$

- $\phi = X; \mathcal{A} \subseteq P(X)$
 $\downarrow$
 subsets of " X "
- (i) $\forall A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$
 - (ii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$
 - (iii) $\{E_i\}_{i=1}^{\infty} \rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$

(ii) If E_1, E_2 are measurable $\Rightarrow E_1 \cup E_2$ is also measurable.

$$\Rightarrow E_1 \cup E_2 \in M$$

(iii) Let $\{E_i\}_{i=1}^{\infty}$ be the sequence of measurable sets in " M ", Then we will show

$$\bigcup_{i=1}^{\infty} E_i = E \in M$$

Constructing: $F_1 = E_1$, $F_2 = E_2 - E_1 \Rightarrow F_2 \cap F_1 = \phi$

$$F_3 = E_3 - (E_1 \cup E_2) \Rightarrow F_3 \cap F_2 = \phi$$

$\vdots$

$$F_n = E_n - (E_1 \cup E_2 \cup \dots \cup E_{n-1}) = E_n - \left(\bigcup_{i=1}^{n-1} E_i \right)$$

$\Rightarrow \{F_i\}_{i=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets.

$$\Rightarrow \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i \longrightarrow \textcircled{1}$$

Let $H_n = \bigcup_{i=1}^n F_i$ - i.e. a measurable as well being finite union of measurable sets.

$$\Rightarrow H_n = \bigcup_{i=1}^n F_i \subseteq \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i = E$$

$$\Rightarrow H_n' \supseteq E' \Rightarrow \forall A \subseteq R, A \cap H_n' \supseteq A \cap E'$$

$$\Rightarrow m^*(A \cap H_n') \geq m^*(A \cap E') \longrightarrow \textcircled{2}$$

$\because$ " H_n " is measurable and taking

$\phi \neq A \subseteq R$ as test set:

$$\Rightarrow m^*(A) = m^*(A \cap H_n) + m^*(A \cap H_n') \quad \text{By defn.}$$

$\left| \begin{array}{l} A \subseteq B \\ A' \supseteq B' \end{array} \right.$

$$\Rightarrow m^*(A) \geq m^*(A \cap H_n) + m^*(A \cap E')$$

$$m^*(A) \geq m^*[A \cap (\bigcup_{i=1}^n F_i)] + m^*(A \cap E')$$

$$m^*(A) \geq \sum_{i=1}^n m^*(A \cap F_i) + m^*(A \cap E')$$

$\therefore$ L.H.S is independent of "n", so for $n \rightarrow \infty$

$$\therefore m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap F_i) + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*[A \cap (\bigcup_{i=1}^{\infty} F_i)] + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*[A \cap (\bigcup_{i=1}^{\infty} E_i)] + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E')$$

$\Rightarrow$ "E" is also measurable.

$$\Rightarrow E \in \mathcal{M}$$

$\Rightarrow$ "M" is a σ -algebra.

Topic #44

Theorem: If "F" is measurable set and $m^*(F \Delta G) = 0$, then "G" is measurable set.

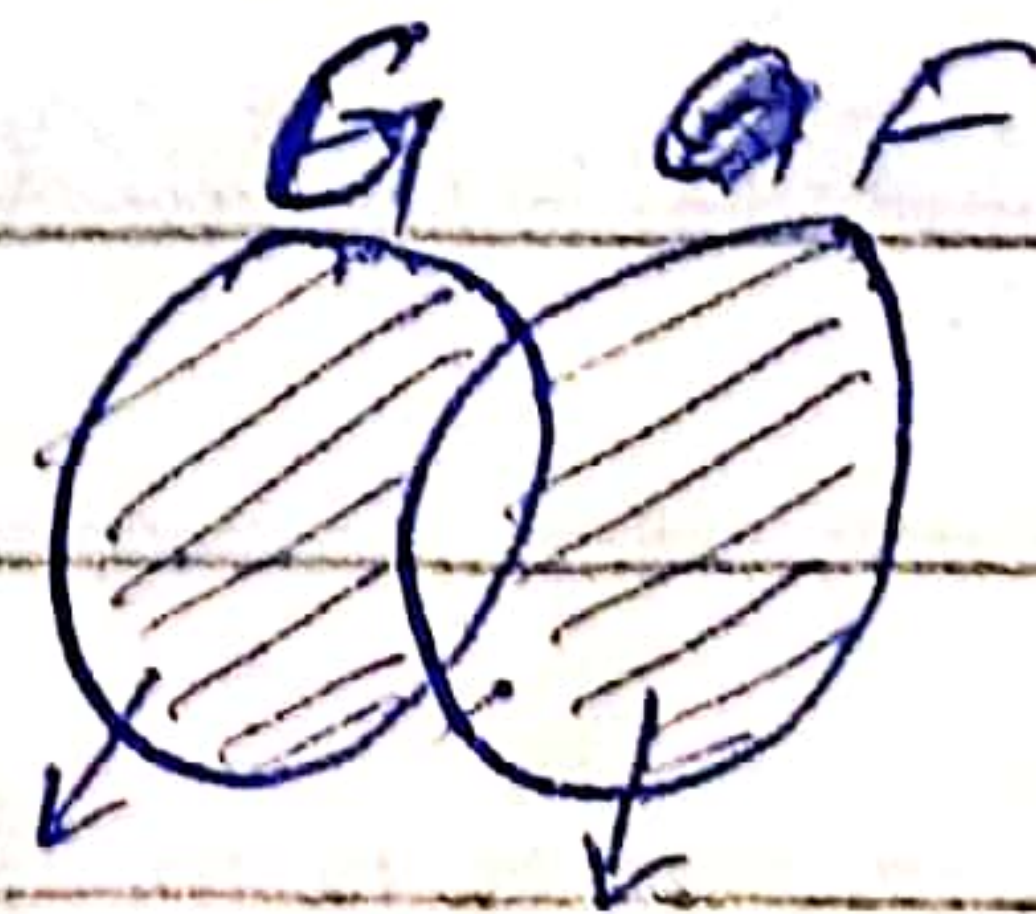
proof: $G - F \subseteq (F \Delta G)$

$$\Rightarrow m^*(G - F) \leq m^*(F \Delta G) = 0$$

$$\Rightarrow m^*(G - F) = 0$$

$\therefore m^*$ is non-negative.

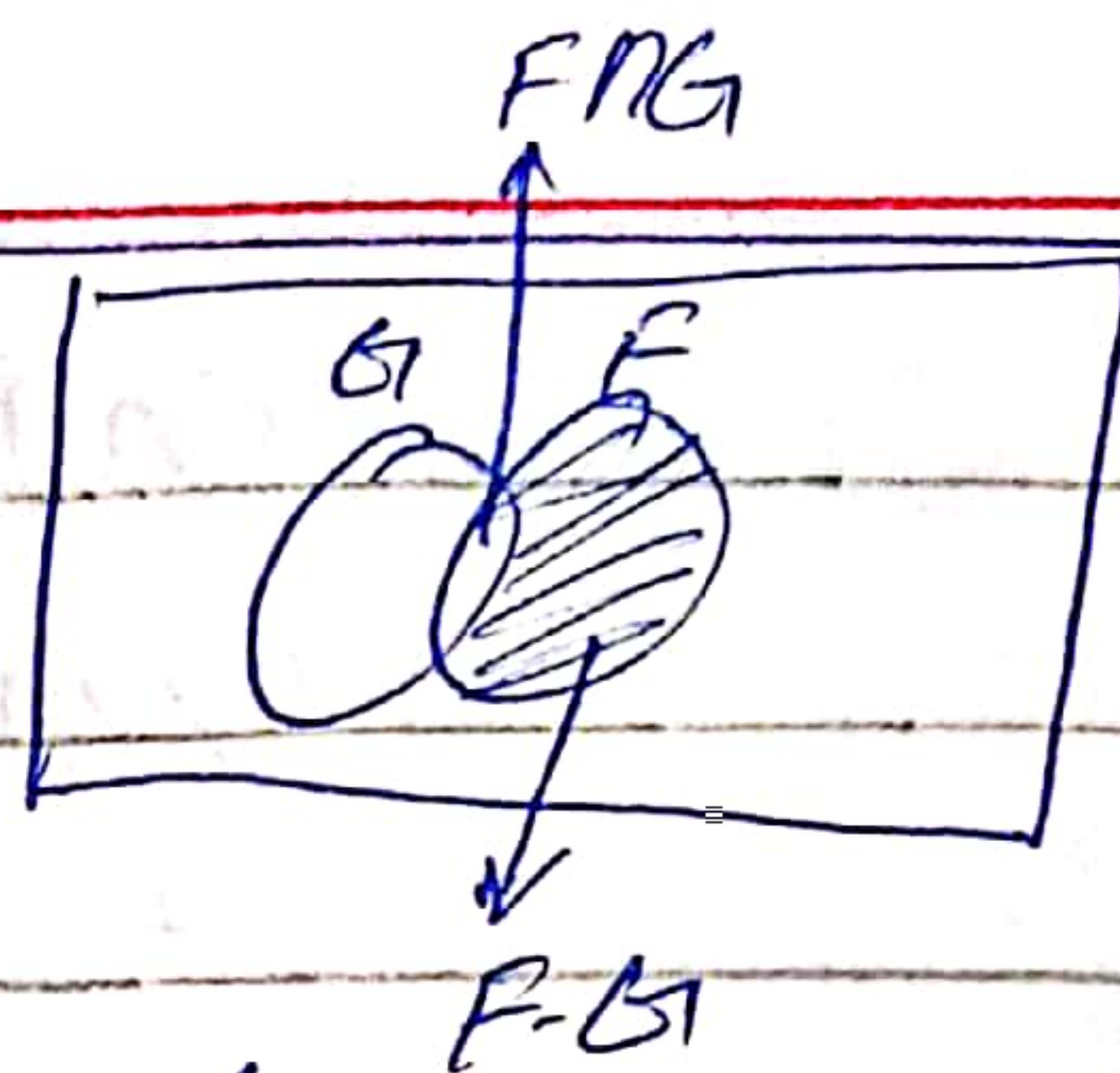
$$F \Delta G = (G - F) \cup (F - G)$$



$\Rightarrow (G - F)$ is measurable.

Similarly, $(F - G)$ is also measurable.

$\therefore F \cap G = F \cap (F - G)'$
 is measurable being
 intersection of two measurable sets.



$\therefore G = (F \cap G) \cup (G - F)$ is also measurable.

Topic # 45

G_δ -Delta and F_σ -Sigma Sets:

G_δ -Set

A $\phi \neq A \subseteq \mathbb{R}$ is said to be G_δ -set if it can be written as countable intersection of open sets.

For Example:

$$\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = (a, b)$$

(a, b) is trivially G_δ -set.

Note: G_δ -set is measurable being the countable intersection of measurable sets.

F_σ -set

A $\phi \neq A \subseteq \mathbb{R}$ is said to be F_σ -set if it can be written as countable union of closed sets.

For Example:

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

(a, b) is also F_σ -set.

Note: F_σ -set is measurable being the countable ~~intersection~~ union of measurable sets.

Note: $G_\delta' = F_\sigma$ and $F_\sigma' = G_\delta$

Topic #46

Theorem: Given any $\phi \neq A \subseteq \mathbb{R}$ and $\varepsilon > 0$, there is a G_δ set G with $A \subseteq G$ and $m^*(A) = m^*(G)$.

Proof: $\because \forall A \subseteq \mathbb{R}$ and for any $\varepsilon > 0$, there exist an open set say " U " such that $A \subseteq U$ and $m^*(U) < m^*(A) + \varepsilon$

$$\begin{array}{|l} A \subseteq B \\ \Rightarrow m^*(A) \leq m^*(B) \\ \hline A \subseteq U \\ \Rightarrow m^*(A) \leq m^*(U) \\ \hline m^*(A) + \varepsilon > m^*(U) \\ \Rightarrow m^*(U) < m^*(A) + \varepsilon \\ \text{where } \varepsilon > 0 \end{array}$$

In particular for $\varepsilon = \frac{1}{n}$, let " U_n " an open set, such that $A \subseteq U_n$

$$\Rightarrow m^*(A) \leq m^*(U_n) \rightarrow \textcircled{1}$$

$$\text{and } m^*(U_n) < m^*(A) + \frac{1}{n} \rightarrow \textcircled{2}$$

$$\text{Since } A \subseteq U_n \Rightarrow A \subseteq \bigcap_{n=1}^{\infty} U_n = G \text{ (say)} \rightarrow \textcircled{3}$$

i.e. G_δ -set being intersection of measurable sets. (open sets)

$$\Rightarrow m^*(G) = m^*\left(\bigcap_{n=1}^{\infty} U_n\right) \leq m^*(U_n)$$

$$\Rightarrow m^*(G) < m^*(A) + \frac{1}{n} \text{ (From } \textcircled{2})$$

$$\text{As for } n \rightarrow \infty \Rightarrow \varepsilon = \frac{1}{n} \rightarrow 0$$

$$\therefore m^*(G) \leq m^*(A) \rightarrow \textcircled{4}$$

$$\text{From } \textcircled{3} \Rightarrow A \subseteq G \Rightarrow m^*(A) \leq m^*(G) \rightarrow \textcircled{5}$$

$$\text{e. From } \textcircled{4} \text{ and } \textcircled{5} \Rightarrow m^*(A) = m^*(G)$$

$$\begin{array}{|l} A \cap B \subseteq A \\ m^*(A \cap B) \leq m^*(A) \end{array}$$



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