

MTH645

Fuzzy Logic and Applications

Written By

Mahar Afaq Saifdar Muhammadi

(MSc Mathematics and MA Islamic Studies)

Lecture #23

Example: $x \Delta y = xy$, $f(x) = e^{-\left(\frac{1-x}{x}\right)}$, $x \Delta_f y = ?$

Sol: $x \Delta_f y = f^{-1}((f(x) \Delta f(y)) \vee a)$

let $y = e^{-\left(\frac{1-x}{x}\right)}$

$$\ln y = \ln e^{-\left(\frac{1-x}{x}\right)} = -\left(\frac{1-x}{x}\right) \cdot \ln e \quad \because \ln e = 1$$

$$\ln y = -\left(\frac{1-x}{x}\right)$$

$$x \cdot \ln y = x - 1$$

$$1 = x - x \ln y = x(1 - \ln y)$$

$$\Rightarrow x = \frac{1}{1 - \ln y} = f^{-1}(y)$$

$$\Rightarrow f^{-1}(x) = \frac{1}{1 - \ln x}$$

$$f(0) = e^{-\left(\frac{1-0}{0}\right)} = e^{-\infty} = \frac{1}{e^{\infty}}$$

$$f(0) = \frac{1}{\infty} = 0$$

$$\Rightarrow a = 0$$

$$x \Delta_f y = f^{-1}((f(x) \Delta f(y)) \vee a)$$

$$= f^{-1}((f(x) \cdot f(y)) \vee a)$$

$$= f^{-1}((f(x) \cdot f(y)) \vee 0)$$

$$= f^{-1}(f(x) \cdot f(y))$$

$$x \Delta_f y = f^{-1} \left[e^{-\left(\frac{1-x}{x}\right)} \cdot e^{-\left(\frac{1-y}{y}\right)} \right]$$

$$= f^{-1} \left[e^{\left[-\left(\frac{1-x}{x}\right) - \left(\frac{1-y}{y}\right)\right]} \right]$$

$$= \frac{1}{1 - \ln \left[e^{\left[-\left(\frac{1-x}{x}\right) - \left(\frac{1-y}{y}\right)\right]} \right]}$$

$$= \frac{1}{1 - \left[-\left(\frac{1-x}{x}\right) - \left(\frac{1-y}{y}\right) \right]}$$

$$= \frac{1}{1 + \frac{1-x}{x} + \frac{1-y}{y}} = \frac{xy}{xy + y(1-x) + x(1-y)}$$

$$= \frac{xy}{xy + y - xy + x - xy}$$

$$x \Delta_f y = \frac{xy}{x+y-xy} = \Delta_4$$

Lecture #24

Example: $f(x) = e^{x-1}$, $x \Delta y = xy$, $x \Delta_f y = ?$

Sol: let $y = e^{x-1}$

$$\ln y = x-1$$

$$x = 1 + \ln y = f^{-1}(y)$$

$$\Rightarrow f^{-1}(x) = 1 + \ln x$$

$$f(0) = e^{0-1} = e^{-1}$$

$$a = f(0) = \frac{1}{e}$$

Now $x \Delta_f y = f^{-1}((f(x) \Delta f(y)) \vee a)$

$$= f^{-1}((f(x) \Delta f(y)) \vee \frac{1}{e})$$

$$= f^{-1}((e^{x-1} \cdot e^{y-1}) \vee \frac{1}{e})$$

$$x \Delta_f y = f^{-1} \left(e^{x+y-2} \vee \frac{1}{e} \right)$$

$$= f^{-1} \left(e^{x+y-2} \vee e^{-1} \right)$$

$$= 1 + \ln(e^{x+y-2} \vee e^{-1})$$

$$= 1 + (\ln e^{x+y-2} \vee \ln e^{-1})$$

$$= 1 + (x+y-2 \vee -1)$$

$$= 1 + x + y - 2 \vee 1 - 1$$

$$x \Delta_f y = x + y - 1 \vee 0 = \Delta,$$

*

Example: $f(x) = \frac{x}{2-x}$, $x \Delta y = xy$, $x \Delta_f y = ?$

Sol: let $y = \frac{x}{2-x}$

$$y(2-x) = x$$

$$2y - xy = x$$

$$2y = x + xy$$

$$2y = x(1+y)$$

$$\Rightarrow x = \frac{2y}{1+y} = f^{-1}(y)$$

$$\Rightarrow f^{-1}(x) = \frac{2x}{1+x}$$

$$f(0) = \frac{0}{2-0} = 0$$

$$\Rightarrow a = 0$$

$$x \Delta_f y = f^{-1}((f(x) \Delta f(y)) \vee a)$$

$$= f^{-1}(f(x) \cdot f(y) \vee 0)$$

$$= f^{-1}(f(x) \cdot f(y))$$

$$= f^{-1} \left(\frac{x}{2-x} \cdot \frac{y}{2-y} \right)$$

$$= f^{-1} \left(\frac{xy}{(2-x)(2-y)} \right)$$

$$= f^{-1} \left(\frac{xy}{4-2x-2y+xy} \right)$$

$$= \frac{2 \left(\frac{xy}{4-2x-2y+xy} \right)}{1 + \frac{xy}{4-2x-2y+xy}}$$

$$= \frac{2xy}{4-2x-2y+xy}$$

$$= \frac{\frac{2xy}{4-2x-2y+xy}}{1(4-2x-2y+xy) + xy} = \frac{2xy}{4-2x-2y+xy+xy}$$

$$= \frac{2xy}{4-2x-2y+2xy} = \frac{2xy}{2(2-x-y+xy)}$$

$$x \Delta_f y = \frac{xy}{2-x-y+xy} = \frac{xy}{2-(x+y-xy)} = \Delta_2$$

Lecture # 25

Nilpotent t-norm:

Δ is nilpotent if for $b \neq 1$, $b^{[n]} = 0$, for some +ve integer "n"; where "n" depends on b.

Strict t-norm:

Δ is strict if for $b \neq 0$, $b^{[n]} > 0$, for some +ve integer "n";

For Example $b^{[3]} = b \Delta b \Delta b$

Exercise:

Q: prove that Δ_1 is nilpotent t-norm.

Sol: $x \Delta_1 y = \max(x+y-1, 0)$

let $x \neq 1$

$$\Rightarrow x \Delta_1 x = \max(x+x-1, 0) = \max(2x-1, 0)$$

$$\text{if } x \leq \frac{1}{2} \Rightarrow x \Delta_1 x = 0$$

$$\text{if } 1 > x > \frac{1}{2} \Rightarrow x > 2x-1$$

$$\begin{aligned} x^{[3]} &= x \Delta_1 x \Delta_1 x = \max(\max(2x-1, 0) + x - 1, 0) \\ &= \max(2x-1+x-1, 0) = \max(3x-2, 0) \end{aligned}$$

$$\Rightarrow 3x-2 < 2x-1 < x$$

$$\Rightarrow 4x-3 < 3x-2 < 2x-1 < x$$

$$nx - (n-1) < (n-1)x - (n-2) < \dots < x$$

$$\Rightarrow x^{[n]} = \max(nx - (n-1), 0)$$

So we get +ve n such that $nx - (n-1)$ becomes -ve.

$$\Rightarrow x^{[n]} = 0$$

$\Rightarrow \Delta_1$ is nilpotent t-norm.

Q: prove that Δ_0 is nilpotent t-norm.

$$\text{Sol: } x \Delta_0 y = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{let } \max(x, y) = 1$$

$$\Rightarrow x \neq 1 \Rightarrow \max(x, x) \neq 1$$

$\exists$ there exists

$$\Rightarrow x \Delta x = 0$$

$$\Rightarrow x^{[2]} = 0$$

$\Rightarrow \Delta$ is nilpotent t-norm.

Zero Divisor:

A fuzzy number $a \in (0, 1)$ is called zero divisor of t-norm Δ , if there exists $b \in (0, 1)$ such that $a \Delta b = 0$.

Theorem: If Δ is strictly monotone, then it has no zero divisor.

proof: let Δ is strictly monotone, we have to show that Δ has no zero divisor.

If $x > 0$ and $y < z$

$$\Rightarrow x \Delta y < x \Delta z$$

let $a \in (0, 1)$ is zero divisor of Δ

If there exists $b \in (0, 1)$ such $a \Delta b = 0$

$$\Rightarrow a \Delta b = 0, \quad a, b \in (0, 1)$$

Take $\frac{b}{2} < b$, $a \leq a$

$$a \Delta \frac{b}{2} \leq a \Delta b = 0$$

$$a \Delta \frac{b}{2} \leq 0$$

$$a \Delta \frac{b}{2} = 0 \quad (\text{because } \text{two fuzzy numbers cannot be } -ve)$$

which is contradiction to strictly monotone.

$\Rightarrow \Delta$ has no zero divisor.

Lecture # 26

Strictly monotone t-norm:

A t-norm Δ is called strictly monotone t-norm if $x \Delta y < x \Delta z$ whenever $x > 0$ and $y < z$

Cancellation law in t-norm:

- ① If $x \Delta y = x \Delta z \Rightarrow x = 0$ or $y = z$
- ② If $x \Delta y = x \Delta z$ and $x = 0 \Rightarrow y = z$

Theorem: Δ is strictly monotone t-norm if and only if it satisfies the cancellation law.

proof: let Δ is strictly monotone t-norm and we show that cancellation law holds.

Take $x \Delta y = x \Delta z \rightarrow$ ①

Suppose on contrary $x \neq 0$ and $y \neq z$

$\Rightarrow x > 0$, let $y < z$

$\Rightarrow x \Delta y < x \Delta z \rightarrow$ ②

From ① and ② $\Rightarrow x \Delta y < x \Delta z$

$\Rightarrow$ which is contradiction to cancellation law

So, If Δ is strictly monotone t-norm then

cancellation law holds.

Conversely, If it satisfies cancellation law then it is strictly monotone:

we have to show that

If $x \Delta y < x \Delta z$ whenever $x > 0$ and $y < z$.

Let Δ is not strictly monotone t-norm.

$x \Delta y \geq x \Delta z$, whenever $x > 0$ and $y < z$.

Now take $x \leq x$ and $y < z$,

$$\Rightarrow x \Delta y \leq x \Delta z \rightarrow \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2} \Rightarrow x \Delta y = x \Delta z$

which is contradiction to cancellation law.

$$\Rightarrow x \Delta y < x \Delta z, \quad x > 0 \text{ and } y < z$$

So Δ is strictly monotone.

which complete the proof.

Idempotent Element:

Let $a \in (0, 1)$ is idempotent element if $a \Delta a = a$.

For example: (i) $0 \Delta 0 = 0 \Rightarrow 0$ is idempotent element.

(ii) $1 \Delta 1 = 1 \Rightarrow 1$ is idempotent element.

Theorem: If T is strictly monotone then it has only trivial idempotent element.

Proof: Let T has non trivial idempotent element x and y .

Let $x, y \in (0, 1)$

$$x \Delta y < x \Delta z \quad ; \quad x > 0, y < z$$

Take $x \Delta x$ and $x < 1$

$$\Rightarrow x \Delta x < x \Delta 1 = x$$

$$\Rightarrow x \Delta x < x$$

which is contradiction.

So T is trivial idempotent element.

Lecture #27

Binary Fuzzy Relation:

A binary fuzzy relation is $R: U \times U \rightarrow [0, 1]$.

(i) Reflexive Relation:

A binary fuzzy relation R is reflexive if $R(x, x) = 1$.

(ii) Symmetric Relation:

A fuzzy binary relation R is symmetric if

$$R(x, y) = 1 \Rightarrow R(y, x) = 1$$

(iii) Transitive Relation:

~~Transit~~ A fuzzy binary relation R is transitive

$$\text{if } R(x, y) = 1, R(y, z) = 1 \text{ and } \Rightarrow R(x, z) = 1$$

where $x, y, z \in$ fuzzy numbers.

• **(iv)** R is equivalence relation if R is reflexive, symmetric and transitive.

(v) Antisymmetric Relation:

A fuzzy binary relation R is antisymmetric

if $R(x, y) = R(y, x) = 1 \Rightarrow x = y$.

More Generalized Definitions:

(i) Reflexive Relation:

A fuzzy binary relation R is reflexive if $R(u, u) = 1$

(ii) Symmetric Relation:

A fuzzy binary relation R is symmetric if $R(u, v) = R(v, u)$.

(iii) Transitive Relation:

A fuzzy binary relation R is transitive if $R(u, w) \geq R(u, v) \wedge R(v, w)$.

(iv) Equivalence Relation:

A fuzzy binary relation R is equivalence if R is reflexive, symmetric and transitive relation.

(v) Antisymmetric Relation:

A fuzzy binary relation R is antisymmetric if $R(u, v) > 0$ and $R(v, u) > 0 \Rightarrow u = v$.

α -cut in fuzzy binary relation:

α -cut in fuzzy binary relation is

$$R_\alpha = R^{-1}([\alpha, 1]) = \{(u, v) : R(u, v) \geq \alpha\}$$

For example: $\alpha = 0.5$, $R = \{(x_1, y_1), 0.3\}, \{(x_1, y_2), 0.7\}, \{(x_2, y_1), 0.9\}$

$$R_\alpha = R_{0.5} = \{(x_1, y_2), 0.7\}, \{(x_2, y_1), 0.9\}$$

$\subseteq$

Theorem: If R is a fuzzy on a set U , then R is an equivalence relation if and only if each α -cut, R_α is an equivalence relation on U .

Proof: Let R_α be a fuzzy equivalence relation.

Then (i) $R(u, u) = 1 \geq \alpha \Rightarrow (u, u) \in R_\alpha$, R_α is reflexive.

(ii) If $R(u, v) = 1 = R(v, u) \Rightarrow \text{~~(u, v)~~ } (u, v) \in R_\alpha$

$\Rightarrow R(u, v) = 1 \geq \alpha$ also $R(v, u) = 1 \geq \alpha$

$\Rightarrow (u, v) \in R_\alpha \Rightarrow R_\alpha$ is symmetric.

(iii) If $(u, v), (v, w) \in R_\alpha$

$\Rightarrow R(u, v) = 1 \geq \alpha$ and $R(v, w) = 1 \geq \alpha$

$\Rightarrow R(u, w) \geq R(u, v) \wedge R(v, w) = 1 \geq \alpha$

$R(u, w) \geq \alpha \Rightarrow R(u, w) \in R_\alpha \Rightarrow R_\alpha$ is transitive.

From (i), (ii) and (iii)

$\Rightarrow R_\alpha$ is an equivalence relation.

Conversely: Let R_α is equivalence relation.

Then (i) If $(u, u) \in R_\alpha \Rightarrow R(u, u) \geq \alpha$ (put $\alpha = 1$)

$\Rightarrow R(u, u) \geq 1$

A fuzzy relation is not greater than 1.

So $R(u, u) = 1 \Rightarrow R$ is reflexive.

(ii) $\because (u, v), (v, u) \in R_\alpha$

$\Rightarrow R(u, v) \geq \alpha$ and $R(v, u) \geq \alpha$, $\forall \alpha \in [0, 1]$

Take $\alpha = 1$, $R(u, v) \geq 1$ and $R(v, u) \geq 1$

A fuzzy number is not greater than 1.

So $R(u,v)=1$ and $R(v,u)=1$

$\Rightarrow R$ is symmetric.

(iii) If $(u,v), (v,w) \in R_\alpha \Rightarrow (u,w) \in R_\alpha$

$\Rightarrow R(u,v) \geq \alpha$ and $R(v,w) \geq \alpha \Rightarrow R(u,w) \geq \alpha$

$\Rightarrow R(u,v) \wedge R(v,w) \geq \alpha$

Take $\alpha=1$, $\Rightarrow R(u,v) \wedge R(v,w) \geq 1$

The fuzzy numbers are not greater than 1.

So $R(u,v) \wedge R(v,w) = 1$ and $R(u,w) = 1$

$\Rightarrow R$ is transitive.

From (i), (ii) and (iii).

$\Rightarrow R$ is equivalence relation.

Lecture # 28

Composition of Relation:

Let R and S be fuzzy relation in $u \times v$ and $v \times w$ respectively and let Δ be a t-norm. The composition $R \circ S$ of R and S with respect to Δ is the fuzzy relation on $u \times w$ with membership function

$$(R \circ S)(u,w) = \bigvee_{v \in V} [R(u,v) \Delta S(v,w)]$$

$x \Delta y = x \wedge y \Rightarrow$ max-min composition.

$x \Delta y = xy \Rightarrow$ max-product composition.

Example: Find the max-min and max-product composition of R and S .

Sol:

$$R = \begin{matrix} & y_1 & y_2 & y_3 \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0.1 & 0.0 & 0.6 \\ 0.2 & 0.3 & 1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \end{matrix} \text{ and } S = \begin{matrix} & s_1 & s_2 & s_3 \\ \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} & \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.9 & 1 & 0.3 \\ 1 & 0.5 & 0.4 \end{bmatrix} \end{matrix}$$

max-min composition:

$$R \circ S = \begin{bmatrix} 0.6 & 0.5 & 0.4 \\ 1 & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.4 \end{bmatrix}$$

$$\left. \begin{array}{l} \min(0.1, 0.2) = 0.1 \\ \min(0.0, 0.9) = 0.0 \\ \min(0.6, 1) = 0.6 \end{array} \right\} \max = 0.6$$

$$\left. \begin{array}{l} \min(0.1, 0.5) = 0.1 \\ \min(0.0, 1) = 0.0 \\ \min(0.6, 0.5) = 0.5 \end{array} \right\} \max = 0.5$$

So on,

max-product composition:

$$R \circ S = \begin{bmatrix} 0.1 & 0 & 0.6 \\ 0.2 & 0.3 & 1 \\ 0 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.9 & 1 & 0.3 \\ 1 & 0.5 & 0.4 \end{bmatrix}$$

$$R \circ S = \begin{bmatrix} 0.6 & 0.3 & 0.24 \\ 1 & 0.5 & 0.4 \\ 0.5 & 0.4 & 0.2 \end{bmatrix}$$

$$\left. \begin{array}{l} 0.1 \times 0.2 = 0.02 \\ 0 \times 0.9 = 0 \\ 0.6 \times 1 = 0.6 \end{array} \right\} \max = 0.6$$

$$\left. \begin{array}{l} 0.1 \times 0.5 = 0.05 \\ 0 \times 1 = 0 \\ 0.6 \times 0.5 = 0.3 \end{array} \right\} \max = 0.3$$

So on,

Example:

$R \circ R \leq R \iff R$ is transitive.

$$\max(\min(R(u,v), R(v,w))) \leq R(u,w)$$

Fuzzy Relation: Example:

$$A = \{x_1, x_2\}, B = \{y_1, y_2, y_3\}$$

$$A \times B = \left\{ \frac{(x_1, y_1)}{0.2}, \frac{(x_1, y_2)}{0.3}, \frac{(x_1, y_3)}{1}, \frac{(x_2, y_1)}{0}, \frac{(x_2, y_2)}{0.5}, \frac{(x_2, y_3)}{0.7} \right\}$$

$$R_1 = \begin{matrix} & y_1 & y_2 & y_3 \\ \begin{matrix} x_1 \\ x_2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{matrix} \Rightarrow R_1 = \{(x_1, y_1), (x_2, y_2)\}$$

$$\tilde{F}: U^2 \rightarrow [0, 1]$$

$$\tilde{R} = \begin{matrix} & y_1 & y_2 & y_3 \\ \begin{matrix} x_1 \\ x_2 \end{matrix} & \begin{bmatrix} 0.2 & 0.3 & 1 \\ 0 & 0.5 & 0.7 \end{bmatrix} \end{matrix}$$

$$\tilde{R} = A \times B \rightarrow [0, 1], \tilde{R} = \{(x_1, y_1), 0.2), ((x_2, y_2), 0.5)\}$$

Complete Relation of $A \times B$ if

$$\tilde{R} = \begin{matrix} & y_1 & y_2 & y_3 \\ \begin{matrix} x_1 \\ x_2 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{matrix}$$

Null Relation of $A \times B$ if

$$\tilde{R} = \begin{matrix} & y_1 & y_2 & y_3 \\ \begin{matrix} x_1 \\ x_2 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Example:

$$R = \begin{bmatrix} 0.2 & 0.3 & 0.7 & 0.9 \\ 0.5 & 0.2 & 0.3 & 0.7 \\ 0.1 & 0.3 & 0.9 & 0.8 \end{bmatrix}, S = \begin{bmatrix} 0 & 0.7 & 0.1 & 0.8 \\ 0.9 & 0.2 & 0.7 & 0.1 \\ 0 & 0.3 & 0.9 & 1 \end{bmatrix}$$

Union:

$$R \cup S = \begin{bmatrix} 0.2 & 0.7 & 0.7 & 0.9 \\ 0.7 & 0.2 & 0.7 & 0.7 \\ 0.1 & 0.3 & 0.9 & 1 \end{bmatrix} \quad R \cup S = \max(R, S)$$

Intersection:

$$R \cap S = \begin{bmatrix} 0 & 0.3 & 0.1 & 0.8 \\ 0.5 & 0.2 & 0.3 & 0.1 \\ 0 & 0.3 & 0.9 & 0.8 \end{bmatrix}$$

$$R \cap S = \min(R, S)$$

Compliment.

$$R^c = 1 - R = \begin{bmatrix} 0.8 & 0.7 & 0.3 & 0.1 \\ 0.5 & 0.8 & 0.7 & 0.3 \\ 0.9 & 0.7 & 0.9 & 0.2 \end{bmatrix}$$

$$R^c = 1 - R$$

Example:

$$\tilde{S} = \begin{bmatrix} 0 & 0.7 & 0.1 & 0.8 \\ 0.9 & 0.2 & 0.7 & 0.1 \\ 0 & 0.3 & 0.9 & 1 \end{bmatrix}$$

Find α -cut of $\tilde{S}$

let $\alpha = 0.4$

$$\tilde{S}_\alpha = \tilde{S}_{0.4} = \begin{bmatrix} 0 & 0.7 & 0 & 0.8 \\ 0.9 & 0 & 0.7 & 0 \\ 0 & 0 & 0.9 & 1 \end{bmatrix}$$

$$\tilde{S}_\alpha = \tilde{S}(u, v) \geq \alpha$$

Lecture #29

Fuzzy Implications:

A fuzzy implications in a map $\Rightarrow: [0, 1]^2 \rightarrow [0, 1]$ satisfying

$\Rightarrow$	1	0
1	1	0
0	1	1

$$(1 \Rightarrow 1) = 1$$

$$(0 \Rightarrow 1) = 1$$

$$(0 \Rightarrow 0) = 1$$

$$(1 \Rightarrow 0) = 0$$

$$(x \Rightarrow y) = x \text{ implication } y$$

R-Implication:

A R-implication is a map $\Rightarrow : [0,1]^2 \rightarrow [0,1]$ such that

$$(x \Rightarrow y) = \max \{ z \in [0,1] : x \Delta z \leq y \}$$

$$= \vee \{ z \in [0,1] : x \Delta z \leq y \}$$

$$\begin{aligned} \text{(i)} \quad (1 \Rightarrow 1) &= \max \{ z \in [0,1] : 1 \Delta z \leq 1 \} \\ &= \max \{ z \in [0,1] : z \leq 1 \} \\ &= \max \{ z \in [0,1] \} = 1 \end{aligned}$$

$$(1 \Rightarrow 1) = 1$$

$$\begin{aligned} \text{(ii)} \quad (1 \Rightarrow 0) &= \max \{ z \in [0,1] : 1 \Delta z \leq 0 \} \\ &= \max \{ z \in [0,1] : z < 0 \} \\ &= \max \{ 0 \} \end{aligned}$$

$$(1 \Rightarrow 0) = 0$$

$$\begin{aligned} \text{(iii)} \quad (0 \Rightarrow 0) &= \max \{ z \in [0,1] : 0 \Delta z \leq 0 \} \\ &= \max \{ z \in [0,1] \} \end{aligned}$$

$$(0 \Rightarrow 0) = 1$$

$$\begin{aligned} \text{(iv)} \quad (0 \Rightarrow 1) &= \max \{ z \in [0,1] : 0 \Delta z \leq 1 \} \\ &= \max \{ z \in [0,1] \} \end{aligned}$$

$$(0 \Rightarrow 1) = 1$$

$$(x \Rightarrow y) = \max \{ z \in [0,1] : z \Delta x \leq y \}$$

$$\text{If } x \leq y; z \leq 1$$

$$\Rightarrow x \Delta z \leq y \Delta 1 = y$$

$$\Rightarrow x \Delta z \leq y$$

Example 1: $x \Delta y = \min(x, y) = x \wedge y$

$$(x \Rightarrow y) = \max \{ z \in [0, 1] : \min(x, z) \leq y \}$$

$$= \begin{cases} 1 & ; x \leq y \\ y & ; x > y \end{cases}$$

Example 2: $x \Delta y = xy$

$$(x \Rightarrow y) = \max \{ z \in [0, 1] : \min(xz, y) \leq y \}$$

$$= \begin{cases} 1 & ; x \leq y \\ y/x & ; x > y \end{cases}$$

Example 3: $x \Delta y = \max(0, x+y-1)$

$$(x \Rightarrow y) = \max \{ z \in [0, 1] : x \Delta z \leq y \}$$

$$= \max \{ z \in [0, 1] : \max(0, x+z-1) \leq y \}$$

$$(x \Rightarrow y) = \begin{cases} 1 & ; x \leq y \\ y-x+1 & ; x > y \end{cases}$$

Theorem: Let Δ be archimedean t-norm and f is generator of t-norm. Then $(x \Rightarrow y) = f^{-1} \left[\frac{f(y)}{f(x)} \wedge 1 \right]$

proof: $x \Delta y = f^{-1}(f(x) \cdot f(y) \vee f(0))$

$$(x \Rightarrow y) = \max \{ z \in [0, 1] : x \Delta z \leq y \}$$

$$= \max \{ z \in [0, 1] : f^{-1}(f(x)f(z) \vee f(0)) \leq y \}$$

$$f: [0, 1] \rightarrow [a, 1] \Rightarrow 0 \leq a < 1$$

Now $f^{-1}(f(x)f(z) \vee f(0)) \leq y$

$$\Rightarrow f(x)f(z) \vee f(0) \leq f(y)$$

$$\Rightarrow f(x)f(z) \vee a \leq f(y)$$

∇ Nebla

η Eta

$$\Rightarrow f(x) f(z) \leq f(y)$$

$$\Rightarrow f(z) \leq f(y)/f(x)$$

$$\Rightarrow z \leq f^{-1}\left(\frac{f(y)}{f(x)} \wedge 1\right)$$

$$(x \Rightarrow y) = f^{-1}\left(\frac{f(y)}{f(x)} \wedge 1\right)$$

(proved)

Lecture #30

∇ -Implication:

A ∇ -implication is a map $\Rightarrow [0,1]^2 \rightarrow [0,1]$ of the form $(x \Rightarrow y) = \eta(x) \nabla y$; $\eta(x) = 1-x$.

∇ is t -conorm.

Example 1: $x \nabla y = \max(x, y) = v(x, y) = x \vee y$

Sol: $(x \Rightarrow y) = \eta(x) \nabla y$
 $= (1-x) \nabla y = \max(1-x, y)$

Example 2: $x \nabla y = x + y - xy$

Sol: $(x \Rightarrow y) = \eta(x) \nabla y$
 $= (1-x) \nabla y$
 $= (1-x) + y - (1-x)y$
 $= 1-x+y-y+xy$

$$(x \Rightarrow y) = 1-x+xy$$

Example 3: $x \nabla y = 1 \wedge (x+y)$

Sol: $(x \Rightarrow y) = \eta(x) \nabla y = (1-x) \nabla y$
 $= 1 \wedge (1-x+y) = \min(1, 1-x+y)$

$$(0 \Rightarrow 0) = 1$$

$$(1 \Rightarrow 1) = 1$$

$$(0 \Rightarrow 1) = 1$$

$$(1 \Rightarrow 0) = \eta(x) \nabla y = (1-1) \nabla 0 = 0 \nabla 0$$

$$(1 \Rightarrow 0) = 0$$

Example: $g(x) = e^{-x^2}$ $\eta(x) = \sqrt{1-x^2}$

Then $(x \Rightarrow y) = \begin{cases} 1 & ; x \leq y \\ \sqrt{1-x^2+y^2} & ; x > y \end{cases}$

Sol: $(x \Rightarrow y) = \eta(x) \nabla y$

$$= g^{-1}(g(\eta(x))g(y) \vee g(1))$$

$$= g^{-1}(g(\sqrt{1-x^2})e^{-y^2} \vee e^{-1})$$

$$= g^{-1}(e^{-(1-x^2)}e^{-y^2} \vee e^{-1})$$

$$= g^{-1}(e^{-1+x^2-y^2} \vee e^{-1})$$

$$(x \Rightarrow y) = \sqrt{\ln(e^{-1+x^2-y^2})^{-1}} \vee \sqrt{\ln(e^{-1})^{-1}}$$

$$= \sqrt{\ln e^{1-x^2+y^2}} \vee \sqrt{\ln e^1}$$

$$= \sqrt{(1-x^2+y^2)\ln e} \vee \sqrt{\ln e}$$

$$= \sqrt{1-x^2+y^2} \vee \sqrt{1}$$

$$(x \Rightarrow y) = \sqrt{1-x^2+y^2} \vee 1$$

$$y = e^{-x^2}$$

$$\ln y = \ln e^{-x^2}$$

$$\ln y = -x^2$$

$$x^2 = -\ln y$$

$$x^2 = \ln y^{-1}$$

$$g^{-1}(y) = x = \sqrt{\ln y^{-1}}$$

$$(x \Rightarrow y) = \begin{cases} 1 & ; x \leq y \\ \sqrt{1-x^2+y^2} & ; x > y \end{cases}$$

$$\therefore \boxed{\ln e = 1}$$

(proved)

Lecture #31

Distance Between Two points:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$, then the distance D between two points is calculated as:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example: $A(-2, 1)$ and $B(2, 4)$, then the D is

$$D = \sqrt{(2 - (-2))^2 + (4 - 1)^2} = \sqrt{4^2 + 3^2}$$
$$= \sqrt{16 + 9} = \sqrt{25}$$

$$D = 5$$

Distance Between Fuzzy Sets:

Let $X = \{x_1, x_2, \dots, x_n\}$ be a discourse set the fuzzy sets (FSs) A on X is represented in terms of a function $m: X \rightarrow [0, 1]$ such as

$$A = \{ \langle x_i, m_A(x_i) \rangle \mid x_i \in X \}$$

where $m_A(x_i)$ is called a membership function and always bounded in the closed interval $[0, 1]$.

Let A and B two fuzzy sets on $X = \{x_1, x_2, \dots, x_n\}$, then the most widely used distances for fuzzy sets are:

(1) Euclidean distance $d_e(A, B)$

$$d_e(A, B) = \sqrt{\sum_{i=1}^n \left(m_A(x_i) - m_B(x_i) \right)^2}$$

(2) Normalized Euclidean distance $d_e^n(A, B)$

$$d_e^n(A, B) = \frac{1}{n} \sqrt{\sum_{i=1}^n (m_A(x_i) - m_B(x_i))^2}$$

(3) Hamming distance $d_h(A, B)$

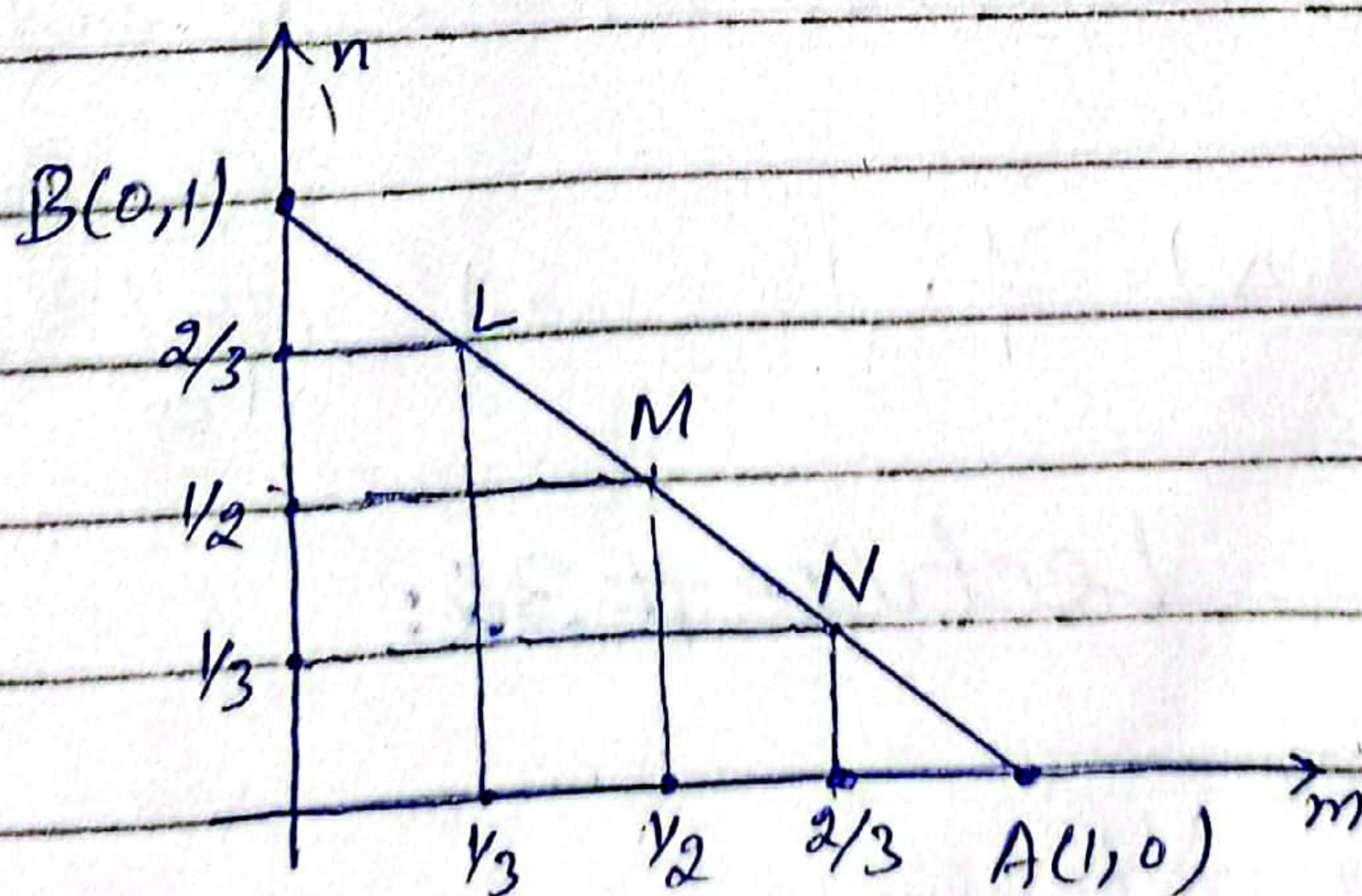
$$d_h(A, B) = \sum_{i=1}^n |m_A(x_i) - m_B(x_i)|$$

(4) Normalized Hamming distance $d_h^n(A, B)$

$$d_h^n(A, B) = \frac{1}{n} \sum_{i=1}^n |m_A(x_i) - m_B(x_i)|$$

Example 1:

Suppose that $A = (1, 0)$, $B = (0, 1)$, $L = (1, \frac{1}{3})$, $M = (1, \frac{1}{2})$, $N = (1, \frac{2}{3})$ are fuzzy sets on $X = \{1\}$ then the geometrical representation of these sets is shown in the diagram below:



(1) Therefore, the Hamming distance between the fuzzy sets L and M is:

$$d(L, M) = \left| \frac{2}{3} - \frac{1}{3} \right| = \frac{1}{3}$$

Similarly, the Hamming distance between the fuzzy sets M and N is:

$$d(M, N) = \left| \frac{1}{2} - \frac{2}{3} \right| = \left| -\frac{1}{6} \right| = \frac{1}{6}$$

(2) Now the normalized Hamming distance between the fuzzy sets M and N is

$$d^n(M, N) = \frac{1}{1} \cdot \left| \frac{1}{2} - \frac{2}{3} \right| = 1 \cdot \left| -\frac{1}{6} \right| = \frac{1}{6}$$

(3) Euclidean distance between M and N is:

$$d_e(M, N) = \sqrt{\left(\frac{1}{2} - \frac{2}{3} \right)^2} = \sqrt{\left(\frac{1}{6} \right)^2} = \frac{1}{6}$$

(4) Normalized Euclidean distance between M and N is:

$$d_e^n(M, N) = \frac{1}{1} \cdot \sqrt{\left(\frac{1}{2} - \frac{2}{3} \right)^2} = 1 \cdot \sqrt{\left(\frac{1}{6} \right)^2} = \frac{1}{6}$$

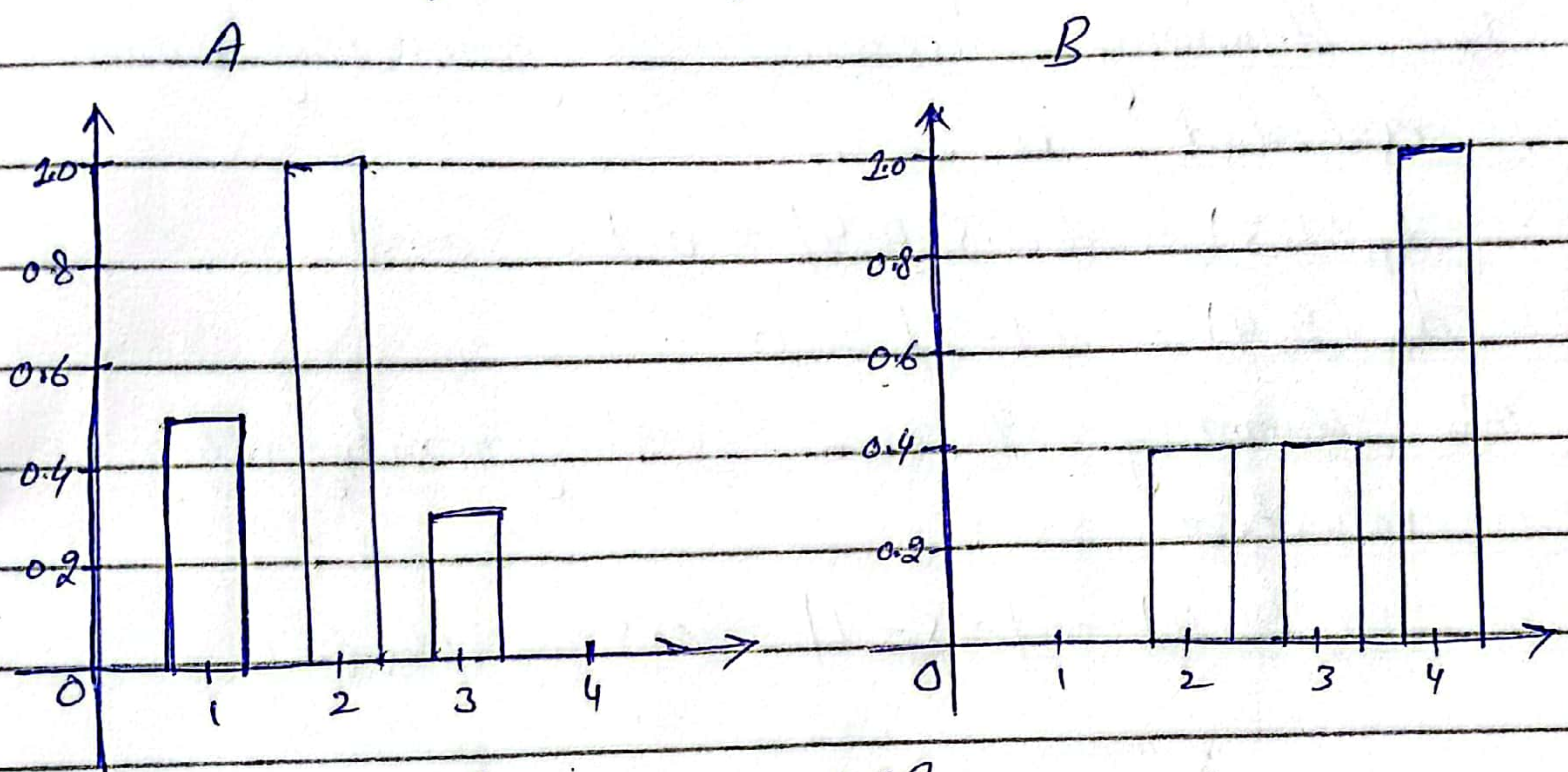
Lecture # 32:

Example 2:

Find and plot the Hamming distance $d_h(A, B)$ between two fuzzy sets A and B on

the discourse set $X = \{1, 2, 3, 4\}$, such that
 $A = \{(1, 0.5), (2, 1), (3, 0.3)\}$ and
 $B = \{(2, 0.4), (3, 0.4), (4, 1)\}$.

Figure: Fuzzy sets A and B



First we find the difference between generating values, that is:

$$|1-2| = 1, |1-3| = 2, |1-4| = 3$$

$$|2-2| = 0, |2-3| = 1, |2-4| = 2$$

$$|3-2| = 1, |3-3| = 0 \text{ and } |3-4| = 1,$$

we get the generating values $(0, 1, 2, 3)$.

Now the Hamming distance with generating value 0 is evaluated as:

$$d_h(2, 2) = |1 - 0.4| = 0.6 \text{ and}$$

$$d_h(3, 3) = |0.3 - 0.4| = 0.1$$

The Hamming distance with generating value 1 is evaluated as:

$$d_h(1,2) = |0.5 - 0.4| = 0.1$$

$$d_h(2,3) = |1 - 0.4| = 0.6 \quad \text{and}$$

$$d_h(3,4) = |0.3 - 1| = 0.7$$

The Hamming distance with generating value 2 is evaluated as:

$$d_h(1,3) = |0.5 - 0.4| = 0.1 \quad \text{and}$$

$$d_h(2,4) = |1 - 1| = 0$$

The Hamming distance with generating value 3 is evaluated as:

$$d_h(1,4) = |0.5 - 1| = 0.5, \quad \text{that is}$$

$$D = 0.6 + 0.1 + 0.1 + 0.6 + 0.7 + 0.1 + 0 + 0.5$$

$$D = 2.7$$

Now the Euclidean distance with generating value 0 is evaluated as:

$$\begin{aligned} d_e(2,2) + d_e(3,3) &= \sqrt{(1-0.4)^2 + (0.3-0.4)^2} \\ &= \sqrt{(0.6)^2 + (-0.1)^2} \\ &= 0.608 \end{aligned}$$

The Euclidean distance with generating value 1 is evaluated as:

$$d_e(1,2) + d_e(2,3) + d_e(3,4) = \sqrt{(0.5-0.4)^2 + (1-0.4)^2 + (0.3-1)^2}$$

$$= \sqrt{(0.1)^2 + (0.6)^2 + (-0.7)^2}$$

$$= 0.9273$$

The Euclidean distance with generating value 2 is evaluated as:

$$d_e(1,3) + d_e(2,4) = \sqrt{(0.5-0.4)^2 + (1-1)^2}$$

$$= \sqrt{(0.1)^2 + (0)^2}$$

$$= 0.1$$

The Euclidean distance with generating value 3 is evaluated as:

$$d_e(1,4) = \sqrt{(0.5-1)^2} = \sqrt{(-0.5)^2}$$

$$= 0.5, \text{ that is}$$

$$D = 0.608 + 0.9273 + 0.1 + 0.5$$

$$D = 2.13.$$

Normalized Hamming Distance.

$$D = \frac{1}{4} \cdot 2.7 = 0.675$$

Normalized Euclidean Distance

$$D = \frac{1}{4} \cdot 2.13 = 0.5325$$

Lecture # 33:

Distance between two fuzzy sets by using max and min operations:

$$d(A, B) = \max [\min (m_A(x_i), m_B(x_i))]$$

Consider the $X = \{1, 2, 3, 4\}$, such that

$$A = \{(1, 0.5), (2, 1), (3, 0.3)\} \text{ and}$$

$$B = \{(2, 0.4), (3, 0.4), (4, 1)\}$$

Distance between fuzzy set A and B

d	x_A	x_B	$m_A(x_i)$	$m_B(x_i)$	$\min(m_A(x_i), m_B(x_i))$	$\max[\min(m_A(x_i), m_B(x_i))]$
0	2	2	1.0	0.4	0.4	0.4
	3	3	0.3	0.4	0.3	-
1	1	2	0.5	0.4	0.4	0.4
	2	3	1.0	0.4	0.4	-
	3	2	0.3	0.4	0.3	-
	3	4	0.3	1.0	0.3	-
2	1	3	0.5	0.4	0.4	1.0
	2	4	1.0	1.0	1.0	-

Therefore $d(A, B) = 0.4 + 0.4 + 1.0 = 1.8$

Now find $d(B, A) = \max[\min(m_B(x_i), m_A(x_i))]$

Consider the $X = \{1, 2, 3, 4\}$, such that

$$B = \{(2, 0.4), (3, 0.4), (4, 1)\} \text{ and}$$

$$A = \{(1, 0.5), (2, 1), (3, 0.3)\}$$

$$\mu = \mu u$$

$$\nu = \nu u$$

Distance between fuzzy set B and A.

d	x_B	x_A	$\mu_B(x_i)$	$\mu_A(x_i)$	$\min(\mu_B(x_i), \mu_A(x_i))$	$\max[\min(\dots)]$
0	2	2	0.4	1.0	0.4	0.4
	3	3	0.4	0.3	0.3	-
1	2	1	0.4	0.5	0.4	0.4
	3	2	0.4	1.0	0.4	-
	2	3	0.4	0.3	0.3	-
	4	3	1.0	0.3	0.3	-
2	3	1	0.4	0.5	0.4	1.0
	4	2	1.0	1.0	1.0	-

Therefore $d(B, A) = 0.4 + 0.4 + 1.0 = 1.8$

Some Important Facts:

1. Let $X = \{x_1, x_2, \dots, x_n\}$ be a universe set, then $D(A, B)$ the distance between two intuitionistic fuzzy sets (IFSs) 'A' and 'B' satisfy the following relationship:

(i) $D_h(A, B) = 2d_h(A, B)$

(ii) $D_h^n(A, B) = 2d_h^n(A, B)$

~~(iii)~~ (i) $D_h(A, B) = 2d_h(A, B)$

Let $A(\mu_A(x), \nu_A(x))$ and $B(\mu_B(x), \nu_B(x))$ be two intuitionistic fuzzy sets (IFSs), then

$$D_h(A, B) = \sum_{i=1}^n |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|$$

$$= \sum_{i=1}^n |\mu_A(x_i) - \mu_B(x_i)| + |1 - \mu_A(x_i) - (1 - \mu_B(x_i))|$$

$$D_h(A, B) = \sum_{i=1}^n |\mu_A(x) - \mu_B(x)| + |1 - \mu_A(x) + \mu_B(x)|$$

$$D_h(A, B) = \sum_{i=1}^n |\mu_A(x) - \mu_B(x)| + |\mu_A(x) - \mu_B(x)|$$

$$D_h(A, B) = 2 \sum_{i=1}^n |\mu_A(x) - \mu_B(x)|$$

$$D_h(A, B) = 2 d_h(A, B).$$

$$(ii) D_h^n(A, B) = 2 d_h^n(A, B)$$

Let $A(\mu_A(x), \nu_A(x))$ and $B(\mu_B(x), \nu_B(x))$ be two IFSs then,

$$D_h^n(A, B) = \frac{1}{n} \sum_{i=1}^n |\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|$$

$$= \frac{1}{n} \sum_{i=1}^n |\mu_A(x) - \mu_B(x)| + |1 - \mu_A(x) - (1 - \mu_B(x))|$$

$$= \frac{1}{n} \sum_{i=1}^n |\mu_A(x) - \mu_B(x)| + |-\mu_A(x) + \mu_B(x)|$$

$$= \frac{1}{n} \sum_{i=1}^n |\mu_A(x) - \mu_B(x)| + |\mu_A(x) - \mu_B(x)|$$

$$= \frac{2}{n} \sum_{i=1}^n |\mu_A(x) - \mu_B(x)|$$

$$= 2 \cdot \frac{1}{n} \sum_{i=1}^n |\mu_A(x) - \mu_B(x)|$$

$$D_h^n(A, B) = 2 d_h^n(A, B)$$

Lecture #34

Properties of Distance

Theorem:

Let $X = \{x_1, x_2, x_3, \dots, x_n\}$ be a universe set, then $d(A, B)$, the distance measure between two fuzzy sets (FSs) A and B satisfy the following conditions:

- (1) $0 \leq d(A, B) < 1$
- (2) $d(A, B) = 0$ iff $A = B$
- (3) $d(A, B) = d(B, A)$
- (4) $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$ for any $A, B, C \in \text{FS}(X)$.

Lecture #35

Lecture #34 and Lecture #35 are same.

Lecture #36

Intuitionistic Fuzzy Sets:

An intuitionistic fuzzy set (IFS) on X is given below:

$$A = \{ \langle x_i, \mu_A(x_i), \nu_A(x_i) \rangle \mid x_i \in X \}$$

where $\mu_A(x) : X \rightarrow [0, 1]$ and $\nu_A(x) : X \rightarrow [0, 1]$

satisfied the following condition,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X$$

The numbers $\mu_A(x), \nu_A(x) \in [0, 1]$ denote the

degree of membership and non-membership of x to A respectively.

Some operations on Intuitionistic Fuzzy sets:

Let $A(\mu_A(x), \nu_A(x))$ and $B(\mu_B(x), \nu_B(x))$ be two IFSs then

- (1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$
- (2) $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x), \forall x \in X$
- (3) $A \cup B = \{ \langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) \rangle \mid x \in X \}$
- (4) $A \cap B = \{ \langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) \rangle \mid x \in X \}$
- (5) $A \oplus B = \{ \langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \times \mu_B(x), \nu_A(x) \times \nu_B(x) \rangle \mid x \in X \}$
- (6) $A \otimes B = \{ \langle x, \mu_A(x) \times \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \times \nu_B(x) \rangle \mid x \in X \}$
- (7) $A - B = \{ \langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x)) \rangle \mid x \in X \}$
- (8) $A^c = \{ \langle x, (\nu_A(x), \mu_A(x)) \rangle \mid x \in X \}$

Example:

Let A and B be two IFSs in a nonempty set X , then,

- (i) $A - B = A \cap B^c$
- (ii) $A - B = B - A$ iff $A = B$
- (iii) $A - B = B^c - A^c$

Sol: Let $A(\mu_A(x), \nu_A(x))$ and $B(\mu_B(x), \nu_B(x))$ be two IFSs then.

$$(i) A - B = \{ \langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x)) \rangle \mid x \in X \}$$

$$B^c = \{ \langle x, (\nu_B(x), \mu_B(x)) \rangle \mid x \in X \}$$

$$A \cap B^c = \{ \langle x, (\mu_A(x), \nu_A(x)) \cap (\nu_B(x), \mu_B(x)) \rangle \mid x \in X \}$$

$$A \cap B^c = \{ \langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x)) \rangle \mid x \in X \}$$

$$A \cap B^c = A - B \quad (\text{proved})$$

$$(ii) \quad A - B = \{ \langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x)) \rangle \mid x \in X \}$$

$$A \subseteq B \text{ if } A = B \text{ then } \mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x)$$

$$A - B = \{ \langle x, \min(\mu_B(x), \nu_A(x)), \max(\nu_B(x), \mu_A(x)) \rangle \mid x \in X \}$$

$$A - B = B - A \quad (\text{proved})$$

$$(iii) \quad A - B = \{ \langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x)) \rangle \mid x \in X \}$$

$$A^c = \{ \langle x, (\nu_A(x), \mu_A(x)) \rangle \mid x \in X \}$$

$$B^c = \{ \langle x, (\nu_B(x), \mu_B(x)) \rangle \mid x \in X \}$$

$$B^c \cap A^c = \{ \langle x, \min(\nu_B(x), \mu_A(x)), \max(\mu_B(x), \nu_A(x)) \rangle \mid x \in X \}$$

$$B^c \cap A^c = A - B \quad (\text{proved})$$

Lecture # 37

~~Example~~ Distance between two IFSs:

Example:

Let us consider two IFSs A and B in $X = \{1, 2, 3, 4, 5, 6, 7\}$, such that

$$A = \frac{(0.7, 0.3)}{1} + \frac{(0.2, 0.8)}{2} + \frac{(0.6, 0.4)}{4} + \frac{(0.5, 0.5)}{5} + \frac{(1, 0)}{6}$$

$$B = \frac{(0.2, 0.8)}{1} + \frac{(0.6, 0.4)}{4} + \frac{(0.8, 0.2)}{5} + \frac{(1, 0)}{7}$$

Find the Hamming distance between A and B.

Sol: $d_h(A, B) = \sum_{i=1}^n |m_A(x_i) - m_B(x_i)|$

$$d_h(A, B) = \sum_{i=1}^7 |m_A(x_i) - m_B(x_i)|$$

$$d_h(A, B) = |0.7 - 0.2| + |0.3 - 0.8| + |0.2 - 0| + |0.8 - 1| \\ + |0.6 - 0.6| + |0.4 - 0.4| + |0.5 - 0.8| + |0.5 - 0.2| \\ + |1 - 0| + |0 - 1| + |0 - 1| + |1 - 0|$$

$$d_h(A, B) = 0.5 + 0.5 + 0.2 + 0.2 + 0 + 0 + 0.3 + 0.3 \\ + 1 + 1 + 1 + 1 = 6$$

The normalized Hamming distance is

$$d_h^n(A, B) = \frac{1}{n} \cdot \sum_{i=1}^n |m_A(x_i) - m_B(x_i)|$$

$$d_h^n(A, B) = \frac{1}{7} \cdot 6 = 0.86$$

The Euclidean distance A and B.

$$d_e(A, B) = \sqrt{\sum_{i=1}^n (m_A(x_i) - m_B(x_i))^2}$$

$$d_e(A, B) = \sqrt{\sum_{i=1}^7 (m_A(x_i) - m_B(x_i))^2}$$

$$\begin{aligned}
 d_e(A, B) &= \sqrt{(0.7-0.2)^2 + (0.3-0.8)^2 + (0.2-0)^2 + (0.8-0.1)^2} \\
 &\quad + (0.6-0.6)^2 + (0.4-0.4)^2 + (0.5-0.8)^2 + (0.5-0.2)^2 \\
 &\quad + (1-0)^2 + (0-1)^2 + (0-1)^2 + (1-0)^2 \\
 &= \sqrt{0.5^2 + (-0.5)^2 + 0.2^2 + (-0.2)^2 + 0^2 + 0^2 + (-0.3)^2 + (0.3)^2 + 1^2 + 1^2 + (-1)^2 + 1^2}
 \end{aligned}$$

$$d_e(A, B) = \sqrt{4.76} = 2.1817$$

The normalized Euclidean distance is

$$d_e^n(A, B) = \frac{1}{n} \cdot \sqrt{\sum_{i=1}^n |m_A(x_i) - m_B(x_i)|^2}$$

$$d_e^n(A, B) = \frac{1}{7} \cdot 2.1817$$

$$d_e^n(A, B) = 0.3117$$

Lecture #38

Picture Fuzzy Sets:

Let $X = \{x_1, x_2, \dots, x_n\}$ be a fixed set, a picture fuzzy set P on X is defined as:

$$P = \{(x, \alpha_P(x), \gamma_P(x), \beta_P(x)) \mid x \in X\}$$

where $\alpha_P(x)$, $\beta_P(x)$, $\gamma_P(x) \in [0, 1]$ are called the acceptance membership, neutral and rejection membership degrees of $x \in X$ to the set P respectively, and $\alpha_P(x)$, $\gamma_P(x)$ and $\beta_P(x)$ fulfil the condition:

$$0 \leq \alpha_P(x) + \gamma_P(x) + \beta_P(x) \leq 1 \quad \forall x \in X.$$

Also $\eta_P(x) = 1 - \alpha_P(x) - \gamma_P(x) - \beta_P(x)$, then $\eta_P(x)$ is said to be a degree of refusal membership

of $x \in X$ in P . For our convenience, the picture fuzzy sets (PFSs) over a fixed set X is written as $PFSs(X)$.

Some operations on PFSs:

let A and B be two PFSs on X , then the union, intersection and complement are described as follows:

$$(1) A \subseteq B \text{ iff } \alpha_A(x) \leq \alpha_B(x), \gamma_A(x) \leq \gamma_B(x) \text{ and } \beta_A(x) \geq \beta_B(x) \text{ such that for all } x \in X.$$

$$(2) A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A.$$

$$(3) A \cup B = \{ \langle x, \max(\alpha_A(x), \alpha_B(x)), \min(\gamma_A(x), \gamma_B(x)), \min(\beta_A(x), \beta_B(x)) \rangle \mid x \in X \}.$$

$$(4) A \cap B = \{ \langle x, \min(\alpha_A(x), \alpha_B(x)), \min(\gamma_A(x), \gamma_B(x)), \max(\beta_A(x), \beta_B(x)) \rangle \mid x \in X \}.$$

$$(5) A^c = \{ \langle x, \beta_A(x), \gamma_A(x), \alpha_A(x) \rangle \mid x \in X \}.$$

Lecture # 39

Example:

$$P_1 = \{ (x_1, (0.6, 0.2, 0.1)), (x_2, (0.4, 0.1, 0.4)) \}$$

$$P_2 = \{ (x_1, (0.7, 0.3, 0)), (x_2, (0.5, 0.2, 0.3)) \}$$

$$(i) P_1 \subseteq P_2 = ?$$

$$P_1 \subseteq P_2 \text{ iff } \alpha_{P_1}(x) \leq \alpha_{P_2}(x), \gamma_{P_1}(x) \leq \gamma_{P_2}(x) \text{ and } \beta_{P_1}(x) \geq \beta_{P_2}(x)$$

$$0.6 \leq 0.7, 0.2 \leq 0.3 \quad \text{and} \quad 0.1 \geq 0$$

$$0.4 \leq 0.5, 0.1 \leq 0.2 \quad \text{and} \quad 0.4 \geq 0.3$$

Hence $P_1 \subseteq P_2$.

$$(ii) P_2 \not\subseteq P_1$$

$P_1 \neq P_2$ because $P_1 \subseteq P_2$ and $P_2 \not\subseteq P_1$

$$(iii) P_1 \cup P_2 = \{ \langle x, \max(\alpha_{P_1}(x), \alpha_{P_2}(x)), \min(\gamma_{P_1}(x), \gamma_{P_2}(x)), \min(\beta_{P_1}(x), \beta_{P_2}(x)) \rangle \mid x \in X \}$$

$$P_1 \cup P_2 = \{ \langle x_1, (0.7, 0.2, 0) \rangle, \langle x_2, (0.5, 0.1, 0.3) \rangle \}$$

$$(iv) P_1 \cap P_2 = \{ \langle x, \min(\alpha_{P_1}(x), \alpha_{P_2}(x)), \min(\gamma_{P_1}(x), \gamma_{P_2}(x)), \max(\beta_{P_1}(x), \beta_{P_2}(x)) \rangle \mid x \in X \}$$

$$P_1 \cap P_2 = \{ \langle x_1, (0.6, 0.2, 0.1) \rangle, \langle x_2, (0.4, 0.1, 0.4) \rangle \}$$

$$(v) P_1^c = \{ \langle x, \beta_{P_1}(x), \gamma_{P_1}(x), \alpha_{P_1}(x) \rangle \mid x \in X \}$$

$$P_1^c = \{ \langle x_1, (0.1, 0.2, 0.6) \rangle, \langle x_2, (0.4, 0.1, 0.4) \rangle \}$$

$$P_2^c = \{ \langle x_1, (0, 0.3, 0.7) \rangle, \langle x_2, (0.3, 0.2, 0.5) \rangle \}$$

Lecture # 40

Distance between two PFSs :

Let P_1 and P_2 be two PFSs, such that

$$P_1 = \{ \langle x_i, (\alpha_{P_1}(x_i), \gamma_{P_1}(x_i), \beta_{P_1}(x_i)) \rangle \}$$

$$P_2 = \{ \langle x_i, (\alpha_{P_2}(x_i), \gamma_{P_2}(x_i), \beta_{P_2}(x_i)) \rangle \}$$

$$\Rightarrow P_1 = (\alpha_{P_1}(x_i), \gamma_{P_1}(x_i), \beta_{P_1}(x_i))$$

$$P_2 = (\alpha_{P_2}(x_i), \gamma_{P_2}(x_i), \beta_{P_2}(x_i))$$

Normalized Hamming distance.

$$d_h^n(A, B) = \frac{1}{n} \sum_{i=1}^n |m_A(x_i) - m_B(x_i)|$$

$$d_h^n(P_1, P_2) = \frac{1}{n} \sum_{i=1}^n [|\alpha_{P_1}(x_i) - \alpha_{P_2}(x_i)| + |\gamma_{P_1}(x_i) - \gamma_{P_2}(x_i)| + |\beta_{P_1}(x_i) - \beta_{P_2}(x_i)|]$$

Example: Let P_1 and P_2 be two PFSs defined on $X = \{x_1, x_2, x_3, x_4\}$, such that

$$P_1 = \{ \langle x_1, (0.56, 0.34, 0.1) \rangle, \langle x_2, (0.9, 0.07, 0.03) \rangle, \langle x_3, (0.4, 0.33, 0.19) \rangle, \langle x_4, (0.09, 0.79, 0.03) \rangle \}$$

$$P_2 = \{ \langle x_1, (0.7, 0.1, 0.09) \rangle, \langle x_2, (0.1, 0.66, 0.2) \rangle, \langle x_3, (0.06, 0.81, 0.12) \rangle, \langle x_4, (0.72, 0.14, 0.09) \rangle \}$$

$$d_h^n(P_1, P_2) = ? \quad , \quad n=4$$

$$\begin{aligned} d_h^n(P_1, P_2) &= \frac{1}{4} [|0.56 - 0.7| + |0.34 - 0.1| + |0.1 - 0.09| + |0.9 - 0.1| + \\ &\quad |0.07 - 0.66| + |0.03 - 0.2| + |0.4 - 0.06| + |0.33 - 0.81| + \\ &\quad |0.19 - 0.12| + |0.09 - 0.72| + |0.79 - 0.14| + |0.03 - 0.09|] \\ &= \frac{1}{4} [0.14 + 0.24 + 0.01 + 0.8 + 0.59 + 0.17 + 0.34 + 0.48 \\ &\quad + 0.07 + 0.63 + 0.65 + 0.06] \end{aligned}$$

$$d_h^n(P_1, P_2) = \frac{1}{4} [4.18] = 1.045 \text{ Ans}$$

Let $P_1 = (\alpha_{P_1}(x_i), \gamma_{P_1}(x_i), \beta_{P_1}(x_i))$ and $P_2 = (\alpha_{P_2}(x_i), \gamma_{P_2}(x_i), \beta_{P_2}(x_i))$ be two PFSs defined on $X = \{x_1, x_2, \dots, x_n\}$

$$D_p(P_1, P_2) = \frac{1}{3n} \sum_{i=1}^n \left[|\alpha_{P_1}(x_i) - \alpha_{P_2}(x_i)| + |\gamma_{P_1}(x_i) - \gamma_{P_2}(x_i)| + |\beta_{P_1}(x_i) - \beta_{P_2}(x_i)| + \max[|\alpha_{P_1}(x_i) - \alpha_{P_2}(x_i)|, |\gamma_{P_1}(x_i) - \gamma_{P_2}(x_i)|, |\beta_{P_1}(x_i) - \beta_{P_2}(x_i)|] \right]$$

Example: Let P_1 and P_2 be two PFSs defined on $X = \{x_1, x_2, x_3\}$, such that

$$P_1 = \{ \langle x_1, (0.8, 0.1, 0) \rangle, \langle x_2, (0.4, 0.2, 0.3) \rangle, \langle x_3, (0.5, 0.3, 0) \rangle \}$$

$$P_2 = \{ \langle x_1, (0.3, 0.3, 0.2) \rangle, \langle x_2, (0.7, 0.1, 0.1) \rangle, \langle x_3, (0.4, 0.3, 0.2) \rangle \}$$

$$D_p(P_1, P_2) = ? \quad , \quad n = 3$$

$$D_p(P_1, P_2) = \frac{1}{3(3)} \left[|0.8 - 0.3| + |0.1 - 0.3| + |0 - 0.2| + \max[|0.8 - 0.3|, |0.1 - 0.3|, |0 - 0.2|] + |0.4 - 0.7| + |0.2 - 0.1| + |0.3 - 0.1| + \max[|0.4 - 0.7|, |0.2 - 0.1|, |0.3 - 0.1|] + |0.5 - 0.4| + |0.3 - 0.3| + |0 - 0.2| + \max[|0.5 - 0.4|, |0.3 - 0.3|, |0 - 0.2|] \right]$$

$$D_p(P_1, P_2) = \frac{1}{9} [0.5 + 0.2 + 0.2 + 0.5 + 0.3 + 0.1 + 0.2 + 0.3 + 0.1 + 0 + 0.2 + 0.2]$$

$$= \frac{1}{9} [2.8] = 0.311 \quad \underline{\text{Ans}}$$

Lecture #41

Hesitant Fuzzy Sets (HFSs):

let X be a reference set, then we define a hesitant fuzzy set on X in terms of a function h that when applied to X returns a subset of $[0,1]$. A HFS H on X is written as:

$$H = \{ \langle x, h(x) \rangle \mid \forall x \in X \}$$

where $h(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$ to H .

Lecture #42

Example:

$$X = \{x_1, x_2\}$$

$$H_1 = \{ \langle x_1, \{0.1, 0.2\} \rangle, \langle x_2, \{0.5, 0.6, 0.8\} \rangle \}$$

$$H_2 = \{ \langle x_1, \{0.8\} \rangle, \langle x_2, \{0.6, 0.7\} \rangle \}$$

Some Operations on HFSs:

For any h, h_1 and h_2 in $H(X)$, some operations on them can be described as follows:

- (1) Lower bound: $h^-(x) = \min h(x)$;
- (2) Upper bound: $h^+(x) = \max h(x)$;
- (3) Complement: $h^c(x) = \{1 - \gamma \mid \gamma \in h(x)\}$;
- (4) Union: $(h_1 \cup h_2)(x) = \{ \gamma \in h_1 \cup h_2 \mid \gamma \geq \max(h_1^-(x), h_2^-(x)) \}$.

- (5) Intersection: $(h_1 \cap h_2)(x) = \{\gamma \in h_1(x) \cap h_2(x) \mid \gamma \leq \min(h_1^+(x), h_2^+(x))\}$;
- (6) $h^\lambda(x) = \{\gamma^\lambda \mid \gamma \in h(x)\}$;
- (7) $\lambda h(x) = \{1 - (1 - \gamma)^\lambda \mid \gamma \in h(x)\}$;
- (8) $(h_1 \oplus h_2)(x) = \{\gamma_1 + \gamma_2 - \gamma_1 \cdot \gamma_2 \mid \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}$;
- (9) $(h_1 \otimes h_2)(x) = \{\gamma_1 \cdot \gamma_2 \mid \gamma_1 \in h_1(x), \gamma_2 \in h_2(x)\}$.

Lecture # 43

Example: Let $X = \{x_1, x_2, x_3\}$ be a universal set.

h_1 and h_2 be two HFSs on X such that

$$h_1 = \{ \langle x_1, \{0.3, 0.4\} \rangle, \langle x_2, \{0.6, 0.8\} \rangle, \langle x_3, \{0.3, 0.4, 0.5, 0.7\} \rangle \}$$

$$h_2 = \{ \langle x_1, \{0.5, 0.6\} \rangle, \langle x_2, \{0.4, 0.5\} \rangle, \langle x_3, \{0.2, 0.3, 0.4, 0.6\} \rangle \}$$

Then find out the following operations:-

(i) $h_1^-(x_3) = ?$

$$h_1^-(x_3) = \min\{0.3, 0.4, 0.5, 0.7\} = 0.3$$

(ii) $h_2^+(x_2) = ?$

$$h_2^+(x_2) = \max\{0.4, 0.5\} = 0.5$$

(iii) $h_1^c(x_2) = ?$

$$h_1^c(x) = \{1 - \gamma \mid \gamma \in h\}$$

$$h_1^c(x_2) = \{1 - 0.6, 1 - 0.8\} = \{0.4, 0.2\}$$

(iv) $h_1^\lambda(x_2) = ?$

$$\text{if } \lambda = 2$$

$$h^\lambda(x) = \{\gamma^\lambda\}$$

$$h_1^\lambda(x_2) = \{0.6^2, 0.8^2\} = \{0.36, 0.64\}$$

(v) $\lambda h_1(x_1) = ?$

$$\text{if } \lambda = 2$$

$$\lambda h(x) = \{1 - (1 - \gamma)^\lambda\}$$

$$\lambda h_1(x_1) = \{1 - (1 - 0.3)^2, 1 - (1 - 0.4)^2\}$$

$$= \{1 - 0.7^2, 1 - 0.6^2\} = \{1 - 0.49, 1 - 0.36\}$$

$$\lambda h_1(x_1) = \{0.51, 0.64\}$$

$$(vi) (h_1 \oplus h_2)(x_2) = ? \quad (h_1 \oplus h_2)(x) = \{ \gamma_1 + \gamma_2 - \gamma_1 \gamma_2 \}$$

$$\begin{aligned} (h_1 \oplus h_2)(x_2) &= \{0.6, 0.8\} \oplus \{0.4, 0.5\} \\ &= \{0.6+0.4-(0.6)(0.4), 0.6+0.5-(0.6)(0.5), 0.8+0.4-(0.8)(0.4), \\ &\quad 0.8+0.5-(0.8)(0.5)\} \\ &= \{1.0-0.24, 1.1-0.30, 1.2-0.32, 1.3-0.40\} \\ &= \{0.76, 0.8, 0.88, 0.9\} \end{aligned}$$

$$(vii) (h_1 \otimes h_2)(x_1) = ? \quad (h_1 \otimes h_2)(x) = \{ \gamma_1 \cdot \gamma_2 \}$$

$$\begin{aligned} (h_1 \otimes h_2)(x_1) &= \{0.3, 0.4\} \otimes \{0.5, 0.6\} \\ &= \{(0.3)(0.5), (0.3)(0.6), (0.4)(0.5), (0.4)(0.6)\} \\ &= \{0.15, 0.18, 0.20, 0.24\} \end{aligned}$$

$$(viii) (h_1 \cup h_2)(x_3) = ?$$

$$\begin{aligned} (h_1 \cup h_2)(x_3) &\geq \max \{ \min \{0.3, 0.4, 0.5, 0.7\}, \min \{0.2, 0.3, 0.4, 0.6\} \} \\ &\geq \max \{0.3, 0.2\} = 0.3 \\ &= \{0.3, 0.4, 0.5, 0.6, 0.7\} \end{aligned}$$

$$(ix) (h_1 \cap h_2)(x_3) = ?$$

$$\begin{aligned} (h_1 \cap h_2)(x_3) &\leq \min \{ \max \{0.3, 0.4, 0.5, 0.7\}, \max \{0.2, 0.3, 0.4, 0.6\} \} \\ &\leq \min \{0.7, 0.6\} = 0.6 \\ &= \{0.2, 0.3, 0.4, 0.5, 0.6\} \end{aligned}$$

Lecture #44

Hesitant Fuzzy Element (HFE):

$$h_1 = \{ \langle x_1, \{0.2, 0.3\} \rangle \} \Rightarrow h_1 = (0.2, 0.3) \rightarrow (HFE)$$

Theorem:

Suppose that h_1 and h_2 are two HFEs, then

$$L_{h_1 \oplus h_2} = L_{h_1} * L_{h_2} \quad \text{and}$$

$$L_{h_1 \otimes h_2} = L_{h_1} \times L_{h_2}$$

Example: let $h_1 = (0.1, 0.2, 0.7)$ and $h_2 = (0.2, 0.4)$ be two HFEs then

$$h_1 \oplus h_2 = \{0.1, 0.2, 0.7\} \oplus \{0.2, 0.4\}$$

$$= \{0.1+0.2-(0.1)(0.2), 0.1+0.4-(0.1)(0.4), 0.2+0.2-(0.2)(0.2), 0.2+0.4-(0.2)(0.4), 0.7+0.2-(0.7)(0.2), 0.7+0.4-(0.7)(0.4)\}$$

$$= \{0.28, 0.46, 0.38, 0.52, 0.76, 0.82\}$$

$$h_1 \otimes h_2 = \{0.1, 0.2, 0.7\} \otimes \{0.2, 0.4\}$$

$$= \{(0.1)(0.2), (0.1)(0.4), (0.2)(0.2), (0.2)(0.4), (0.7)(0.2), (0.7)(0.4)\}$$

$$= \{0.02, 0.04, 0.04, 0.08, 0.14, 0.28\}$$

$$\therefore L_{h_1 \oplus h_2} = 6 = 3 \times 2 = L_{h_1} * L_{h_2}$$

$$L_{h_1 \otimes h_2} = 6 = 3 \times 2 = L_{h_1} \times L_{h_2} \quad (\text{proved})$$

Distance between HFEs:

$$De(h_1, h_2) = \sqrt{\sum_{i=1}^n (h_{1i} - h_{2i})^2}$$

$$D_h(h_1, h_2) = \sum_{i=1}^n |h_{1i} - h_{2i}|$$

$$De^n(h_1, h_2) = \frac{1}{n} \sqrt{\sum_{i=1}^n (h_{1i} - h_{2i})^2}$$

$$D_h^n(h_1, h_2) = \frac{1}{n} \sum_{i=1}^n |h_{1i} - h_{2i}|$$

Example 1: Let $h_1 = (0.7, 0.8, 0.9)$ and $h_2 = (0.5, 0.6, 0.8)$ be two HFEs, then evaluate (i) $D_h(h_1, h_2)$ (ii) $D_e(h_1, h_2)$

Sol: (i) $D_h(h_1, h_2) = |0.7 - 0.5| + |0.8 - 0.6| + |0.9 - 0.8|$
 $= 0.2 + 0.2 + 0.1 = 0.5$

(ii) $D_e(h_1, h_2) = \sqrt{(0.7 - 0.5)^2 + (0.8 - 0.6)^2 + (0.9 - 0.8)^2}$
 $= \sqrt{(0.2)^2 + (0.2)^2 + (0.1)^2} = \sqrt{0.04 + 0.04 + 0.01} = \sqrt{0.09}$
 $= 0.3$

Example 2: Let $h_1 = (0.3, 0.4)$ and $h_2 = (0.6, 0.7, 0.8)$ be two HFEs, then evaluate the normalized Hamming distance and normalized Euclidean distance between h_1 and h_2 .

Sol: $h_1 = (0.3, 0.3, 0.4)$, $h_2 = (0.6, 0.7, 0.8)$ (Add minimum element for normalized HFEs)

$D_h^n(h_1, h_2) = \frac{1}{3} [|0.3 - 0.6| + |0.3 - 0.7| + |0.4 - 0.8|]$
 $= \frac{1}{3} [0.3 + 0.4 + 0.4] = \frac{1}{3} (1.1)$

$D_h^n(h_1, h_2) = 0.367$

$D_e^n(h_1, h_2) = \frac{1}{3} \sqrt{(0.3 - 0.6)^2 + (0.3 - 0.7)^2 + (0.4 - 0.8)^2}$

$= \frac{1}{3} \sqrt{(0.3)^2 + (0.4)^2 + (0.4)^2}$

$= \frac{1}{3} \sqrt{0.09 + 0.16 + 0.16}$

$= \frac{1}{3} \sqrt{0.41} = \frac{1}{3} (0.64)$

$D_e^n(h_1, h_2) = 0.213$

Lecture # 45

Similarity:

Two objects are said to be similar if both have same shape.

e.g. Two bags have same shape and ~~not~~ have not same size.

Similarity Measure (SM):

Similarity measure is a process through which we can evaluate resemblance between two objects.

There are many kinds of similarity measures like

- (i) Similarity ^{measures} based on distance.
- (ii) Cosine similarity measures.
- (iii) Exponential similarity measures.
- (iv) Jaccard and Dice similarity measures.
- (v) William and Steel similarity measures: etc.

Similarity Measures based on distance:

Let S be a Similarity Measures between two fuzzy sets is calculated as:-

Suppose that A and B are two fuzzy sets then the similarity measure $S(A, B)$ based on distance $d_h(A, B)$ is written as:

$$S(A, B) = 1 - d_h(A, B)$$

$$\text{or } S(A, B) = 1 - d_e(A, B)$$

$$\text{or } S(A, B) = 1 - d_e^n(A, B)$$

$$\text{or } S(A, B) = 1 - d_h^n(A, B)$$

Similarity Measures ~~always~~ satisfied the condition

$$0 \leq S(A, B) \leq 1$$

Theorem: The similarity measures $S(A, B)$ between two fuzzy sets satisfied the following axioms:-

- (i) $0 \leq S(A, B) \leq 1$
- (ii) $S(A, B) = 1$ if and only if $A = B$
- (iii) $S(A, B) = S(B, A)$ i.e. commutative.

End

Always remember in prayers.