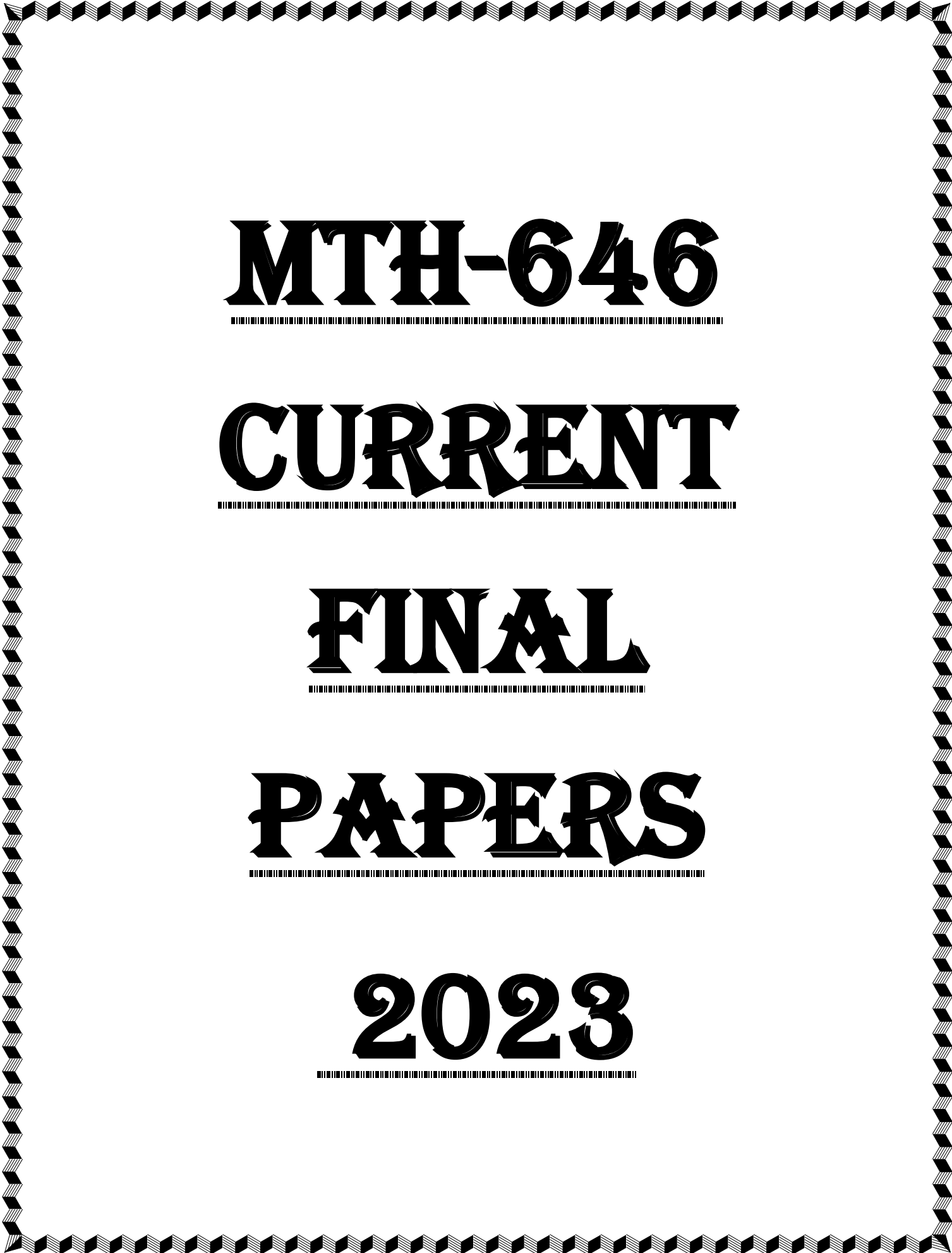




MTH-646

CURRENT

FINAL

PAPERS

2023

MIT-646

Final Term

Q#2: Write expression of Jacobi's Iterative Scheme for 1D heat eq. (3-marks)

As 1D heat eq is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Jacobi's iterative scheme for heat eq is given as

$$u_i^{(n+1)} = \frac{\lambda}{2(1+\lambda)} [u_{i-1}^{(n)} + u_{i+1}^{(n)}] + \frac{C_i}{1+\lambda} \rightarrow (A)$$

where, $C_i = u_{i,j} + \lambda/2 [u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$

Q#2: If for Gauss-Seidal method then follow the whole method and just eq (A) will be changed as. (3-marks)

As 1D heat eq is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Gauss-Seidal Iterative Scheme for heat eq is given as:

$$u_i^{(n+1)} = \frac{\lambda}{2(1+\lambda)} [u_{i-1}^{(n+1)} + u_{i+1}^{(n)}] + \frac{C_i}{1+\lambda} \rightarrow (B)$$

where, $C_i = u_{i,j} + \lambda/2 [u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$

Q#3: Write the Jacobi's SOR iterative Scheme for Poisson PDE. (3-M)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

Solution:

Jacobi's OR method for Poisson eq can be written as

$$u_{i,j}^{(n+1)} = \frac{1}{4} \omega [u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)} + h^2 f_{i,j}] + (1-\omega) u_{i,j}^{(n)}$$

Q#4: ^{1m} SOR / OR Scheme for Gauss-Seidel method :-

$$u_{i,j}^{(n+1)} = \frac{1}{4} \omega [u_{i+1,j}^{(n+1)} + u_{i-1,j}^{(n)} + u_{i,j+1}^{(n+1)} + u_{i,j-1}^{(n)} + h^2 f_{i,j}] + (1-\omega) u_{i,j}^{(n)}$$

Q#5: ^{1m} Find value of "l" from $l = i + (m-1-j)(n-1)$ for interval mesh points $w_{1,7}$ and $w_{10,8}$ use $m=n=10$, (3-marks)
(5-marks)

As $w_{1,7}$ and $w_{10,8}$ are given and

$$w_{i,j} = w_l \Rightarrow i=1 \text{ and } j=7,8$$

So, for $i=1, j=7$ or $w_{1,7}$

$$\Rightarrow l = i + (m-1-j)(n-1) = 1 + (10-1-7)(10-1)$$

$$l = 1 + (2)(9) = 19 \Rightarrow w_{1,7} = w_{19}$$

And for $i=1, j=8$ or $w_{1,8}$

$$l = 1 + (10-1-8)(10-1) = 1 + (1)(9) = 1+9=10$$

$$\Rightarrow l=10 \Rightarrow w_{1,8} = w_{10}$$

∴ formula for finding mesh point

$$\Rightarrow w_{i,j} = w_l$$

Q#6: ^{1m} Consider the amplification factor $C_r = \frac{(1-2c^2\lambda^2\sin^2\beta/2) \pm i2c\lambda\sin\beta/2}{\sqrt{1-c^2\lambda^2\sin^2\beta/2}}$

of a PDE - Find Mode of C_r i.e. $|C_r| = ?$ (3-marks)
(5-marks)

There are three cases

Case-I: $c^2\lambda^2\sin^2\beta/2 > 1$

$$|C_r| = |1 + 2c\lambda\sin\beta/2 (\sqrt{1-c^2\lambda^2\sin^2\beta/2} - c\lambda\sin\beta/2)|$$

These opposite signs make this equation unstable.

So, $|C_r| > 1$ - (Unstable)

Case-II: $c^2\lambda^2\sin^2\beta/2 = 1$

→ From given eq, we will put $c^2\lambda^2\sin^2\beta/2 = 1$. So, we have $C_r = -1$.

$$\Rightarrow |C_r| = |-1| \Rightarrow |C_r| = 1 \text{ (Stable)}$$

Case-III: $c^2\lambda^2\sin^2\beta/2 < 1$

$|C_r| \leq 1$ - (Conditionally Stable).

Q#7: Discretize the following 1D wave eq, apply central difference method / approximation. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (3-marks)

→ Apply Central Difference approximation on both types of derivative on the given wave equation,

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = c^2 \left(\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2} \right)$$

These two steps are the Ans.

$$u_i^{n+1} - 2u_i^n + u_i^{n-1} = \left(\frac{\Delta t^2 c^2}{\Delta x^2} \right) (u_{i-1}^n - 2u_i^n + u_{i+1}^n)$$

let $r = \Delta t / \Delta x$, so above eq becomes

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + r^2 c^2 (u_{i-1}^n - 2u_i^n + u_{i+1}^n)$$

→ Skip this process. Its just for better understanding :)

And exact equation is,

$$u_i^{n+1} = 2(1 - r^2 c^2) u_i^n + r^2 c^2 (u_{i-1}^n + u_{i+1}^n) - u_i^{n-1}$$

(3-marks)

(5-marks)

Q#8: ^{Imp} Label the internal mesh points $w_e = w_{ij}$ of the solution of domain for $n=m=10$, and using fixed values of $j=8$ and $i=1,2,3$.

As $w_{ij} = w_e \rightarrow (1)$

and $l = i + (m-1-j)(n-1)$

Given $i=1,2,3$ and $j=8$ and $m=n=10$

Now, fix $j=8$ in (1)

* for $i=1$ and $w_{1,8} = w_e \Rightarrow l = 1 + (10-1-8)(10-1)$

$$l = 1 + (1)(9) = 10 \Rightarrow \boxed{w_{1,8} = w_{10}}$$

* for $i=2 \Rightarrow w_{2,8} = w_e \Rightarrow l = 2 + (10-1-8)(10-1) = 11$

$$\Rightarrow l=11 \Rightarrow \boxed{w_{2,8} = w_{11}}$$

* for $i=3 \Rightarrow w_{3,8} = w_e \Rightarrow l = 3 + (10-1-8)(10-1) = 12$

$$\Rightarrow l=12 \Rightarrow \boxed{w_{3,8} = w_{12}}$$

Q9: Calculate value of u_i^2 for $i=1, 2$ from following discretized hyperbolic: $u_i^2 = \frac{1}{2}(u_{i-1}^0 + u_{i+1}^0) + 0.2$, $i=1, 2$. Assume initial condition $u_i^0 = x_i^0$ and $\Delta x = \Delta t = 0.2$ (5-marks)

From given eq

$$u_i^2 = \frac{1}{2}[u_{i-1}^0 + u_{i+1}^0] + 0.2 \rightarrow A$$

as $\Delta x = \Delta t = 0.2$

from initial condition $u_i^0 = x_i^0$

for $i=1$, $u_1^0 = x_1^0$

$$u_1^0 = (0.2)^0 = 0.04$$

It is initial point for $i=1$.

Now,

$$\text{for } i=2, \Rightarrow u_2^0 = x_2^0 = (0.4)^0 = 0.16$$

Now, using eq (A)

$$\text{for } i=1 \Rightarrow u_1^1 = \frac{1}{2}[u_0^0 + u_2^0] + 0.2 = \frac{1}{2}(0 + 0.16) + 0.2 = 0.28 = u_1^1$$

$$\text{for } i=2 \Rightarrow u_2^1 = \frac{1}{2}[u_1^0 + u_3^0] + 0.2 = \frac{1}{2}(0.04 + 0) + 0.2 = 0.22 = u_2^1$$

Ans

Q10: Write expression for Jacobi's and Gauss Seidel Method SOR iterative Scheme for rotated five points formula for following 2D poisson eq. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ (5-marks).

⇒ Jacobi's SOR scheme for Poisson eq using rotated five points formula is given as

$$u_{ij}^{(n+1)} = \frac{1}{4} \omega [u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)} + h^2 f_{ij}] + (1-\omega) u_{ij}^{(n)}$$

where as,

⇒ Gauss Seidel's SOR Scheme for poisson eq using rotated five points formula is,

$$u_{ij}^{(n+1)} = \frac{1}{4} \omega [u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)} + h^2 f_{ij}] + (1-\omega) u_{ij}^{(n)}$$

Q#11: ^{Imp} Consider amplification factor of upwind scheme as follows: $|G|^2 = (1 - \lambda + \lambda \cos \beta)^2 + (\lambda \sin \beta)^2 < 1$
 where $\lambda = \frac{c \Delta t}{\Delta x}$ and $\beta = k \Delta x$. Show that $\lambda < 1$. (5-marks)

As,

$$(1 - \lambda + \lambda \cos \beta)^2 + (\lambda \sin \beta)^2 < 1$$

$$(1 - \lambda)^2 + (\lambda \cos \beta)^2 + 2(1 - \lambda)(\lambda \cos \beta) + \lambda^2 \sin^2 \beta < 1$$

$$1 - 2\lambda + \lambda^2 + \lambda^2 \cos^2 \beta + 2\lambda \cos \beta - 2\lambda^2 \cos \beta + \lambda^2 \sin^2 \beta < 1$$

$$1 - 2\lambda + \lambda^2 + \lambda^2 (\cos^2 \beta + \sin^2 \beta) + 2\lambda \cos \beta - 2\lambda^2 \cos \beta < 1$$

$$2\lambda^2 - 2\lambda + 2\lambda \cos \beta - 2\lambda^2 \cos \beta \leq 0 \quad \because \cos^2 \beta + \sin^2 \beta = 1$$

$$2\lambda(\lambda - 1 + \cos \beta - \lambda \cos \beta) \leq 0 \rightarrow \lambda(1 - \lambda \cos \beta) - (1 - \cos \beta)$$

$$\lambda(1 - \cos \beta) - (1 - \cos \beta) \leq 0 \quad \because 2\lambda \leq 0$$

$$\lambda(1 - \cos \beta) \leq (1 - \cos \beta)$$

$$\boxed{\lambda \leq 1}$$

Q#12: Write the discretized form of following 2D (Laplace PDE) for $\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$ (3-marks)
 by taking $h=0.5, k=0.25$.
 Applying CD approximation on Laplace eq, we have

$$2\left(\frac{h^2 + 1}{k^2}\right) u_{ij} - (u_{i-1,j} + u_{i+1,j}) - (u_{i,j} + u_{i,j+1}) = 0$$

$$2\left(\frac{(0.5)^2 + 1}{(0.25)^2}\right) u_{ij} - (u_{i-1,j} + u_{i+1,j}) - (u_{i,j} + u_{i,j+1}) = 0$$

$$2(4+1) u_{ij} - (u_{i-1,j} + u_{i+1,j}) - (u_{i,j} + u_{i,j+1}) = 0$$

$$10 u_{ij} - (u_{i-1,j} + u_{i+1,j}) - (u_{i,j} + u_{i,j+1}) = 0$$

Q#13: Decompose given matrix A, find L and U triangular matrix.
(3-Marks)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

As $A = LU$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Q#14: Determine matrix $A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & 2 \\ -1 & 2 & 4 \end{bmatrix}$ is diagonally dominant or not.
(3-Marks)

Since $|a_{ij}| \geq \sum_{i=1}^n |a_{ij}|$

Case-I: $|a_{11}| \geq |a_{12}| + |a_{13}| \Rightarrow |3| \geq |-2| + |1| \Rightarrow |3| \geq 3 \Rightarrow 3=3$

Case-II: $|a_{21}| \geq |a_{22}| + |a_{23}| \Rightarrow |-3| \geq |1| + |2| \Rightarrow 3=3$

Case-III: $|a_{33}| \geq |a_{31}| + |a_{32}| \Rightarrow |4| \geq |-1| + |2| \Rightarrow 4 \geq 3$

So, the matrix A is diagonally dominant.

Q#15: Write the five points formula for poisson PDE $u_{xx} + u_{yy} = 2$ for uniform grid step size in both direction i.e. $h = \Delta x = \Delta y$.

As poisson eq is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$

Apply CD on both derivatives, $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$ we have

$$2 \left(\frac{h^2}{k^2} + 1 \right) u_{ij} - (u_{i-1,j} + u_{i+1,j}) - \left(\frac{h^2}{k^2} \right) (u_{i,j+1} + u_{i,j-1}) = h^2 f_{ij}$$

as $h=k$, So,

$$2(2) u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j+1} - u_{i,j-1} = 2h^2$$

$$4 u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j+1} - u_{i,j-1} = 2h^2$$

Q#16: For the LU decomposition matrix eq: (3-Mark)

$$\begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{bmatrix} \begin{bmatrix} 1 & u_1 & 0 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -4 & 0 \\ -4 & 10 & -4 \\ 0 & -4 & 10 \end{bmatrix}$$

Compute value of l_1, b_2 and b_3 from above matrix eq.

$$\rightarrow \boxed{l_1=10}, \boxed{b_2=-4}, \boxed{b_3=-4}$$

Q#17: Compute the value of l_i .

$$A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}, \quad l_i = a_i - \frac{C_{i-1} b_i}{l_{i-1}}$$

In this matrix

$$\begin{bmatrix} a_1 & c_1 & & \\ 3 & -1 & 0 & 0 \\ b_1 & a_2 & c_2 & \\ -1 & 3 & -1 & 0 \\ 0 & b_2 & a_3 & c_3 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & b_3 & a_4 \\ & & -1 & 3 \end{bmatrix}$$

$$\Rightarrow l_1 = a_1 = 3$$

$$\Rightarrow l_2 = a_2 - \frac{C_{2-1} b_2}{l_{2-1}} = \frac{3 - (-1)(-1)}{3}$$

$$\Rightarrow l_2 = 3 + \frac{(-1)}{3} = 3 - \frac{1}{3} = \frac{9-1}{3} = \frac{8}{3}$$

$$\Rightarrow \boxed{l_2 = 8/3}$$

$\therefore$ In same process, we can find l_3, l_4 .

Remember Me In Your Prayers ★

Imp
Q#18: Find value of l from $l = i + (m-1-j)(n-1)$ for internal mesh point $w_{2,4}$ and $w_{3,5}$ of solution domain. Given that $m=n=10$.

$$l = i + (m-1-j)(n-1)$$

As $w_{i,j} = w_p$ and $n=m=10$ at $w_{2,4} \Rightarrow i=2, j=4$

$$\text{So, } l = 2 + (10-1-4)(10-1) = 47 \Rightarrow \boxed{w_{2,4} = w_{47}}$$

$$\text{at } w_{3,5} \Rightarrow i=3, j=5 \Rightarrow l = 3 + (10-1-5)(10-1) = 39 \Rightarrow \boxed{w_{3,5} = 39}$$

Imp
Q#19: $A = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}$, $l_i = a_i - \frac{c_{i-1}}{l_{i-1}} b_i$ (5-marks)
 $l_1, l_2, l_3, l_4, l_5 = ?$

$$l_1 = a_1 \Rightarrow l_1 = 4$$

$$\Rightarrow l_2 = a_2 - \frac{c_1}{l_1} b_2 = 4 - \frac{(-1)}{4} (-1) = 4 - \frac{1}{4} = \frac{16-1}{4} = \frac{15}{4}$$

$$\Rightarrow \boxed{l_2 = \frac{15}{4}} \Rightarrow \boxed{l_2 = 3.75}$$

$$\Rightarrow l_3 = a_3 - \frac{c_2}{l_2} b_3 = 4 - \frac{(-1)}{3.75} (-1) = 4 - \frac{1}{3.75} = \frac{56}{15} = 3.73 \Rightarrow \boxed{l_3 = 3.73}$$

$$\Rightarrow l_4 = a_4 - \frac{c_3}{l_3} b_4 = 4 - \frac{(-1)}{56/15} (-1) = 4 - \frac{15}{56} = 3.73, \Rightarrow \boxed{l_5 = 3.73}$$

Imp
Q#20: Derive an expression for 1D heat eq by applying Duxfort-Frankel- (3-marks)

$$T_i^{n+1} = \frac{2\lambda}{1+2\lambda} \left[T_{i-1}^n + T_{i+1}^n \right] + \frac{1-2\lambda}{1+2\lambda} T_i^n$$

Imp
Q#21: Find Numerical Solution of $\left(\frac{\partial u_1}{\partial t} \right)$, $\left(\frac{\partial u_2}{\partial t} \right)$ and $\left(\frac{\partial u_3}{\partial t} \right)$ with help of method of line (MOL) on the 1D heat eq.
 $\left(\frac{\partial u(x,t)}{\partial t} \right) = \frac{\partial^2 u(x,t)}{\partial x^2}$, $0 \leq x \leq 1$, $t > 0$ (5-marks)

Subject to initial & boundary condition.

$$BC \rightarrow u(0,t) = 0 = u(1,t), \quad \forall t > 0$$

$$IC \rightarrow u(x,0) = \sin \pi x, \quad 0 \leq x \leq 1$$

Assume $h=0.25$.

Since formula for method of line (MOL) for heat eq is

$$\left(\frac{\partial u}{\partial t} \right)_{i,j} = \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \right)$$

for $j=0$ and $i=1,2,3$.

$$\frac{\partial u_1}{\partial t} = \frac{u_{0,0} - 2u_1 + u_2}{h^2} = \frac{u_2 - 2u_1}{h^2} \quad \text{--- (1)}$$

$$\frac{\partial u_2}{\partial t} = \frac{u_1 - 2u_2 + u_3}{h^2} \quad \text{--- (2)}$$

$$\frac{\partial u_3}{\partial t} = \frac{u_2 - 2u_3 + u_4^0}{h^2} = \frac{u_2 - 2u_3}{h^2} \quad \text{--- (3)}$$

for initial condition

$$u_1 = \sin \pi(0.25) = 0.7071$$

$$u_2 = \sin \pi(0.5) = 1, \quad u_3 = \sin \pi(0.75) = 0.7071$$

putting values of u_1, u_2 and u_3 in eq (1), (2), (3), we get

$$\frac{\partial u_1}{\partial t} = -6.6272, \quad \frac{\partial u_2}{\partial t} = -9.376, \quad \frac{\partial u_3}{\partial t} = -6.624$$

Q # 22. Find value of labelled l of the internal mesh points $w_l = w_{ij}$ of the solution domain for fixed value of $j=4$ and $i=3,6,8$. Given that $m=n=10$.

$$\text{As } w_l = w_{ij} \Rightarrow l = i + (m-1-j)(n-1)$$

Fix $j=4, i=3,6,8$.

$$i=3 \Rightarrow l = 3 + (10-1-4)(10-1) = 48$$

$$i=6 \Rightarrow l = 6 + (10-1-4)(10-1) = 51$$

$$i=8 \Rightarrow l = 8 + (10-1-4)(10-1) = 53.$$

So,

$$w_{3,4} = w_{48}$$

$$w_{6,4} = w_{51}$$

$$w_{8,4} = w_{53}.$$

V-V-IMP

Q # 23. Discretize the following 1-D wave eq $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

by applying central difference approximation and hence find value of $u_{i,j}$.

Applying CD approx. on both type of derivatives, we have

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}$$

Let $\gamma^2 = (\Delta t / \Delta x c^*)^2$ then, we get

$$u_i^{n+1} = 2(1 - \gamma^2) u_i^n + \gamma^2 c^2 (u_{i-1}^n + u_{i+1}^n) - u_i^{n-1}$$

Q#24. Find amplification factor G of following PDE, by Fourier method.

$$\frac{\partial f(x,t)}{\partial t} + c \frac{\partial f(x,t)}{\partial x} = 0, \quad \begin{matrix} c > 0 \\ \text{mcs} \\ \text{also} \end{matrix}$$

after applying FWD on $\partial f / \partial t$ and BWD on $\partial f / \partial x$.

After applying FWD on $\partial f / \partial t$ and BWD on $\partial f / \partial x$, we have

$$\frac{f_i^{j+1} - f_i^j}{\Delta t} + c \frac{f_i^j - f_{i-1}^j}{\Delta x} = 0$$

$$f_i^{j+1} = f_i^j - \left(\frac{c \Delta t}{\Delta x} \right) (f_i^j - f_{i-1}^j) \quad \text{as } f_i^j = G^j e^{ikx}$$

So, $G_1 G^j e^{ikx} = G^j e^{ikx} (1 - \lambda(1 - e^{-ik\Delta x}))$

after simplification $\Rightarrow G = 1 - \lambda + \lambda \cos \beta - \lambda i \sin \beta$

$$|G| = \sqrt{(1 - \lambda + \lambda \cos \beta)^2 + (\lambda \sin \beta)^2}$$

Q#25. Consider, two dimensional Laplace eq in region

$$R = \{(x,y) | 0 \leq x \leq 0.5, 0 \leq y \leq 0.5\} \text{ with boundary conditions.}$$

$$u(0,y) = 0, \quad u(x,0) = 0$$

$$u(x,0.5) = 200x, \quad u(0.5,y) = 400y$$

Calculate the value of all boundary points of solution domain by assuming $h = k = 0.125$.

Since $h = k = 0.125$

Boundary Conditions are

for $x=0 \Rightarrow u(x_i, 0.5) = 200x_i, \quad i=1,2,3$

$$u(1, 0.5) = 200(0.125) = 25$$

$$u(2, 0.5) = 200(0.25) = 50$$

$$u(3, 0.5) = 200(0.375) = 75$$

for y -axis $j=1,2,3$

$$u(0.5, y_j) = 400y_j$$

$$u(0.5, 1) = 400(0.125) = 50$$

$$u(0.5, 2) = 400(0.25) = 100$$

$$u(0.5, 3) = 400(0.375) = 150$$

(3-Marks)

Q# 26: The discretized form of 1D wave eq at first time level is $w_{i,1} = w_{i,0} + k g_i + \frac{1}{2} [f_{i-1} - 2f_i + f_{i+1}]$ subject to boundary condition and initial condition.

$$BC \rightarrow u(0,t) = 0 = u(1,t) \quad \forall, t > 0$$

$$IC \rightarrow u(x,0) = \sin \pi x = f(x)$$

$$u_t(x,0) = 0 = g(x) \quad \forall, 0 \leq x \leq 1$$

$$\text{Since } h = \frac{1-0}{4} \Rightarrow h = 0.25$$

$$w_{0,j} = w_{4,j} = 0$$

$$\rightarrow w_{i,0} = \sin \pi x_i = f_i \quad \text{--- (A)}$$

$$g_i = \frac{\partial u}{\partial t}(x_i, 0) = 0$$

So, for $i=1$

$$w_{1,1} = w_{1,0} + k g_1 + \frac{1}{2} [f_0 - 2f_1 + f_2] \quad \text{--- (1)}$$

$$\text{from (A), } w_{1,0} = \sin \pi (0.25)$$

$$w_{1,0} = 0.7071 = f_1, \quad w_{2,0} = 1 = f_2, \quad w_{3,0} = 0.7071 = f_3$$

putting in eq (1)

$$\Rightarrow w_{1,1} = 0.7071 + 0 + \frac{1}{2} [0 - 2(0.7071) + 1]$$

$$w_{1,1} = 0.5$$

$$\Rightarrow w_{2,1} = w_{2,0} + \frac{1}{2} (f_1 - 2f_2 + f_3)$$

$$= 1 + \frac{1}{2} (0.7071 - 2(1) + 0.7071)$$

$$\Rightarrow w_{2,1} = 0.7071$$

→ Crout Algorithm

$$1) l_{ii} = a_{ii} \text{ ; } l_{ij} = \frac{c_i - \sum_{k=1}^{j-1} l_{ik} b_{kj}}{l_{ii}}$$

$$2) y_1 = d_1 / l_{11} \Rightarrow y_i = \frac{1}{l_{ii}} (d_i - \sum_{k=1}^{i-1} l_{ik} y_k)$$

$$3) x_n = y_n \text{ ; } x_i = y_i - \frac{c_i}{l_{ii}} (x_{i+1} + 1)$$

→ How to write LU matrix in Crout's and doolittle's method?

(A) Crout's method

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

(B) Doolittle's method:

$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

→ Apply FWD and Crank Nicolson

on given heat eq $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

FWD of time derivative and

Crank Nicolson of space derivative

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} = \alpha^2 \left[\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{(\Delta x)^2} + \frac{U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}}{(\Delta x)^2} \right]$$

→ Apply Fourier discretization factor (3 marks)

$U_{i,j} = C_i e^{iKx}$ in following discretization form of PDE

$$-\frac{\lambda}{2} U_{i+1,j+1} + (1+\lambda) U_{i,j+1} - \frac{\lambda}{2} U_{i-1,j+1} = \frac{\lambda}{2} U_{i-1,j} + (1-\lambda) U_{i,j} + \frac{\lambda}{2} U_{i+1,j}$$

Solution. Since $U_{i,j} = C_i e^{iKx}$

So eq becomes,

$$-\frac{\lambda}{2} C_i e^{iK(x-\Delta x)} + (1+\lambda) C_i e^{iKx} - \frac{\lambda}{2} C_i e^{iK(x+\Delta x)} = \frac{\lambda}{2} C_i e^{iK(x-\Delta x)} + (1-\lambda) C_i e^{iKx} - \frac{\lambda}{2} C_i e^{iK(x+\Delta x)}$$

→ Iterative formula for Jacobis.

$$U_i^{n+1} = \frac{\lambda}{1+\lambda} [U_{i-1}^n + U_{i+1}^n] + C_i$$

where

$$C_i = U_{i,j} + \frac{\lambda}{2} [U_{i-1,j} - 2U_{i,j} + U_{i+1,j}]$$

→

→ Rotate Standard Five Point formula and generate a new formula:

S-F-P formula is

$$w_{i,j} = \frac{1}{4} [w_{i-1,j} + w_{i+1,j} + w_{i,j-1} + w_{i,j+1}]$$

If we rotate this formula upto 45° ,

then, we have Diagonal 5-point formula

$$\Rightarrow w_{i,j} = \frac{1}{4} [w_{i-1,j} + w_{i+1,j} + w_{i,j-1} + w_{i,j+1}]$$

→ Wave Eq:

→ Heat Eq:

→ Elliptical Eq:

→ Poisson Eq:

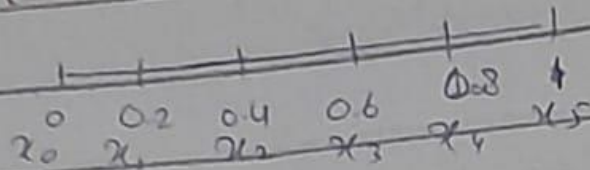
→ Laplace Eq:

→ Gauss Seidel Over Relaxation (SOR) Poisson + Laplace

$$\rightarrow U_i^0 = \sin \pi x$$

$$i = 3, j = 3, m = n = 4.$$

$$U_i^0 = ?$$



$$U_3^0 = \sin \pi x_3$$

$$U_3^0 = \sin \pi (0.6) = \text{---}$$

$\rightarrow$ For applying Fourier Series formula, we have.

$$f_i^j = C_j^i e^{i k x}$$

Question would be asked as
Example # 20.

$\rightarrow$ Apply FWD and Central Differ.