

MTH-646

FINAL TERM

FORMULAS

2023

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MTH-646

FORMULAS FOR FINAL TERMS

Lecture #19 Crout's Algorithm

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7:06 AM Friday

$$l_1 = a_{11}, i=1 \Rightarrow l_i = a_{i1} - \frac{C_{i-1}}{l_{i-1}} b_i$$

$$y_i = d_i / l_i \Rightarrow y_i = \frac{1}{l_i} (d_i - b_i y_{i-1}) \quad \forall i=1, 2, 3, \dots, n$$

$$x_n = y_n \Rightarrow x_i = y_i - \frac{C_i}{l_i} x_{i+1} \quad \forall i=n-1, \dots, 1$$

heat Equation : $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$

Wave Equation : $\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}$

Laplace Equation : $\frac{\partial^2 u(x,y)}{\partial x^2} = \frac{\partial^2 u(x,y)}{\partial y^2}$

Forward Difference : $\frac{u_{i+1,j} - u_{i,j}}{h}$ ~~error~~, and $\frac{u_{i,j+1} - u_{i,j}}{k}$ w.r.t time.

Backward difference : $\frac{u_{i,j} - u_{i-1,j}}{h}$ and $\frac{u_{i,j} - u_{i,j-1}}{k}$ w.r.t time.

Central Difference Method : $\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$ and $\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$ w.r.t 't'.

LU-Decomposition: $A=LU \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Lecture # 20: Doolittle Method of lower upper formula.

here diagonal matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

of 'l' will be 1. d.s

Crank-Nicolson Method: In this method, $\frac{\partial^2 u}{\partial x^2}$ is replaced by average of central finite difference approx as jth and j+1th terms in time steps as,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2} \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} \right)$$

$$\rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{\partial h^2} [u_{i-1,j} + 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}]$$

$\therefore$ Crank Nicolson is also said as Implicit Scheme.

formulas for λ, n, h, m .

$$\lambda = \frac{\alpha^2 k}{h^2}, \quad n = \frac{b-a}{h}, \quad h = \frac{b-a}{n}, \quad m = \frac{T-t_0}{K}$$

$\therefore$ If FWD is not mention in a question related to heat equation, other than Crank-Nicolson Method then we will use FWD formula -

Lecture # 23 Schmidt Method

$$T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{(\Delta x)^2} (T_{i-1}^n - 2T_i^n + T_{i+1}^n)$$

$\therefore$ its two level explicit methods by $n, n+1$.

Durfort-Frankel Method:

$$T_i^{n+1} = \frac{2\lambda}{1+2\lambda} (T_{i-1}^n + T_{i+1}^n) + \frac{1-2\lambda}{1+2\lambda} T_i^{(n-1)}$$

$\therefore$ It's three time level finite difference eq. by $n, n-1, n+1$.

Jacobi's Iterative method: $|a_{ii}| \geq \sum_{j \neq i}^n |a_{ij}|$

a_{ij} denotes entry in i th row and j th column.

Lecture #25: Jacobi's Iterative Method for heat diffusion Equation.

✓ FWD ✓

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} = \frac{1}{2h^2} [U_{i-1,j} - 2U_{i,j} + U_{i+1,j} + U_{i,j+1} - 2U_{i,j} + U_{i,j+1}]$$

may

Crank Nicolson ↑

Q1 Iterative formula for Jacobi:

$$U_i^{n+1} = \frac{\lambda}{2(1+\lambda)} [U_{i-1}^n + U_{i+1}^n] + \frac{C_i}{1+\lambda}$$

where, $C_i = U_{i,j} + \frac{\lambda}{2} [U_{i-1,j} - 2U_{i,j} + U_{i+1,j}]$

Gauss-Seidal Method:

In general, $x_1^{n+1} = \frac{1}{a_{11}} (b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)})$

$$x_2^{n+1} = \frac{1}{a_{22}} (b_2 - a_{21}x_1^{(n+1)} - a_{23}x_3^{(n)})$$

$$x_3^{(n+1)} = \frac{1}{a_{33}} (b_3 - a_{31}x_1^{(n+1)} - a_{32}x_2^{(n+1)})$$

may Write Gauss Seidal^{Q33} iterative method of heat diffusion Equation.

Q1 Iterative Formula for Gauss-Seidal Method:

$$U_i^{(n+1)} = \frac{\lambda}{2(1+\lambda)} [U_{i+1}^{(n)} + U_{i-1}^{(n+1)}] + \frac{C_i}{1+\lambda}$$

where $C_i = U_{i,j} + \frac{\lambda}{2} [U_{i-1,j} - 2U_{i,j} + U_{i+1,j}]$

Lecture #26: Stability Analysis

may $F_w(i, x) = w(x) \exp(i \Omega x_i)$

i is imaginary unit

Ω is Fourier mode.

$w(x)$ is amplitude mod.

Stability Analysis of upwind Scheme:

BWD Scheme: Consider PDE $\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = 0$, $c > 0$ } $\left. \begin{array}{l} \text{inf} \\ \text{max} \end{array} \right\}$

FWD Scheme: $\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = 0$, $c < 0$

Formula: In case $c > 0$, $\frac{\partial f}{\partial t}$ replace by 1st order FWD at $\frac{\partial f(i,j)}{\partial t}$
and BWD for $\frac{\partial f}{\partial x}$
$$\frac{f_i^{j+1} - f_i^j}{\Delta t} + c \frac{f_i^j - f_{i-1}^j}{\Delta x} = 0 \Rightarrow f_i^{j+1} = f_i^j - \frac{c \Delta t}{\Delta x} (f_i^j - f_{i-1}^j)$$

Fourier Method (Van-Neuman): $f_i^j = U_{i,j} = G^j e^{iKx}$

$\bar{I} = \sqrt{-1}$, K = wave no, generally real no.
 $\downarrow$
imaginary mode

$U_{i,j} = G^j e^{iKx}$ is also called amplification factor.

Unstability:

Stability:

$|G| > 1$
unbounded, unstable

$|G| \leq 1$
stable, bounded

$$\therefore 1 - \cos \beta = 2 \sin^2 \beta/2, \quad \text{IK} \Delta x = \beta$$

Lecture #29: Elliptical PDEs

2D Laplace DE: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\text{or } U_{xx} + U_{yy} = 0$$

Imp
SFP
Standard 5-Point formula:

$$U_{i,j} = \frac{1}{4} [U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}]$$

Diagonal 5-Point formula: when we rotate SFP formula upto 45° , then we have a new formula, as

$$U_{i,j} = \frac{1}{4} [U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i+1,j+1}]$$

Lecture #35:

me $\rightarrow$ 2D, 2nd order DE.
Elliptical Equation has two types.

Poisson Equation

The equation is called Poisson eq if $\nabla^2 U \neq 0$ as

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x)$$

Laplace Equation.

The eq is called Laplace eq if $\nabla^2 U = 0$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

Lecture #36:

Standard 5-Point formula for Laplace Eq: $h=k$ (uniform)

$$4W_{i,j} - (W_{i-1,j} + W_{i+1,j}) - (W_{i,j-1} + W_{i,j+1}) = 0$$

Mesh point formula.

$$I = i + (m-1-j)(n-1)$$

Standard 5-Point formula for Laplace Eq: $k \neq h$ (non-uniform).

$$\frac{2}{k^2} \left(\frac{h^2}{k^2} + 1 \right) W_{i,j} - (W_{i-1,j} + W_{i+1,j}) - (W_{i,j-1} + W_{i,j+1}) = 0$$

Lecture #37: Part (U)

Standard 5-point formula for ~~me~~ for Poisson Eq:

$$2 \left(\frac{h^2}{k^2} + 1 \right) W_{i,j} - (W_{i-1,j} + W_{i+1,j}) - (W_{i,j-1} + W_{i,j+1}) = -h^2 f_{i,j}, \quad h \neq k$$

$$4W_{i,j} - (W_{i-1,j} + W_{i+1,j}) - (W_{i,j-1} + W_{i,j+1}) = -\frac{1}{4} f_{i,j}, \quad h=k$$

Lecture # 37 Part (II) LU-Decomposition Example.

msg Non-Uniform Grid Point:

$$2 \left(\frac{h^2}{k^2} + 1 \right) w_{i,j} - (w_{i-1,j} + w_{i+1,j}) - \frac{h^2}{k^2} (w_{i,j-1} + w_{i,j+1}) = -h^2 f_{i,j}$$

Lecture #38: Diagonal 5 point formula for Laplace Eq:

$\therefore h=k$ (uniform grid point).

$$\begin{aligned} & 4w_{i,j} - (w_{i-1,j} + w_{i+1,j}) - (w_{i,j-1} + w_{i,j+1}) = 0 \\ & \text{or } w_{i,j} = \frac{1}{4} [w_{i-1,j} + w_{i+1,j} + w_{i,j-1} + w_{i,j+1}] \end{aligned} \quad \left. \vphantom{\begin{aligned} & 4w_{i,j} - (w_{i-1,j} + w_{i+1,j}) - (w_{i,j-1} + w_{i,j+1}) = 0 \\ & \text{or } w_{i,j} = \frac{1}{4} [w_{i-1,j} + w_{i+1,j} + w_{i,j-1} + w_{i,j+1}] \end{aligned}} \right\} \text{Standard}$$

and,

$$\text{msg } w_{i,j} = \frac{1}{4} [w_{i-1,j-1} + w_{i+1,j-1} + w_{i-1,j+1} + w_{i+1,j+1}] \quad \checkmark \text{ Rotated}$$

$\therefore$ If $h \neq k$, then it will be non-uniform and it will be a poisson equation.

✓ Gauss-Seidal method on Standard 5-point method:

$$w_{i,j}^{(n+1)} = \frac{1}{4} [w_{i-1,j}^{(n+1)} + w_{i+1,j}^{(n)} + w_{i,j-1}^{(n+1)} + w_{i,j+1}^{(n)}]$$

Lecture #39: SOR (Successive Over Relaxation)

Gauss Seidal Method for Laplace Eq:

msg

$$u_{i,j}^{(n+1)} = \frac{1}{4} \omega [u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}]$$

$\therefore \omega = \text{angular Acceleration}$

SOR for Poisson Equation: $\nabla^2 U = f(x, y)$

$$U_{i,j} = \frac{1}{4} [U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} + h^2 f_{i,j}(x, y)]$$

SOR for Jacobi's method:

$$U_{i,j}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n)} + U_{i,j+1}^{(n)}] + h^2 f_{i,j}(x, y)$$

SOR for Gauss Seidel Method:

$$U_{i,j}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n+1)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n+1)} + U_{i,j+1}^{(n)}] + h^2 f_{i,j}(x, y)$$

Iterative form of SOR: $U_{i,j}^{(n+1)} = U_{i,j}^{(n)} + \frac{1}{4} R_{i,j}$, $\omega \in [1, 2]$

$$U_{i,j}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n+1)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n+1)} + U_{i,j+1}^{(n)}] + (1-\omega) U_{i,j}^{(n)} \leftarrow \text{Gauss Seidel for Laplace eq}$$

SOR for Jacobi's method for Poisson eq:

$$U_{i,j}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n)} + U_{i,j+1}^{(n)} + h^2 f_{i,j}(x, y)] + (1-\omega) U_{i,j}^{(n)}$$

SOR for Gauss Seidel method for Poisson eq:

$$U_{i,j}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n+1)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n+1)} + U_{i,j+1}^{(n)} + h^2 f_{i,j}(x, y)] + (1-\omega) U_{i,j}^{(n)}$$

Internal mesh points by latest (Hx)th iteration:

For Jacobi's

$$U_{i,j}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n)} + U_{i,j+1}^{(n)} + h^2 f_{i,j}(x, y)]$$

Lecture #40:

Jacobi's SOR using Rotated/Diagonal five point formula for Poisson eq:

$$U_{ij}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n)} + U_{i,j+1}^{(n)} + h^2 f_{ij}(x,y)] + (1-\omega)U_{ij}^{(n)}$$

Gauss Seidel SOR using Diagonal/Rotated 5-point formula for Poisson eq:

$$U_{ij}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n+1)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n+1)} + U_{i,j+1}^{(n)} + h^2 f_{ij}(x,y)] + (1-\omega)U_{ij}^{(n)}$$

ADI (Alternating Direction Implicit) Method.

⇒ Standard 5-point formula for L-B:

$$\text{Step I: } 4w_{ij} - w_{i-1,j} - w_{i+1,j} - w_{i,j-1} - w_{i,j+1} = 0$$

$$\Rightarrow 4w_{ij} - w_{i-1,j} - w_{i+1,j} = w_{i,j-1} + w_{i,j+1} \quad \text{--- (1)}$$

$$\Rightarrow 4w_{ij} - w_{i,j-1} - w_{i,j+1} = w_{i-1,j} + w_{i+1,j} \quad \text{--- (2)}$$

eq. (1), (2) can be written as

$$\textcircled{1} \Rightarrow w_{i-1,j} - 4w_{ij} + w_{i+1,j} = -w_{i,j-1} - w_{i,j+1} \quad \text{--- (3)}$$

$$\textcircled{2} \Rightarrow w_{i,j-1} - 4w_{ij} + w_{i,j+1} = -w_{i-1,j} - w_{i+1,j} \quad \text{--- (4)}$$

can be used as iterative formula.
Ans.

$$\text{jth term: } j=1, 2, 3, \dots, n-1$$

$$w_{i,j}^{(n+1)} = 4w_{ij}^{(n)} + w_{i+1,j}^{(n+1)} = -w_{i,j-1}^{(n)} - w_{i,j+1}^{(n)} \quad \left. \begin{array}{l} \text{--- (3)} \\ \text{--- (4)} \end{array} \right\} \Rightarrow \text{write ADI formula}$$

$$\text{ith term: } i=1, 2, 3, \dots, n-1$$

$$w_{i,j-1}^{(n+1)} - 4w_{ij}^{(n+1)} + w_{i,j+1}^{(n+1)} = -w_{i-1,j}^{(n+1)} - w_{i+1,j}^{(n+1)}$$

lect #20 Algorithms for Doolittle method.

$$L_{11} = a_{11} \Rightarrow L_{12} = a_{12}/L_{11}$$

$$\text{For } i=1, 2, 3, \dots, n-1, \Rightarrow L_{i,i-1} = a_{i,i-1} \Rightarrow L_{ii} = a_{ii} - L_{i,i-1}U_{i-1,i}$$

$$\Rightarrow U_{i,i+1} = a_{i,i+1}/L_{ii} \Rightarrow \text{End.} \Rightarrow L_{n,n-1} = a_{n,n-1}$$

$$\Rightarrow L_{n,n} = a_{n,n} - L_{n,n-1}U_{n-1,n}$$

Jacobi's SOR method using SFPP For Poisson Eq.

$$U_{ij}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n)} + U_{i,j+1}^{(n)} + h^2 f_{ij}(x,y)] + (1-\omega) U_{ij}^{(n)}$$

Gauss Seidal " " " " "

$$U_{ij}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j}^{(n+1)} + U_{i+1,j}^{(n)} + U_{i,j-1}^{(n)} + U_{i,j+1}^{(n)} + h^2 f_{ij}(x,y)] + (1-\omega) U_{ij}^{(n)}$$

Jacobi's SOR method using Rotated 5-point formula for P.E:

$$U_{ij}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j-1}^{(n)} + U_{i+1,j-1}^{(n)} + U_{i-1,j+1}^{(n)} + U_{i+1,j+1}^{(n)} + h^2 f_{ij}(x,y)] + (1-\omega) U_{ij}^{(n)}$$

Gauss Seidal " " " " "

$$U_{ij}^{(n+1)} = \frac{1}{4} \omega [U_{i-1,j+1}^{(n+1)} + U_{i+1,j-1}^{(n+1)} + U_{i-1,j-1}^{(n)} + U_{i+1,j+1}^{(n)} + h^2 f_{ij}(x,y)] + (1-\omega) U_{ij}^{(n)}$$

In these four eq if we remove $h^2 f_{ij}(x,y)$, then the above eqs will be Laplace eqs-

Lecture#40: Numerical Solution of Hyperbolic PDE by Finite Difference Method

$$U_i^{n+1} = 2(1-\gamma^2 c^2) U_i^n + \gamma^2 c^2 (U_{i-1}^n + U_{i+1}^n) - U_i^{n-1}$$

Iterative Scheme for it:

$$U_i^1 = (1-\gamma^2 c^2) U_i^0 + \frac{1}{2} \gamma^2 c^2 (U_{i-1}^0 + U_{i+1}^0)$$

where

$$h = \frac{l-0}{n}, \quad K = \frac{l-0}{m},$$

Method of Lines for Heat Equation:

$$\left(\frac{du_i}{dt}\right) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + o(h^2) \quad i=1, 2, \dots, n-1$$

$$\Rightarrow \left(\frac{d\bar{u}}{dt}\right) = \bar{A}\bar{u} = \bar{A}(t)\bar{u}(t)$$

$$\text{or } \left(\frac{1}{u}\right) du = A dt$$

Integrating both sides

$$\int \frac{1}{u} du = \int A dt \Rightarrow \ln u = \int A dt + C$$



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