

VIRTUAL UNIVERSITY OF PAKISTAN

MTH647

Methods in Mathematical Physics

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LECTURE NO. 60

Fourier Transform of Unit Step Function:

Find the Fourier series of

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Solution: Fourier Transform of $f(x)$

$$F(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

For given problem,

$$F(\alpha) = \int_{-a}^a 1 \cdot e^{-i\alpha u} du \quad \dots (1)$$

$$F(\alpha) = \left[\frac{e^{-i\alpha u}}{-i\alpha} \right]_{-a}^a \Rightarrow \frac{e^{-i\alpha a} - e^{-i\alpha(-a)}}{-i\alpha} \Rightarrow \frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha}$$

Divide and multiply by 2,

$$\frac{2}{\alpha} \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i} \right]$$

Using relation " $\sin(x) = \frac{e^{i(x)} - e^{-i(x)}}{2i}$ " we have

$$F(\alpha) = \frac{2}{\alpha} \sin \alpha a$$

It is the required function when $\alpha \neq 0$.

If we put $\alpha = 0$ in equation (1), we get

$$F(\alpha) = \int_{-a}^a 1 du \Rightarrow 2a$$

LECTURE NO. 61

Fourier Transform of Exponential Functions:

Fourier function of

$$e^{-px^2}, p > 0$$

Solution: As we know from previous lecture,

$$F(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

For given problem,

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-pu^2} \cdot e^{-i\alpha u} du$$

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-p\left\{u^2 - \frac{i\alpha}{p}u\right\}} du \Rightarrow \int_{-\infty}^{\infty} e^{-p\left\{u^2 - \frac{i\alpha}{p}u + \left(\frac{i\alpha}{2p}\right)^2 - \left(\frac{i\alpha}{2p}\right)^2\right\}} du$$

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-p\left\{\left(u - \frac{i\alpha}{2p}\right)^2 + \frac{\alpha^2}{4p^2}\right\}} du \Rightarrow e^{\frac{\alpha^2}{4p^2}} \int_{-\infty}^{\infty} e^{-p\left(u - \frac{i\alpha}{2p}\right)^2} du$$

Let

$$\sqrt{p} \left(u - \frac{i\alpha}{2p} \right) = y$$

$$dy = \sqrt{p} du \Rightarrow du = \frac{1}{\sqrt{p}} dy$$

By putting in above relation, we have

$$F(\alpha) = \frac{e^{\frac{\alpha^2}{4p^2}}}{\sqrt{p}} \int_{-\infty}^{\infty} e^{-y^2} dy \Rightarrow \frac{e^{\frac{\alpha^2}{4p^2}}}{\sqrt{p}} \sqrt{\pi} \Rightarrow \sqrt{\frac{\pi}{p}} e^{\frac{\alpha^2}{4p^2}}$$

This is our required result.

LECTURE NO. 62

Conjugate of a Fourier Series:

Theorem: For a real valued function $f(x)$ on $(-\infty, \infty)$, show that $\overline{F(\alpha)} = F(-\alpha)$

Proof: As we know

$$F(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

Taking conjugate, we have

$$\begin{aligned} \overline{F(\alpha)} &= \overline{\int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du} \Rightarrow \int_{-\infty}^{\infty} \overline{f(u)} (\overline{e^{-i\alpha u}}) du \Rightarrow \int_{-\infty}^{\infty} f(u) e^{i\alpha u} du \\ \overline{F(\alpha)} &= \int_{-\infty}^{\infty} f(u) e^{-i(-\alpha)u} du \Rightarrow \int_{-\infty}^{\infty} f(u) e^{-i(-\alpha)u} du \Rightarrow F(-\alpha) \end{aligned}$$

Hence proved that

$$\overline{F(\alpha)} = F(-\alpha)$$

LECTURE NO. 63

Fourier Transforms of Even and Odd Functions:

Let $f(x)$ be an odd function

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du = F(\alpha)$$

Here put $u = -t \Rightarrow du = -dt$

As $u \rightarrow \infty$, $t \rightarrow \infty$ and $u \rightarrow -\infty \Rightarrow t \rightarrow \infty$

$$F(\alpha) = - \int_{-\infty}^{\infty} f(-t) e^{-i\alpha(-t)} dt \Rightarrow - \int_{-\infty}^{\infty} f(t) e^{-i\alpha(-t)} dt = - \int_{-\infty}^{\infty} f(t) e^{-i(-\alpha)t} dt$$

Hence proved that

$$F(\alpha) = -F(-\alpha) = -\overline{F(\alpha)}$$

LECTURE NO. 64

Attenuation Property of Fourier Transforms:

Theorem:

$$F\{f(x) e^{-px}\} = F(\alpha - pi) \quad \text{where} \quad F(\alpha) = F\{f(x)\}$$

Proof: As we know

$$F\{f(x)\} = F(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u} du$$

For given function,

$$\begin{aligned} F\{f(x) e^{-px}\} &= \int_{-\infty}^{\infty} f(u) e^{-pu} e^{-i\alpha u} du \\ &= \int_{-\infty}^{\infty} f(u) e^{i^2 pu} e^{-i\alpha u} du \Rightarrow \int_{-\infty}^{\infty} f(u) e^{i^2 pu - i\alpha u} du \end{aligned}$$

$$F\{f(x)e^{-px}\} = \int_{-\infty}^{\infty} f(u)e^{-i(\alpha-ip)u} du = F(\alpha - ip)$$

Hence proved that

$$F\{f(x)e^{-px}\} = F(\alpha - pi)$$

LECTURE NO. 65

Shifting Property of Fourier Transform:

Theorem:

$$\text{If } F\{f(x)\} = F(\alpha) \quad \text{then} \quad F\{f(x-p)\} = e^{-iap} F(\alpha)$$

Proof: As we know

$$F(\alpha) = F\{f(x)\} = \int_{-\infty}^{\infty} f(u)e^{-i\alpha u} du$$

For given function,

$$F\{f(x-p)\} = \int_{-\infty}^{\infty} f(u-p)e^{-i\alpha u} du$$

Put $u-p=t \Rightarrow u=p+t$ and $dt=du$

$$\begin{aligned} F\{f(x-p)\} &= \int_{-\infty}^{\infty} f(t)e^{-i\alpha(p+t)} dt \Rightarrow \int_{-\infty}^{\infty} f(t)e^{-i\alpha p} e^{-i\alpha t} du \\ &= e^{-i\alpha p} \int_{-\infty}^{\infty} f(t)e^{-i\alpha t} du \Rightarrow e^{-i\alpha p} F\{f(x)\} = e^{-i\alpha p} F(\alpha) \end{aligned}$$

This is our required result.

LECTURE NO. 66

Fourier Transform of Derivatives:

Theorem: For a function $f(x)$ on $(-\infty, \infty)$, show that if $f(x)$ is n -times differentiable and $f^{n-1}(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, then

$$F\{f^{(n)}(x)\} = (i\alpha)^n F(u)$$

Proof:

$$F\left\{\frac{d^n y}{du^n}\right\} = \int_{-\infty}^{\infty} \frac{d^n y}{du^n} e^{-i\alpha u} du$$

By integrating,

$$\begin{aligned} &= e^{-i\alpha u} \left[\frac{d^{n-1}y}{du^{n-1}} \right]_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} \frac{d^{n-1}y}{du^{n-1}} e^{-i\alpha u} du \\ &= 0 + (i\alpha)^1 \left[\left\{ e^{-i\alpha u} \frac{d^{n-2}y}{du^{n-2}} \right\}_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} \frac{d^{n-2}y}{du^{n-2}} e^{-i\alpha u} du \right] \\ &= (i\alpha)^1 \left[0 + (i\alpha) F\left\{ \frac{d^{n-2}y}{du^{n-2}} \right\} \right] \Rightarrow (i\alpha)^2 F\left\{ \frac{d^{n-2}y}{du^{n-2}} \right\} \end{aligned}$$

Now by induction,

$$F\left\{\frac{d^n y}{du^n}\right\} = (i\alpha)^n F\{f(x)\} = (i\alpha)^n F(u)$$

Hence proved that

$$F\{f^{(n)}(x)\} = (i\alpha)^n F(u)$$

LECTURE NO. 67

Parseval's Theorems:

Theorem: If $f(x)$ and $g(x)$ are real valued on $(-\infty, \infty)$, then show

$$I) \int_{-\infty}^{\infty} F(\alpha) G(-\alpha) d\alpha = \int_{-\infty}^{\infty} f(u) g(u) du$$

$$II) \int_{-\infty}^{\infty} |f(u)|^2 du = \int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha$$

Proof: I)

$$\begin{aligned} \int_{-\infty}^{\infty} F(\alpha) G(-\alpha) d\alpha &= \int_{-\infty}^{\infty} F(\alpha) \left\{ \int_{-\infty}^{\infty} g(u) e^{-i(-\alpha)u} du \right\} \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha u} d\alpha \right\} g(u) du = \int_{-\infty}^{\infty} f(u) g(u) du \end{aligned}$$

II): Putting $f = g$

$$\begin{aligned} \int_{-\infty}^{\infty} F(\alpha) G(-\alpha) d\alpha &= \int_{-\infty}^{\infty} F(\alpha) \overline{G(\alpha)} d\alpha = \int_{-\infty}^{\infty} F(\alpha) G(\alpha) d\alpha = \int_{-\infty}^{\infty} f(u) g(u) du \\ \Rightarrow \int_{-\infty}^{\infty} |G(\alpha)|^2 d\alpha &= \int_{-\infty}^{\infty} |g(u)|^2 du \text{ Or } \int_{-\infty}^{\infty} |f(u)|^2 du = \int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha \end{aligned}$$

Required result proved.

LECTURE NO. 68

Convolution (Definition and Related Theorem):

$$F(f) = F(\alpha) \quad , \quad G(g) = G(\alpha)$$

$$F(f \cdot g) \neq F(\alpha) G(\alpha)$$

$$F(f * g) = F(\alpha) G(\alpha)$$

Convolution: of function $f(x)$ and $g(x)$ is defined and given,

$$f * g = \int_{-\infty}^{\infty} f(u) g(x - u) du$$

Now we have to prove some theorems.

1) Commutative ($f * g = g * f$): By definition

$$f * g = \int_{-\infty}^{\infty} f(u) g(x - u) du$$

Put $x - u = t, u = x - t \Rightarrow du = -dt$ and as $\begin{cases} u \rightarrow \infty \Rightarrow t \rightarrow \pm\infty \\ u \rightarrow -\infty \Rightarrow t \rightarrow \infty \end{cases}$

So, we get

$$f * g = \int_{+\infty}^{-\infty} f(x - t) g(t) (-dt) \Rightarrow \int_{-\infty}^{\infty} g(t) f(x - t) (dt) = g * f$$

Hence proved

$$f * g = g * f$$

2) $F\{f * g\} = F(\alpha) G(\alpha)$ Or $F^{-1}[F(\alpha) G(\alpha)] = f * g$

By definition,

$$F^{-1}[F(\alpha) G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) G(\alpha) e^{i\alpha x} d\alpha \Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \cdot G(\alpha)$$

As we know (from previous lectures)

$$G(\alpha) = \int_{-\infty}^{\infty} g(t) e^{-iat} dt$$

By putting, we have

$$F^{-1}[F(\alpha) G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} d\alpha \int_{-\infty}^{\infty} g(t) e^{-iat} dt$$

$$F^{-1}[F(\alpha) G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) dt \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} e^{-iat} dt$$

$$F^{-1}[F(\alpha) G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) dt \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha(x-t)} dt$$

By using inverse Fourier transformation,

$$F^{-1}[F(\alpha) G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) f(x-t) dt \Rightarrow g * f$$

As " $g * f = f * g$ " so

$$F^{-1}[F(\alpha) G(\alpha)] = f * g$$

Applying " F " on both sides, we get

$$F\{f * g\} = F(\alpha) G(\alpha)$$

Examples:

$$\begin{aligned} i): \quad f(x) &= e^{-x^2} ; \quad g(x) = \begin{cases} 1 & |x| \leq p \\ 0 & |x| \geq p \end{cases} \\ ii) \quad f(x) &= e^{-\alpha x^2} ; \quad g(x) = e^{-\beta x^2} ; \quad \alpha, \beta > 0 \end{aligned}$$

Do Your Self.

LECTURE NO. 69

Solving Integral Equation by Convolution:

Solve the integral equation,

$$y(x) = g(x) + \int_{-\infty}^{\infty} y(u) r(x-u) du$$

Solution: For the given integral equation, taking fourier transformation on both sides,

$$F(y(x)) = F\{g(x)\} + F\left\{\int_{-\infty}^{\infty} y(u) r(x-u) du\right\}$$

Convolution of last function,

$$\int_{-\infty}^{\infty} y(u) r(x-u) du = y(x) * r(x)$$

So, above relation can be writtana as

$$F(y(x)) = F\{g(x)\} + F\{y(x) * r(x)\} \dots (1)$$

$$\text{Let } \begin{cases} F(y(x)) = Y(\alpha) \\ F\{g(x)\} = G(\alpha) \\ F\{r(x)\} = R(\alpha) \end{cases} \quad \text{By putting equation (1) becomes}$$

$$Y(\alpha) = G(\alpha) + F\{y(x) * r(x)\} \Rightarrow G(\alpha) + Y(\alpha)R(\alpha)$$

By solving we have

$$Y(\alpha) = \frac{G(\alpha)}{1 - R(\alpha)}$$

Taking inverse fourier transformation,

$$F^{-1}\{Y(\alpha)\} = F^{-1}\left\{\frac{G(\alpha)}{1-R(\alpha)}\right\} \Rightarrow \int_{-\infty}^{\infty} \left(\frac{G(\alpha)}{1-R(\alpha)}\right) e^{i\alpha x} dx$$

Since $F \rightarrow 1$ then $F^{-1} \rightarrow \frac{1}{2\pi}$ and here $F^{-1}\{Y(\alpha)\} = y(x)$

So the above equation becomes,

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{\frac{G(\alpha)}{1-R(\alpha)}\right\} e^{i\alpha x} dx$$

This is our required result.

LECTURE NO. 70

Fourier Sine and Cosine Transforms:

I): If $f(x)$ is an odd function, then Fourier *sine* transform is defined and given by

$$F_s(\alpha) = \int_0^{\infty} f(u) \sin \alpha u du$$

And its inverse Fourier *sine* transform is given by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\alpha) \sin \alpha u d\alpha$$

II): If $f(x)$ is an even function, then Fourier *cosine* transform of $f(x)$ is defined and given by

$$F_c(\alpha) = \int_0^{\infty} f(u) \cos \alpha u du$$

And its inverse Fourier *cosine* transform is given by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\alpha) \cos \alpha u d\alpha$$

Example:

$$\text{If; } f(x) = e^{-mx}, \quad m > 0 \quad \text{then} \quad F_c(\alpha) = ?$$

As we know

$$F_c(\alpha) = \int_0^{\infty} f(u) \cos \alpha u du$$

For given function, it can be written as

$$F_c(\alpha) = \int_0^{\infty} e^{-mu} \cos \alpha u du \quad \dots (1)$$

Using integral formula, which is given as

$$\left[\int e^{px} \cos qx dx = \frac{e^{px}}{p^2 + q^2} \{p \cos qx + q \sin qx\} \right]$$

Here $p \rightarrow -m$, $q \rightarrow \alpha$ and $x \rightarrow u$

So, by applying integral formula, equation (1) becomes

$$F_c(\alpha) = \left[\frac{e^{-mu}}{m^2 + \alpha^2} \{-m \cos \alpha u + \alpha \sin \alpha u\} \right]_0^{\infty}$$

By applying limits, we get

$$F_c(\alpha) = \frac{m}{m^2 + \alpha^2} = F_c\{e^{-mx}\}$$

Now its inverse, *cosine* transform;

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\alpha) \cos \alpha u d\alpha$$

For given function, it can be written as (Putting $F_c(\alpha)$ calculated value)

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{m}{m^2 + \alpha^2} \cos \alpha u \, d\alpha$$

As $f(x) = e^{-mx}$,

$$e^{-mx} = \frac{2m}{\pi} \int_0^{\infty} \frac{\cos \alpha u}{m^2 + \alpha^2} \, d\alpha$$

$$\frac{\pi e^{-mx}}{2m} = \int_0^{\infty} \frac{\cos \alpha u}{m^2 + \alpha^2} \, d\alpha$$

This is our required result.

LECTURE NO. 71

Integral Equations Solution by Fourier Sine Transform:

Solve the integral equation

$$\int_0^{\infty} f(x) \sin \alpha x \, dx = \begin{cases} 1 - \alpha & 0 \leq \alpha \leq 1 \\ 0 & \alpha > 1 \end{cases}$$

Solution: The given function can be written as

$$F_s\{f(\alpha)\} = F_s(\alpha) = \int_0^{\infty} f(x) \sin \alpha x \, dx = \begin{cases} 1 - \alpha & 0 \leq \alpha \leq 1 \\ 0 & \alpha > 1 \end{cases}$$

Its F^{-1} for given function,

$$F_s^{-1}\{f(x)\} = f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\alpha) \sin \alpha x \, d\alpha$$

$$f(x) = \frac{2}{\pi} \int_0^1 (1 - \alpha) \sin \alpha x \, d\alpha$$

Integrating and applying limits, we have

$$= \frac{2}{\pi} \left[\left\{ (1 - \alpha) \left(-\frac{\cos \alpha x}{x} \right) \right\}_0^1 + \int_0^1 \frac{\cos \alpha x}{x} (-1) \, d\alpha \right]$$

$$= \frac{2}{\pi} \left[0 - (1) \left(-\frac{1}{x} \right) - \frac{1}{x} \left| \frac{\sin \alpha x}{x} \right|_0^1 \right] \Rightarrow \frac{2}{\pi} \left[\frac{1}{x} - \frac{\sin x}{x^2} \right]$$

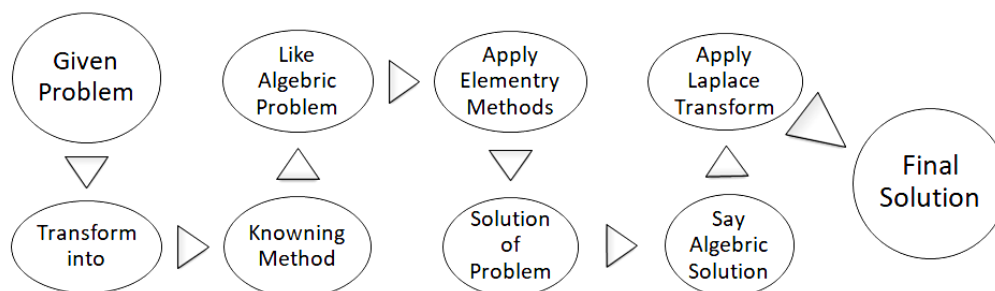
$$f(x) = \frac{2}{\pi} \left(\frac{x - \sin x}{x^2} \right)$$

This is our required result.

LECTURE NO. 72

Laplace Transforms:

Objectives/Aim/Steps:



Usefulness:

Deals with discontinuous functions (like electrical signals, mechanical forces etc) and sine, cosine forms (like Fourier analysis).

It is Direct method to solve differential equations.

Applications:

Physics, Engineering Problems → Transform into → Differential Equations (Like ODEs, PDEs, BVPs, etc.)

LECTURE NO. 73

Integral Transform:

For given function $f(t)$ defined on $[0, \infty)$ or $t \geq 0$, if improper function

$$\int_0^{\infty} K(s, t) f(t) dt$$

Is convergent, then it is called an integral transformation of $f(t)$ where $s \in C$. And if chosen

$$K(s, t) = e^{-st} \quad \text{then} \quad F(s) = L(f(t)) = \int_0^{\infty} e^{-st} f(t) dt, \quad s \in C.$$

Is said to be Laplace transform of $f(t)$ provided that it is convergent. Here $K(s, t) = e^{-st}$ is known as Kernel.

Q: Why $K(s, t) = e^{-st}$? Why we choose/use Kernel?

Ans: 1) Kernel smooth out the given functions (Like sharp edges, corners etc.)

2) Easy to integrate or differentiate.

3) By kernel, differential equation transform into algebraic equation. So that it can be easily solved.

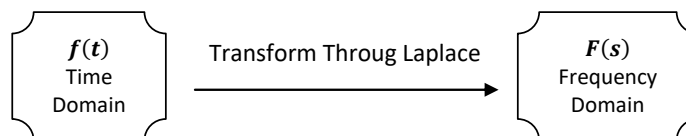
LECTURE NO. 74

Geometrical Interpretation of Laplace Transform:

Since we know

$$L(f(t)) = F(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad t \geq 0, \quad s \in C, R.$$

Transformation of $f(t)$ into $f(s)$:



$F(s)$ gives us information about the different frequency components that makes the $f(t)$.

Laplace relations:

$$\begin{aligned} L(f(t)) &= F(s) \\ f(t) &= L^{-1}(F(s)) \\ L^{-1}[L(f(t))] &= f(t) \\ L[L^{-1}(f(t))] &= F(s) \end{aligned}$$

LECTURE NO. 75

Laplace Transform of "1":

Since we know

$$L(f(t)) = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

For given function,

$$L(1) = \int_0^{\infty} 1 \cdot e^{-st} dt \Rightarrow \lim_{T \rightarrow \infty} \int_0^T e^{-st} dt$$

Integrating and applying limits, we have

$$\begin{aligned} L(1) &= \left[\lim_{T \rightarrow \infty} \frac{e^{-st}}{-s} \right]_0^T \Rightarrow \left[-\frac{1}{s} \lim_{T \rightarrow \infty} e^{-st} \right]_0^T \Rightarrow -\frac{1}{s} \left[\lim_{T \rightarrow \infty} \frac{1}{e^{sT}} \right]_0^T \dots (1) \\ L(1) &= -\frac{1}{s} \left[\lim_{T \rightarrow \infty} \frac{1}{e^{sT}} - \frac{1}{e^{s(0)}} \right] \Rightarrow -\frac{1}{s} \left[\frac{1}{e^{\infty}} - \frac{1}{1} \right] \\ L(1) &= \frac{1}{s} \end{aligned}$$

This is our required result for $s > 0$. What if $s < 0$?

$$\text{If; } s < 0 \Rightarrow -s > 0 \Rightarrow -s = m \text{ (say)} \in \mathbb{R}^+$$

If so, then equation (1) becomes

$$\lim_{T \rightarrow \infty} \frac{1}{e^{sT}} = \lim_{T \rightarrow \infty} e^{-st} = \lim_{T \rightarrow \infty} e^{mT} \Rightarrow e^{\infty} = \infty$$

So, whenever $s < 0$, we cannot calculate the transform of "1". Hence, we can take only $s > 0$.

LECTURE NO. 76

Laplace Transform of Exponential Function:

Let $f(t) = e^{at}$. Here "a" is a constant, then evaluate $L(e^{at})$

Solution: Laplace transform for given function,

$$L(e^{at}) = \int_0^{\infty} e^{-st} e^{at} dt \Rightarrow \int_0^{\infty} e^{-(s-a)t} dt$$

Integrating and applying limits, we have

$$\begin{aligned} L(e^{at}) &= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} \Rightarrow \frac{-1}{(s-a)} [e^{-(s-a)t}]_0^{\infty} \\ L(e^{at}) &= \frac{-1}{(s-a)} \left[\frac{1}{e^{(s-a)t}} \right]_0^{\infty} = \frac{-1}{(s-a)} \left[\frac{1}{e^{\infty}} - \frac{1}{e^0} \right] \\ L(e^{at}) &= \frac{1}{(s-a)} \end{aligned}$$

This is our required result. (For $s - a > 0 \Rightarrow s > a$)

LECTURE NO. 77

Linearity of Laplace Transform:

Theorem: If $L(f(t))$ and $L(g(t))$ exist, then for any constants "α" and "β", transform of

$$\alpha L(f(t)) + \beta L(g(t))$$

Also exist and further

$$L[\alpha (f(t)) + \beta (g(t))] = \alpha L(f(t)) + \beta L(g(t))$$

Also holds.

Proof: Applying Laplace transform on L.H.S.

$$\begin{aligned} &= L[\alpha (f(t)) + \beta (g(t))] \\ &= \int_0^{\infty} e^{-st} [\alpha (f(t)) + \beta (g(t))] dt \Rightarrow \int_0^{\infty} [\alpha \{e^{-st} f(t)\} + \beta \{e^{-st} g(t)\}] dt \end{aligned}$$

∴ Integration is a linear transformation on $\mathbb{R}$.

$$= \int_0^{\infty} \alpha e^{-st} f(t) dt + \int_0^{\infty} \beta e^{-st} g(t) dt \Rightarrow \alpha \int_0^{\infty} e^{-st} f(t) dt + \beta \int_0^{\infty} e^{-st} g(t) dt$$

$$= \alpha L(f(t)) + \beta L(g(t)) = R.H.S$$

Required result proved.

$$\begin{aligned} &\therefore L(f(t)) \text{ and } L(g(t)) \text{ exist.} \\ \Rightarrow \alpha L(f(t)) + \beta L(g(t)) < \infty &\text{ also exist. (Improper Integral)} \end{aligned}$$

LECTURE NO. 78

Corollary:

$$L[\alpha \{f(t)\} + \beta \{g(t)\}] = \alpha L\{f(t)\} + \beta L\{g(t)\}$$

If $\beta = 0$;

$$L[\alpha \{f(t)\}] = \alpha L\{f(t)\}$$

Here we use "Scalar Composition" property.

This property also used in calculus. For example:

$$\begin{cases} \text{Linear Transformation: } T(\alpha v) = \alpha(Tv) \\ \text{Derivative: } \frac{d}{dx}(\alpha \sin x) = \alpha \frac{d}{dx} \sin x \\ \text{Integration: } \int c f(x) dx = c \int f(x) dx \end{cases}$$

LECTURE NO. 79

Laplace Transform of Hyperbolic Function:

As we know

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

Calculate: $L(\cosh at)$, where "a" is a constant

Solution: As given

$$L(\cosh at) = L\left[\frac{1}{2}(e^{at} + e^{-at})\right] \Rightarrow \frac{1}{2}[L(e^{at} + e^{-at})]$$

$$L(\cosh at) = \frac{1}{2}[L(e^{at}) + L(e^{-at})] \Rightarrow \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$\therefore L(e^{at}) = \frac{1}{s-a} \text{ and } L(e^{-at}) = \frac{1}{s+a}, \text{ so}$$

$$L(\cosh at) = \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] \Rightarrow \frac{s}{s^2-a^2}$$

This is our required result.

LECTURE NO. 80

Laplace Transform of Cosine and Sine:

Show that

$$L(\cos at) = \frac{s}{s^2 + a^2} \quad \text{and} \quad L(\sin at) = \frac{a}{s^2 + a^2}$$

Proof: Suppose that

$$L(\cos at) = F_c \quad \text{and} \quad L(\sin at) = F_s$$

$L(\cos at)$: Applying Laplace Transform

$$F_c = \int_0^{\infty} e^{-st} (\cos at) dt$$

Integrating and applying limits, we have

$$F_c = \left[\cos at \left(\frac{e^{-st}}{-s} \right) \right]_0^{\infty} - \int_0^{\infty} \left(\frac{e^{-st}}{-s} \right) (-a \sin at) dt$$

$$F_c = -\frac{1}{s} \left[\frac{\cos at}{e^{st}} \right]_0^{\infty} - \frac{a}{s} \int_0^{\infty} e^{-st} (\sin at) dt$$

$$F_c = -\frac{1}{s} \left[\frac{\cos \infty}{e^\infty} - \frac{\cos(0)}{e^0} \right] - \frac{a}{s} (L(\sin at))$$

$$F_c = -\frac{1}{s} [0 - 1] - \frac{a}{s} F_s$$

$$F_c = \frac{1}{s} - \frac{a}{s} F_s \quad \dots (1)$$

$L(\sin at)$:

$$F_s = \int_0^\infty e^{-st} (\sin at) dt \Rightarrow \left| \sin at \left(\frac{e^{-st}}{-s} \right) \right|_0^\infty - \int_0^\infty \left(\frac{e^{-st}}{-s} \right) (a \cos at) dt$$

$$F_s = -\frac{1}{s} \left[\frac{\sin at}{e^{st}} \right]_0^\infty + \frac{a}{s} \int_0^\infty e^{-st} (\cos at) dt$$

$$F_s = -\frac{1}{s} \left[\frac{\sin \infty}{e^\infty} - \frac{\sin(0)}{e^0} \right] + \frac{a}{s} (L(\cos at))$$

$$F_s = -\frac{1}{s} [0 - 0] + \frac{a}{s} F_c$$

$$F_s = \frac{a}{s} F_c \quad \dots (2)$$

By putting F_s value in equation (1), we get

$$F_c = \frac{1}{s} - \frac{a}{s} \frac{a}{s} F_c \Rightarrow \frac{1}{s} - \frac{a^2}{s^2} F_c$$

$$F_c + \frac{a^2}{s^2} F_c = \frac{1}{s} \Rightarrow F_c \left(\frac{s^2 + a^2}{s^2} \right) = \frac{1}{s}$$

$$F_c = \frac{1}{s} \left(\frac{s^2}{s^2 + a^2} \right) \Rightarrow \frac{s}{s^2 + a^2}$$

As $L(\cos at) = F_c$, so

$$L(\cos at) = \frac{s}{s^2 + a^2}$$

This is our required result.

Now, from equation (2)

$$F_c = \frac{s}{a} F_s$$

By putting in equation (1) we get

$$\frac{s}{a} F_s = \frac{1}{s} - \frac{a}{s} F_s \Rightarrow \left(\frac{s}{a} + \frac{a}{s} \right) \frac{s}{a} F_s = \frac{1}{s}$$

$$\left(\frac{s^2 + a^2}{as} \right) F_s = \frac{1}{s} \Rightarrow F_s = \left(\frac{a}{s^2 + a^2} \right)$$

As $L(\sin at) = F_s$, so

$$L(\sin at) = \frac{a}{s^2 + a^2}$$

This is our required result.

LECTURE NO. 81

Laplace Transform of t^n ($n \in N$):

Theorem: Prove that

$$L(t^n) = \frac{n!}{s^{n+1}} ; \quad n \in N \quad \dots (1)$$

Proof: We'll prove it by mathematical induction.

For $n = 0$;

$$L(t^0) = \frac{0!}{s^{0+1}} \Rightarrow L(1) = \frac{1}{s}$$

Suppose equation (1) is true for fixed "n" and we will prove it for "n + 1".

For n + 1;

$$L(t^{n+1}) = \frac{(n+1)!}{s^{n+2}}$$

Applying Laplace Transform,

$$L(t^{n+1}) = \int_0^{\infty} e^{-st} t^{n+1} dt$$

Integrating and applying limits, we have

$$L(t^{n+1}) = \left[t^{n+1} \left(\frac{e^{-st}}{-s} \right) \right]_0^{\infty} - \int_0^{\infty} \left(\frac{e^{-st}}{-s} \right) (n+1)t^n dt$$

$$L(t^{n+1}) = -\frac{1}{s} \left[\frac{t^{n+1}}{e^{st}} \right]_0^{\infty} + \frac{n+1}{s} \int_0^{\infty} e^{-st} t^n dt$$

$$L(t^{n+1}) = -\frac{1}{s} \left[\lim_{t \rightarrow \infty} \frac{t^{n+1}}{e^{st}} - 0 \right] + \frac{n+1}{s} L(t^n)$$

As given $L(t^n) = \frac{n!}{s^{n+1}}$, so

$$L(t^{n+1}) = -\frac{1}{s} [0 - 0] + \frac{n+1}{s} \left(\frac{n!}{s^{n+1}} \right)$$

$$L(t^{n+1}) = \frac{(n+1)!}{s^{n+2}}$$

i.e. it is true for "n + 1". Hence, proved our required result.

LECTURE NO. 82

Laplace Transform of t^α ($\alpha > -1$):

Applying Laplace Transform,

$$L(t^\alpha) = \int_0^{\infty} e^{-st} t^\alpha dt$$

Put $\begin{cases} v = st \Rightarrow dv = sdt \Rightarrow dt = \frac{1}{s} dv \text{ and } t = \frac{v}{s} \\ \text{As } t \rightarrow 0 \Rightarrow v \rightarrow 0 \text{ and as } t \rightarrow \infty \Rightarrow v \rightarrow \infty \end{cases}$

$$L(t^\alpha) = \int_0^{\infty} e^{-v} \left(\frac{v}{s} \right)^\alpha \frac{1}{s} dv \Rightarrow \int_0^{\infty} e^{-v} \left(\frac{v^\alpha}{s^{\alpha+1}} \right) dv$$

$$L(t^\alpha) = \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-v} v^\alpha dv$$

Here

$$\int_0^{\infty} e^{-v} v^\alpha dv = \text{Gamma Function} = \gamma(\alpha + 1)$$

So,

$$L(t^\alpha) = \frac{\gamma(\alpha + 1)}{s^{\alpha+1}}$$

This is our required result.

LECTURE NO. 83

S-Shifting Theorem:

If $f(t)$ has the transform $F(s)$ where $s > k$, then " $e^{at} f(t)$ " has the transform $F(s - \alpha)$ where $(s - \alpha) > k$ i.e.

$$L\{e^{at} f(t)\} = F(s - \alpha)$$

In term of inverse Laplace transform,

$$L^{-1}\{F(s - \alpha)\} = e^{\alpha t} f(t)$$

Proof: As we know Laplace Transform,

$$L(f(t)) = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

For given function,

$$F(s - \alpha) = \int_0^{\infty} e^{-(s-\alpha)t} f(t) dt \Rightarrow \int_0^{\infty} e^{-st} e^{\alpha t} f(t) dt \Rightarrow \int_0^{\infty} e^{-st} (e^{\alpha t} f(t)) dt$$

$$F(s - \alpha) = L(e^{\alpha t} f(t))$$

It can also be written as,

$$L^{-1}\{F(s - \alpha)\} = e^{\alpha t} f(t)$$

This is our required result.

LECTURE NO. §4

Application of S-Shifting Theorem:

Evaluate

$$i): L(e^{\alpha t} \cos \omega t) \quad ii): L(e^{\alpha t} \sin \omega t) \quad iii) L^{-1}\left(\frac{2s - 27}{s^2 + 2s + 401}\right)$$

Solution: (Here we are going to use previous results)

i): $L(e^{\alpha t} \cos \omega t)$

$$L(\cos \omega t) = \frac{s}{s^2 + \omega^2} \Rightarrow \frac{s - 0}{(s - 0)^2 + \omega^2}$$

$$L(e^{\alpha t} \cos \omega t) = \frac{s - \alpha}{(s - \alpha)^2 + \omega^2}$$

ii): $L(e^{\alpha t} \sin \omega t)$

$$L(e^{\alpha t} \sin \omega t) = \frac{\omega}{(s - \alpha)^2 + \omega^2}$$

iii) $L^{-1}\left(\frac{2s-27}{s^2+2s+401}\right)$

$$L^{-1}\left(\frac{2s - 27}{s^2 + 2s + 401}\right) = L^{-1}\left(\frac{2s + 2 - 2 - 27}{s^2 + 2s + 1 - 1 + 401}\right) \Rightarrow L^{-1}\left(\frac{2(s + 1) - 29}{(s + 1)^2 + 400}\right)$$

$$= L^{-1}\left(\frac{2(s + 1)}{(s + 1)^2 + (20)^2}\right) - L^{-1}\left(\frac{29}{(s + 1)^2 + (20)^2}\right)$$

$$= L^{-1}\left(\frac{2(s + 1)}{(s + 1)^2 + (20)^2}\right) - L^{-1}\left(\frac{29}{(s + 1)^2 + (20)^2} \times \frac{20}{20}\right)$$

$$= 2L^{-1}\left(\frac{(s + 1)}{(s + 1)^2 + (20)^2}\right) - 29L^{-1}\left(\frac{20}{20\{(s + 1)^2 + (20)^2\}}\right)$$

$$= 2L^{-1}\left(\frac{s - (-1)}{((s - (-1))^2 + (20)^2)}\right) - \frac{29}{20}L^{-1}\left(\frac{20}{((s - (-1))^2 + (20)^2)}\right)$$

Using method of calculations i) and ii)

$$= 2e^{-t} \cos 20t - \frac{29}{20} e^{-t} \sin 20t$$

$$= e^{-t} \left[2 \cos 20t - \frac{29}{20} \sin 20t \right]$$

This is our required result.

LECTURE NO. §5

Piecewise Continuous Function:

Infomlly, a function is a piecewise continuous on an interval if it has finite jumps (discontinuities) on that interval.

A function $f(t)$ is a piecewise continuous on a finite interval $[a, b]$ if satisfy the following conditions.

- 1) Function $f(t)$ defined on an/that interval. (i.e. $[a, b]$)
- 2) Interval can be sub-divided into finitely many sub-intervals in each of which function $f(t)$ is continuous.
- 3) Function $f(t)$ has finite limits as " t " approaches either end points of sub-intervals from interior of $[a, b]$. $\{f(a_0 + 0) < \infty, f(a_0 - 0) < \infty\}$; Finite

Example:

$$f(t) = \begin{cases} \sin x & \text{if } 0 \leq x \leq \pi/2 \\ e^x & \text{if } \pi/2 \leq x \leq \pi \\ -2 & \text{if } \pi \leq x \leq 2\pi \end{cases} \quad \text{Overall interval } [0, 2\pi]$$

LECTURE NO. 86

Existence Theorem for Laplace Transform:

Let $f(t)$ be a piecewise continuous function on every finite interval in $[0, \infty)$ and satisfies $|f(t)e^{-kt}| \leq m$, $\forall t \geq 0$ and for some " k and m " then Laplace Transform of $f(t)$ exists for all $s > k$.

Answer: As given

$$|f(t)e^{-kt}| \leq m \quad \dots (1) \quad \forall t \geq 0 \text{ \& for some "k and m"}$$

Here equation (1) is known as "Growth Restriction". Equation (1) implies,

$$|f(t)| \leq m e^{kt}$$

$$f(t)e^{-kt} \rightarrow 0 \text{ as } t \rightarrow \infty \Rightarrow \lim_{t \rightarrow \infty} \frac{f(t)}{e^{kt}} \rightarrow 0$$

Proof: As given that $f(t)$ be a piecewise continuous function. $f(t)e^{-kt}$ is integrable over any finite interval on $[0, \infty)$. Laplace Transform,

$$|L(f(t))| = \left| \int_0^{\infty} e^{-st} f(t) dt \right| \leq \int_0^{\infty} |f(t)| e^{-st} dt$$

For given function,

$$|L(f(t))| \leq \int_0^{\infty} m e^{kt} e^{-st} dt \Rightarrow m \int_0^{\infty} e^{-(s-k)t} dt \quad \dots (2)$$

Integrating and applying limits, (without " m ") , we have

$$\begin{aligned} \int_0^{\infty} e^{-(s-k)t} dt &= -\frac{1}{s-k} \left| e^{-(s-k)t} \right|_0^{\infty} \Rightarrow -\frac{1}{s-k} \left| \frac{1}{e^{(s-k)t}} \right|_0^{\infty} \\ &= -\frac{1}{s-k} \left[\frac{1}{e^{\infty}} - \frac{1}{e^0} \right] \Rightarrow \frac{1}{s-k} ; \text{ for } s > k \end{aligned}$$

So, equation (2) becomes,

$$|L(f(t))| \leq \frac{m}{s-k} \Rightarrow L(f) \text{ exist if growth restriction is satisfied.}$$

Hence prove our required result.

LECTURE NO. 87

Counter Example of Existence Theorem:

Example:

$$f(t) = \frac{1}{\sqrt{t}} \quad \text{As } t \rightarrow 0 \text{ then } f(t) \rightarrow \infty$$

Here $f(t)$ is not a piecewise continuous function on $[0, \infty)$. Also

$$|f(t)| \leq m e^{kt} \quad \text{If } m = 1, k = 0$$

Applying Laplace transform on given function,

$$L(f(t)) = \int_0^{\infty} e^{-st} \frac{1}{\sqrt{t}} dt$$

Put $\begin{cases} \text{Let } \sqrt{st} = x \Rightarrow st = x^2 \\ x = \sqrt{s} \frac{1}{2\sqrt{t}} dt \Rightarrow \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{s}} dx \end{cases}$

$$L(f(t)) = \int_0^{\infty} e^{-x^2} \frac{2}{\sqrt{s}} dx \Rightarrow \frac{2}{\sqrt{s}} \int_0^{\infty} e^{-x^2} dx \dots (1)$$

Here

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \text{ (Gaussian Integral)}$$

Hence relation (1) becomes,

$$L(f(t)) = \frac{2}{\sqrt{s}} \times \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{s}} ; s > 0$$

Hence $\frac{1}{\sqrt{t}}$ is not a piecewise continuous function but its Laplace Transform exists. This is our required result.

LECTURE NO. 88

Laplace Transform of Derivatives:

Prove that

$$i) L(f') = sL(f) - f(0) \quad ii) L(f'') = s^2L(f) - sf(0) - f'(0)$$

Proof: since we know

$$\text{If } f \text{ is differentiable} \Rightarrow f \text{ is continuous.}$$

I):

Laplace transform for given function

$$L(f') = L(f'(t)) = \int_0^{\infty} e^{-st} f'(t) dt$$

Here $f(t)$ is continuous. Integrating and applying limits, we have

$$L(f') = [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} f(t) (-s)e^{-st} dt$$

$$L(f') = \left[\lim_{t \rightarrow \infty} \frac{f(t)}{e^{st}} - \frac{f(0)}{e^0} \right] + s \int_0^{\infty} e^{-st} f(t) dt$$

$$L(f') = 0 - f(0) + sL(f)$$

$$L(f') = sL(f) - f(0) \dots (1)$$

II):

Taking derivative of equation (1),

$$L(f'') = sL(f') - f'(0)$$

Putting value from equation (1),

$$L(f'') = s[sL(f) - f(0)] - f'(0)$$

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$

Hence prove.

LECTURE NO. 89

Laplace Transform of nth Derivatives:

$$L(f^n) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) \dots - s^0 f^{n-1}(0)$$

Watch Lecture(s).

LECTURE NO. 90

Applications of Laplace of Derivatives:

Find $L(\cos \omega t)$ y using derivative expression of Laplace.

Solution: since we know from previous lectures

$$L(f') = sL(f) - f(0) \quad \dots (1)$$

$$L(f'') = s^2L(f) - sf(0) - f'(0) \quad \dots (2)$$

Here

$$f(t) = \cos \omega t \Rightarrow f'(t) = -\omega \sin \omega t \quad \& \quad f''(t) = -\omega^2 \cos \omega t$$

$$f(0) = 1 \Rightarrow f'(0) = 0$$

By putting values in equation 2,

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$

$$L(-\omega^2 \cos \omega t) = s^2L(f) - s$$

$$-\omega^2 L(\cos \omega t) = s^2L(f) - s$$

$$-\omega^2 L(f) = s^2L(f) - s$$

$$s = L(f)\{s^2 + \omega^2\}$$

$$L(f) = \frac{s}{s^2 + \omega^2}$$

LECTURE NO. 91

Evaluate:

$$L(t \sin \omega t)$$

Solution: Here

$$f(t) = t \sin \omega t \Rightarrow f(0) = 0$$

$$f'(t) = t(\omega \cos \omega t) + 1 \sin \omega t \Rightarrow f'(0) = 0$$

$$f''(t) = \omega[t(-\omega \sin \omega t) + 1 \cos \omega t] + \omega \cos \omega t$$

$$f''(t) = -\omega^2(t \sin \omega t) + 2\omega \cos \omega t$$

By putting values in equation 2,

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$

$$L\{-\omega^2(t \sin \omega t) + 2\omega \cos \omega t\} = s^2L(f)$$

$$-\omega^2 L(t \sin \omega t) + 2\omega L(\cos \omega t) = s^2L(f)$$

$$-\omega^2 L(f) + 2\omega L\left(\frac{s}{s^2 + \omega^2}\right) = s^2L(f)$$

$$\frac{2\omega s}{s^2 + \omega^2} = L(f)\{s^2 + \omega^2\}$$

$$L(f) = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

This is our required result.

LECTURE NO. 92

Laplace Transform of Integral:

Theorem: let $f(t)$ is a piecewise continuous function for $t \geq 0$, & $d[f(t)] = F(s)$. Further $|f(t)| \leq me^{kt}$ for some $m > 0$ and $k > 0$, then

$$L\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s) \quad \text{and} \quad L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^\infty f(\tau) d\tau$$

Proof: Let

$$g(t) = \int_0^t f(\tau) d\tau$$

$$\therefore f(t) \text{ is piecewise continuous} \Rightarrow g(t) \text{ is continuous}$$

$$|g(t)| = \left| \int_0^t f(\tau) d\tau \right| \leq \int_0^t |f(\tau)| d\tau \leq \int_0^t m e^{kt} d\tau$$

$$|g(t)| = m \left| \frac{e^{kt}}{k} \right|_0^t \Rightarrow \frac{m}{k} (e^{kt} - 1) \leq \frac{m}{k} e^{kt}$$

$\Rightarrow g(t)$ satisfies growth restriction. Now

$$g'(t) = \frac{d}{dt} \left(\int_0^t f(\tau) d\tau \right) = f(t)$$

Except at point of discontinuities. This implies $g(t)$ is continuous.

$$L(g'(t)) = L(f(t)) \quad \text{and} \quad sL(g(t)) - g(0) = F(s)$$

$$sL \left\{ \int_0^t f(\tau) d\tau \right\} = F(s) \Rightarrow L \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{F(s)}{s}$$

LECTURE NO. 93

Evaluate:

$$i) L^{-1} \left\{ \frac{1}{s(s^2 + \omega^2)} \right\} \quad \text{and} \quad ii) L^{-1} \left\{ \frac{1}{s^2(s^2 + \omega^2)} \right\}$$

I):

$$L^{-1} \left\{ \frac{1}{s(s^2 + \omega^2)} \right\} = L^{-1} \left\{ \frac{1}{\omega} \cdot \frac{\omega}{s^2 + \omega^2} \right\} \Rightarrow \frac{1}{\omega} L^{-1} \left(\frac{\omega}{s^2 + \omega^2} \right)$$

$$L^{-1}(F(s)) = \frac{1}{\omega} \sin \omega t \quad ; \quad \therefore L^{-1} \left(\frac{\omega}{s^2 + \omega^2} \right) = \sin \omega t$$

Now, (Going to calculate required function)

$$L^{-1} \left(\frac{1}{s} F(s) \right) = \int_0^t f(\tau) d\tau$$

$$L^{-1} \left\{ \frac{1}{s(s^2 + \omega^2)} \right\} = \int_0^t \frac{1}{\omega} \sin \omega \tau d\tau$$

Integrating and applying limits, we have

$$L^{-1} \left\{ \frac{1}{s(s^2 + \omega^2)} \right\} = -\frac{1}{\omega^2} [\cos \omega \tau]_0^t$$

$$L^{-1} \left\{ \frac{1}{s(s^2 + \omega^2)} \right\} = \frac{1}{\omega^2} (1 - \cos \omega t)$$

This is our required result.

II):

$$L^{-1} \left\{ \frac{1}{s^2(s^2 + \omega^2)} \right\} = L^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s(s^2 + \omega^2)} \right\} \Rightarrow L^{-1} \left\{ \frac{1}{s} F(s) \right\}$$

$$L^{-1} \left\{ \frac{1}{s^2(s^2 + \omega^2)} \right\} = \int_0^t f(\tau) d\tau \Rightarrow \int_0^t \frac{1}{\omega^2} (1 - \cos \omega \tau) d\tau$$

$$L^{-1} \left\{ \frac{1}{s^2(s^2 + \omega^2)} \right\} = \frac{1}{\omega^2} \left[\tau - \frac{\sin \omega \tau}{\omega} \right]$$

$$L^{-1} \left\{ \frac{1}{s^2(s^2 + \omega^2)} \right\} = \frac{1}{\omega^2} \left[t - \frac{\sin \omega t}{\omega} \right]$$

This is our required result.

LECTURE NO. 94

Solving ODE with Laplace:

Solve,

$$y'' - y = t \quad ; \quad y(0) = 1, y'(0) = 1$$

Solution: Applying Laplace on given function,

$$L(y'' - y) = L(t)$$

Using formula $\left\{L\left(t^n = \frac{n!}{s^{n+1}}\right)\right\}$ on $L(t^1)$, we get

$$L(y'') - L(y) = \frac{1}{s^2}$$

Putting $L(y'')$ from previous lectures, we have

$$s^2 L(y) - s y(0) - y'(0) - L(y) = \frac{1}{s^2}$$

Putting values from initial conditions,

$$(s^2 - 1)L(y) - s - 1 = \frac{1}{s^2}$$

$$(s^2 - 1)L(y) = \frac{1}{s^2} + (1 + s)$$

$$L(y) = \frac{(1 + s)}{(s^2 - 1)} + \frac{1}{s^2(s^2 - 1)}$$

$$L(y) = \frac{1}{s - 1} + \left[\frac{1}{s^2 - 1} - \frac{1}{s^2} \right]$$

$$L(y) = \frac{1}{s - 1} + \frac{1}{s^2 - 1} - \frac{1}{s^2}$$

$$y = L^{-1}\left(\frac{1}{s - 1} + \frac{1}{s^2 - 1} - \frac{1}{s^2}\right)$$

$$y = L^{-1}\left(\frac{1}{s - 1}\right) + L^{-1}\left(\frac{1}{s^2 - 1}\right) - L^{-1}\left(\frac{1}{s^2}\right)$$

Using results $\begin{cases} L^{-1}\left(\frac{1}{s-1}\right) = e^{-t} \\ L^{-1}\left(\frac{1}{s^2-1}\right) = \sinh t, \\ L^{-1}\left(\frac{1}{s^2}\right) = t \end{cases}$ we get

$$y = e^{-t} + \sinh t - t$$

This is our required result.

LECTURE NO. 95

Unit Step Functions:

It is defined and given as;

$$u(t - a) = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$

Applying Laplace Transform,

$$L(u(t - a)) = \int_0^{\infty} e^{-st} u(t - a) dt$$

$$L(u(t - a)) = \int_0^a e^{-st} (0) dt + \int_a^{\infty} e^{-st} (1) dt \quad ; \quad a > 0$$

$$L(u(t - a)) = 0 + \frac{1}{-s} [e^{-st}]_a^{\infty} \Rightarrow -\frac{1}{s} [0 - e^{-as}]$$

$$L(u(t - a)) = \frac{e^{-as}}{s}$$

This is our required result.

LECTURE NO. 96

Plot the Following:

$$i): A(t) = k[u(t-1) - 2u(t-4) + u(t-6)] \quad k > 0$$

$$ii): B(t) = \sin\left(\frac{\pi}{2}t\right)[u(t) - u(t-2) + u(t-4) - \dots]$$

Watch Lecture.

LECTURE NO. 97

Time Shifting Theorem:

Let $L(f(t)) = F(s)$, then the Laplace Transform of shifted function is given by;

$$L\{f(t-a)u(t-a)\} = \begin{cases} 0 & \text{if } t < a \\ e^{-as} F(s) & \text{if } t > a \end{cases}$$

And

$$L^{-1}\{e^{-as} F(s)\} = f(t-a)u(t-a)$$

Proof: As we know from previous lectures,

$$e^{-as} F(s) = e^{-as} \int_0^{\infty} e^{-st} f(t) dt \Rightarrow \int_0^{\infty} e^{-(a+t)s} f(t) dt \quad \dots (1)$$

$$\text{Put } \begin{cases} a+t=\tau \Rightarrow t=\tau-a, dt=d\tau \\ \text{As } t \rightarrow 0 \Rightarrow \tau \rightarrow a, t \rightarrow \infty \Rightarrow \tau \rightarrow \infty \end{cases}$$

$$e^{-as} F(s) = \int_0^{\infty} e^{-s\tau} f(\tau-a) d\tau \Rightarrow \int_0^{\infty} e^{-st} f(t-a) dt$$

$$e^{-as} F(s) = \int_0^a e^{-st} f(t-a)(0) dt + \int_a^{\infty} e^{-st} f(t-a)(1) dt$$

$$e^{-as} F(s) = \int_0^a e^{-st} f(t-a)u(t-a) dt + \int_a^{\infty} e^{-st} f(t-a)u(t-a) dt$$

$$e^{-as} F(s) = \int_0^a e^{-st} \{f(t-a)u(t-a)\} dt + \int_a^{\infty} e^{-st} f(t-a)u(t-a) dt$$

By combining,

$$e^{-as} F(s) = \int_0^{\infty} e^{-st} \{f(t-a)u(t-a)\} dt$$

$$e^{-as} F(s) = L\{f(t-a)u(t-a)\}$$

$$L^{-1}\{e^{-as} F(s)\} = f(t-a)u(t-a)$$

This is our required result. Example:

$$L\{\sin(t-2)u(t-2)\} = ?$$

$$\text{Using results } \begin{cases} \sin(t-2) = \frac{1}{s^2+1} \\ u(t-2) = e^{-2s} \end{cases}$$

$$L\{\sin(t-2)u(t-2)\} = \frac{1}{s^2+1} e^{-2s} = \frac{e^{-2s}}{s^2+1}$$

This is our required answer.

LECTURE NO. 98

Problem:

Express the following function in terms of Unit Step Function and find its Laplace Transform.

Function is given as;

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{\pi}{2} \\ \cos t & \text{if } t > \frac{\pi}{2} \end{cases}$$

Solution: As we know about Unit Step Function from previous lecture (95), applying on the given function,

$$f(t) = 2[u(t-0) - u(t-1)] + \frac{1}{2}t^2 \left[u(t-1) - u\left(t - \frac{\pi}{2}\right) \right] + \cos t \left[u\left(t - \frac{\pi}{2}\right) \right] \dots (A)$$

Applying Laplace Transform on the given function,

$$L\{f(t)\} = 2L[u(t-0) - u(t-1)] + \frac{1}{2}L\left[t^2 \left(u(t-1) - u\left(t - \frac{\pi}{2}\right) \right)\right] + L\left[\cos t \left(u\left(t - \frac{\pi}{2}\right) \right)\right] \dots (B)$$

Finding values for the equation (B) one by one,

$$L[u(t-0) - u(t-1)] = L(u(t-0)) - L(u(t-1))$$

Using formula $\left\{L(u(t-a)) = \frac{e^{-as}}{s}\right\}$ from previous lecture (95), we get

$$L[u(t-0) - u(t-1)] = \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$L[u(t-0) - u(t-1)] = \frac{1}{s}(1 - e^{-s}) \dots (i)$$

Now, for

$$L\left[t^2 \left(u(t-1) - u\left(t - \frac{\pi}{2}\right) \right)\right] = L\{t^2 u(t-1)\} - L\left\{t^2 u\left(t - \frac{\pi}{2}\right)\right\} \dots (z)$$

Since we know from previous lecture (97),

$$f(t-a)u(t-a) = L^{-1}\{e^{-as} F(s)\}$$

$$L\{f(t-a)u(t-a)\} = e^{-as} F(s)$$

$$L\{f(t-a)u(t-a)\} = e^{-as} L\{f(t)\}$$

Put $t-a = \tau \Rightarrow t = a + \tau$

$$L\{f(\tau)u(t-a)\} = e^{-as} L\{f(a+\tau)\}$$

It can also be written as,

$$L\{f(t)u(t-a)\} = e^{-as} L\{f(a+t)\}$$

Using this expression for equation (z), becomes ($\because a = 1$)

$$\begin{aligned} &= e^{-s} L\{(1+t)^2\} - e^{-\frac{\pi}{2}s} L\left(\frac{\pi}{2} + t\right)^2 \\ &= e^{-s} L\{t^2 + 2t + 1\} - e^{-\frac{\pi}{2}s} L\left\{t^2 + \pi t + \frac{\pi^2}{4}\right\} \end{aligned}$$

Using formula $\left\{L(t^n) = \frac{n!}{s^{n+1}}\right\}$, so we get

$$L\left[t^2 \left(u(t-1) - u\left(t - \frac{\pi}{2}\right) \right)\right] = e^{-s} \left\{ \frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s} \right\} - e^{-\frac{\pi}{2}s} \left\{ \frac{2}{s^3} + \pi\frac{1}{s^2} + \frac{\pi^2}{4}\frac{1}{s} \right\} \dots (ii)$$

Now, for

$$L\left[\cos t \left(u\left(t - \frac{\pi}{2}\right) \right)\right] = e^{-\frac{\pi}{2}s} L\left\{\cos\left(\frac{\pi}{2} + t\right)\right\} \Rightarrow e^{-\frac{\pi}{2}s} L(-\sin t)$$

$$L\left[\cos t \left(u\left(t - \frac{\pi}{2}\right) \right)\right] = -e^{-\frac{\pi}{2}s} L(\sin t)$$

$$L\left[\cos t \left(u\left(t - \frac{\pi}{2}\right) \right)\right] = -e^{-\frac{\pi}{2}s} \frac{1}{1+s^2} \dots (iii)$$

Now, finally putting calculated values (i,ii,iii) in equation (B)

$$L\{f(t)\} = \frac{2}{s}(1 - e^{-s}) + \frac{1}{2}\left[e^{-s}\left\{\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}\right\} - e^{-\frac{\pi}{2}s}\left\{\frac{2}{s^3} + \pi\frac{1}{s^2} + \frac{\pi^2}{4}\frac{1}{s}\right\}\right] - e^{-\frac{\pi}{2}s} \frac{1}{1+s^2}$$

This is our required result.

LECTURE NO. 99

Problem:

Find the Inverse Laplace Transform of;

$$\frac{e^{-2s}}{s^2 + n^2} + \frac{e^{-3s}}{(s + 2)^2}$$

Solution: Since we know

$$\begin{aligned} L^{-1} \left\{ \frac{e^{-2s}}{s^2 + n^2} + \frac{e^{-3s}}{(s + 2)^2} \right\} &= L^{-1} \left(\frac{e^{-2s}}{s^2 + n^2} \right) + L^{-1} \left(\frac{e^{-3s}}{(s + 2)^2} \right) \\ &= \frac{1}{\pi} L^{-1} \left(e^{-2s} \frac{\pi}{s^2 + n^2} \right) + L^{-1} \left(e^{-3s} \frac{1}{(s + 2)^2} \right) \dots (1) \end{aligned}$$

Using results $\begin{cases} L^{-1} \left(e^{-2s} \frac{\pi}{s^2 + n^2} \right) = \sin \pi(t - 2) u(t - 2) \\ L^{-1} \left(e^{-3s} \frac{1}{(s + 2)^2} \right) = (t - 3) e^{-2(t - 3)} u(t - 3) \end{cases}$ (For better understanding, watch lecture)

Equation (1) becomes

$$L^{-1} \left\{ \frac{e^{-2s}}{s^2 + n^2} + \frac{e^{-3s}}{(s + 2)^2} \right\} = \frac{1}{\pi} [\sin \pi(t - 2)] u(t - 2) + (t - 3) e^{-2(t - 3)} u(t - 3)$$

This is our required result.

LECTURE NO. 100

Dirac's Delta Function:

Let

$$f_k(t) = \begin{cases} \frac{1}{k} & \text{if } 0 \leq t \leq k \\ 0 & \text{, Otherwise} \end{cases}$$

In general,

$$f_k(t - a) = \begin{cases} \frac{1}{k} & \text{if } a \leq t \leq a + k \\ 0 & \text{if Otherwise} \end{cases}$$

If $k \rightarrow 0$, then

$$T_k = \lim_{k \rightarrow 0} f_k(t - a) = s(t - a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{, Otherwise} \end{cases}$$

Such that

$$\text{Integraal} = \int_0^{\infty} s(t - a) dt = 1$$

Example:

$$\int_a^{a+k} f_k(t - a) dt = \int_a^{a+k} \frac{1}{k} dt \Rightarrow \frac{1}{k} |t|_a^{a+k} \Rightarrow \frac{1}{k} [a + k - a] = 1$$

Note:

$$\int_0^{\infty} g(t) s(t - a) dt = g(a)$$

These types of function are known as special class of function. i.e. Distributions.

LECTURE NO. 101

Laplace Transform of Dirac's Delta Function:

Since we know

$$f_k(t - a) = \begin{cases} \frac{1}{k} & \text{if } a \leq t \leq a + k \\ 0 & \text{if Otherwise} \end{cases}$$

$$f_k(t-a) = \frac{1}{k} [u(t-a) - u(t-(a+k))]$$

Applying Laplace Transform,

$$L\{f_k(t-a)\} = L\left\{\frac{1}{k} [u(t-a) - u(t-(a+k))]\right\}$$

$$L\{f_k(t-a)\} = \frac{1}{k} \{L[u(t-a)] - L[u(t-(a+k))]\}$$

Using result $\{L[u(t-a)] = \frac{e^{-as}}{s}\}$

$$L\{f_k(t-a)\} = \frac{1}{k} \left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s} \right]$$

$$L\{f_k(t-a)\} = \frac{1}{k} e^{-as} \left[\frac{1 - e^{-ks}}{s} \right]$$

$\therefore k \rightarrow 0$, apply Laplace Transform of Dirac Delta Function (for above relation)

$$L(s(t-a)) = \lim_{k \rightarrow 0} L\{f_k(t-a)\} = \lim_{k \rightarrow 0} e^{-as} \left[1 - \left\{ 1 - ks + \frac{k^2 s^2}{2} - \frac{k^3 s^3}{6} + \dots \right\} \right]$$

$$L(s(t-a)) = \lim_{k \rightarrow 0} e^{-as} \left[1 - 1 + ks - \frac{k^2 s^2}{2} + \frac{k^3 s^3}{6} - \dots \right]$$

$$L(s(t-a)) = \lim_{k \rightarrow 0} e^{-as} \left[ks \left(1 - \frac{ks}{2} + \frac{k^2 s^2}{6} - \dots \right) \right]$$

After applying limits,

$$L(s(t-a)) = e^{-as}$$

This is our required result.

LECTURE NO. 102

Problem:

Find the response of damped spring system given by initial value problem;

$$y'' + 3y' + 2y = u(t-1) - u(t-2) \quad ; \quad y(0) = y'(0) = 0$$

Solution: Applying Laplace Transform on given function,

$$L\{y'' + 3y' + 2y\} = L\{u(t-1) - u(t-2)\}$$

$$Ly'' + 3Ly' + 2Ly = L\{u(t-1)\} - L\{u(t-2)\} \dots (1)$$

Since we know

$$\begin{cases} Ly'' = s^2 L(y) - sy(0) - y'(0) \\ Ly' = sL(y) - y(0) \\ L(u(t-a)) = \frac{e^{-as}}{s} \end{cases}$$

By using, equation (1) becomes

$$\{s^2 L(y) - sy(0) - y'(0)\} + 3\{sL(y) - y(0)\} + 2Ly = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$(s^2 + 3s + 2) L(y) = \frac{1}{s} (e^{-s} - e^{-2s})$$

$$L(y) = \frac{1}{s(s^2 + 3s + 2)} (e^{-s} - e^{-2s})$$

Putting $\left\{ \frac{1}{s(s^2 + 3s + 2)} = F(s) \right\}$ we have

$$L(y) = F(s) (e^{-s} - e^{-2s}) \dots (A)$$

Now, by applying partial fraction on F(s)

$$F(s) = \frac{1}{s(s^2 + 3s + 2)} \Rightarrow \frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

By solving, we get

$$A = \frac{1}{2}, \quad B = \frac{1}{2}, \quad C = -1$$

So,

$$\begin{aligned} F(s) &= \frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1} \\ L^{-1}(F(s)) &= \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left(\frac{1}{s+1}\right) \\ L^{-1}(F(s)) &= \frac{1}{2}(1) + \frac{1}{2}(e^{-2t}) - (e^{-t}) \end{aligned}$$

Now, equation (A) implies,

$$L(y) = F(s)(e^{-s} - e^{-2s}) \Rightarrow F(s)e^{-s} - F(s)e^{-2s}$$

Applying Inverse Laplace,

$$y(t) = L^{-1}(F(s)e^{-s}) - L^{-1}(F(s)e^{-2s})$$

Since $\{L^{-1}(e^{-s} F(s)) = f(t-a)u(t-a)$

$$y(t) = f(t-1)u(t-1) - f(t-2)u(t-2)$$

Note that; Unit Step Function Step corresponds to piecewise continuous function. By applying,

$$y(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ \frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)} & \text{if } 0 < t < 2 \\ \frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)} - \left\{\frac{1}{2} + \frac{1}{2}e^{-2(t-2)} - e^{-(t-2)}\right\} & t > 2 \end{cases}$$

This is our required result. (For better understanding, watch lecture)

LECTURE NO. 103

Hammerblow Response of Mass Spring System:

Solve the given problem;

$$y'' + 3y' + 2y = \delta(t-1); \quad y(0) = y'(0) = 0$$

Solution: As given that

$$y'' + 3y' + 2y = \delta(t-1)$$

Applying Laplace Transform,

$$\begin{aligned} L[y'' + 3y' + 2y] &= L[\delta(t-1)] \\ Ly'' + 3Ly' + 2Ly &= e^{-s} \\ s^2L(y) + 3sL(y) + 2L(y) &= e^{-s} \\ L(y)[s^2 + 3s + 2] &= e^{-s} \\ L(y) &= \frac{e^{-s}}{s^2 + 3s + 2} \Rightarrow \frac{1}{(s+1)(s+2)}e^{-s} \dots (1) \end{aligned}$$

By applying partial fraction on function, we get

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

So relation (1) becomes,

$$L(y) = \left[\frac{1}{s+1} - \frac{1}{s+2} \right] e^{-s} = L^{-1}(F(s)) = f(t)$$

Applying Inverse Laplace Transform,

$$y(t) = L^{-1}\left\{\frac{1}{s+1}e^{-s}\right\} - L^{-1}\left\{\frac{1}{s+2}e^{-s}\right\}$$

$$\text{Using results } \begin{cases} L\left(\frac{1}{s+1}\right) = e^{-t} \\ L\left(\frac{1}{s+2}\right) = e^{-2t} \end{cases}$$

$$\begin{aligned} L^{-1}\{F(s)e^{-as}\} &= f(t-a)u(t-a) \\ f(t) &= e^{-t} - e^{-2t} \end{aligned}$$

So, there exist two answers

$$y(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & t > 1 \end{cases}$$

LECTURE NO. 104

Convolution:

Let

$$f(t) = e^t, \quad g(t) = 1$$

Then

$$L(f(t)) = L(e^t) = \frac{1}{s-1}, \quad L(g(t)) = L(1) = \frac{1}{s}$$

Product;

$$L(f g) = L(e^t 1) \Rightarrow L(e^t) = \frac{1}{s-1}$$

As

$$L(f g) = L(g f) = \frac{1}{s-1}, \quad L(g) L(f) = L(f) L(g) = \frac{1}{s(s-1)}$$

So,

$$L(g) L(f) \neq L(gf) \quad \text{Or} \quad L(f) L(g) \neq L(fg)$$

Sum;

$$L(f) + L(g) = L(f + g)$$

Q: Is there any function whose Laplace gives $L(f) L(g)$?

Ans: Yes, it is convolution function.

$$f * g = \int_0^t f(\tau) g(t - \tau) d\tau$$

$$L(f * g) = L\left(\int_0^t f(\tau) g(t - \tau) d\tau\right) \Rightarrow L(f) L(g)$$

LECTURE NO. 105

Theorem:

If $L(f_1) = F_1$, $L(f_2) = F_2$ and $f_1(t), f_2(t)$ satisfy the Growth Restrictions then,

$$F_1(s) \cdot F_2(s) = L\left\{\int_0^t f_1(t) f_2(t - \tau) d\tau\right\}$$

Proof:

$$\begin{aligned} R.H.S &= L\left\{\int_0^t f_1(t) f_2(t - \tau) d\tau\right\} \\ L\left\{\int_0^t f_1(t) f_2(t - \tau) d\tau\right\} &= \int_0^\infty e^{-s\tau} \left\{\int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t - \tau) d\tau\right\} dt \quad \dots (1) \end{aligned}$$

Here R.H. integral is the form of " $\iint_D \varphi(t, \tau) dt d\tau$ " which is taken over the region bounded by $\tau = 0, \tau = t$.

Changing the order of integration on R.H.S, we have

$$L\left\{\int_0^t f_1(t) f_2(t - \tau) d\tau\right\} = \int_0^\infty \left\{f_1(\tau) \int_{t=\tau}^\infty e^{-s\tau} f_2(t - \tau) dt\right\} d\tau$$

$$\text{Put } \begin{cases} t - \tau = z \Rightarrow t = \tau + z \Rightarrow dt = dz \\ t \rightarrow \tau \Rightarrow z \rightarrow 0, \text{ As } t \rightarrow \infty \Rightarrow z \rightarrow \infty \end{cases}$$

∴

$$\int_{\tau}^{\infty} e^{-s\tau} f_2(t - \tau) d\tau = \int_0^{\infty} e^{-s(\tau+z)} f_2(z) dz \Rightarrow e^{-s\tau} \int_0^{\infty} e^{-sz} f_2(z) dz \Rightarrow e^{-s\tau} F_2(s)$$

So,

$$L \left\{ \int_0^t f_1(t) f_2(t - \tau) d\tau \right\} = \int_0^t f_1(t) e^{-s\tau} F_2(s) d\tau = \int_0^t (f_1(t) e^{-s\tau} d\tau) (F_2(s)) \\ \Rightarrow (F_1(s))(F_2(s)) = L.H.S$$

Hence prove our required result.

LECTURE NO. 106

Problem:

Find $h(t)$, provided that

$$H(s) = \frac{1}{(s-a)s} \Rightarrow L^{-1}(H(s)) = h(t) = ?$$

Solution: If

$$L(f) = F(s) \quad \& \quad L(g) = G(s)$$

Then

$$F(s) G(s) = L \left\{ \int_0^t f(\tau) g(t - \tau) d\tau \right\}$$

Applying Inverse Laplace,

$$L^{-1}(F(s) G(s)) = \int_0^t f(\tau) g(t - \tau) d\tau$$

Given that,

$$H(s) = \frac{1}{(s-a)s} = F(s) G(s)$$

Using result $\begin{cases} F(s) = \frac{1}{s-a} \Rightarrow f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s-a}\right) = e^{at} \Rightarrow f(\tau) \\ G(s) = \frac{1}{s} \Rightarrow g(t) = L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s}\right) = 1 \Rightarrow g(t - \tau) \end{cases}$, we have

$$L^{-1}(H(s)) = L^{-1}(F(s) G(s)) = \int_0^t e^{at} \cdot 1 d\tau$$

Integrating and applying limits,

$$L^{-1}(H(s)) = \frac{1}{a} [e^{at}]_0^t \Rightarrow \frac{1}{a} [e^{at} - 1] = h(t)$$

This is our required result.

LECTURE NO. 107

Properties of Convolution:

$$\begin{aligned} i): f * g &= g * f & ii): f * (g_1 + g_2) &= f * g_1 + f * g_2 \\ iii): (f * g) * \alpha &= f * (g * \alpha) & iv): f * 0 &= 0 * f = 0 \end{aligned}$$

Some Unusual Properties of Convolution:

$$i): f * 1 \neq f$$

$$\text{Let } f(t) = t$$

$$f(t) * 1 = \int_0^t f(\tau) \cdot 1 d\tau = \int_0^t \tau d\tau = \frac{1}{2} [\tau^2]_0^t = \frac{1}{2} t^2 \neq t = f(t)$$

ii): $(f * f)(t) \neq 0$

$$\text{Let } f(t) = \sin t$$

$$f * f = \int_0^t \sin \tau \sin(t - \tau) d\tau = \frac{1}{2} \int_0^t [\cos(2\tau - t) - \cos t] d\tau$$

$$f * f = \frac{1}{2} \left[\frac{1}{2} \sin(2\tau - t) - \tau \cos t \right]_0^t = \frac{1}{2} \left[\frac{1}{2} \sin t - t \cos t + \frac{1}{2} \sin t \right]$$

$$f * f = \frac{1}{2} [\sin t - t \cos t] \neq 0$$

This is our required result.

LECTURE NO. 108

Example:

$$y'' + y = g(t); \quad y(0) = y'(0) = 0$$

Solution: Applying Laplace,

$$L(y'' + y) = L(g(t)) \Rightarrow L(y'') + L(y) = G(s)$$

$$\text{Using results } \begin{cases} L(y'') = s^2 L(y) - sy(0) - y'(0); & y(0) = 0 \Rightarrow L(y'') = s^2 L(y) \\ L(y') = sL(y) - y(0); & y(0) = 0 \Rightarrow L(y') = sL(y) \end{cases}$$

$\therefore$

$$s^2 L(y) + L(y) = G(s) \Rightarrow L(y) = \frac{1}{1 + s^2} G(s) = F(s) G(s) \dots (1)$$

Where

$$F(s) = \frac{1}{1 + s^2} \Rightarrow L^{-1}(F(s)) = L^{-1}\left(\frac{1}{1 + s^2}\right) = \sin t = f(t)$$

$$L^{-1}(G(s)) = g(t)$$

$$L(f * g) = F(s)G(s) \Rightarrow L^{-1}(F(s)G(s)) = f * g$$

So, equation (1) implies,

$$y(t) = L^{-1}(F(s)G(s)) = f * g$$

$$y(t) = \int_0^t f(\tau) g(t - \tau) d\tau \Rightarrow \int_0^t \sin \tau g(t - \tau) d\tau$$

This is our required result.

LECTURE NO. 109

Integral Equation:

It is type of equation that involves unknown function which appears inside and outside an integral.

Example:

$$y(t) - \int_0^t y(t) \sin(t - \tau) d\tau = t$$

These types of equations are known as "Volterra Integral Equations".

Above relation can be written as,

$$y(t) - y(t) * \sin t = t$$

Applying Laplace Transform,

$$L[y(t) - y(t) * \sin] = L(t)$$

Using formula $\left\{ L\left(t^n = \frac{n!}{s^{n+1}}\right) \right\}$ on $L(t)$,

$$L(y) - L(y * \sin t) = \frac{1}{s^2} \Rightarrow L(y) - L(y) L(\sin t) = \frac{1}{s^2}$$

$$L(y) \left[1 - \frac{1}{1 + s^2} \right] = \frac{1}{s^2} \Rightarrow L(y) \left[\frac{s^2}{1 + s^2} \right] = \frac{1}{s^2}$$

$$L(y) = \frac{1+s^2}{s^4} \Rightarrow L(y) = \frac{1}{s^2} + \frac{1}{s^4}$$

Applying Inverse Transform, (Using Formula $\left\{L\left(t^n = \frac{n!}{s^{n+1}}\right)\right\}$)

$$y(t) = L^{-1}\left(\frac{1}{s^2} + \frac{1}{s^4}\right) \Rightarrow L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{6}L^{-1}\left(\frac{3!}{s^4}\right)$$

$$y(t) = t + \frac{1}{6}t^3$$

This is our required result.

LECTURE NO. 110

Volterra Integral Equation of 2nd Kind:

$$y(t) - \int_0^t (1+t) y(t-\tau) d\tau = 1 - \sinh t$$

Solution: Given relation can be written as,

$$y(t) - (1+t) * y(t) = 1 - \sinh t$$

Applying Laplace Transform,

$$L\{y(t) - (1+t) * y(t)\} = L\{1 - \sinh t\}$$

$$L(y) - L(1+t) L(y) = L(1) - L(\sinh)$$

Using formula $\left\{L\left(t^n = \frac{n!}{s^{n+1}}\right)\right\}$ ($\therefore L(\sinh t) = \frac{1}{s^2-1}$)

$$L(y) \left[1 - \frac{1}{s} - \frac{1}{s^2}\right] = \frac{1}{s} - \frac{1}{s^2-1} \Rightarrow L(y) \left[\frac{s^2-s-1}{s^2}\right] = \frac{s^2-s-1}{s(s^2-1)}$$

$$L(y) = \frac{s}{s^2-1}$$

Applying Inverse Transform, ($\therefore L(\cosh at) = \frac{s}{s^2-a^2}$)

$$y(t) = L^{-1}\left(\frac{s}{s^2-1}\right) = \cosh t$$

This is our required result.

LECTURE NO. 111

Differentiation of Laplace Transform:

Let $L\{f(t)\} = F(s)$, where ' $f(t)$ ' is a function satisfying the growth restriction; then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} [F(s)] \dots (1)$$

Proof: We prove it by mathematical induction.

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt \Rightarrow \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t) dt$$

$$\frac{d}{ds} F(s) = \int_0^\infty (-t) e^{-st} f(t) dt \Rightarrow (-1) \int_0^\infty e^{-st} \{t f(t)\} dt$$

$$(-1)^1 \frac{d}{ds} F(s) = \int_0^\infty e^{-st} \{t f(t)\} dt \Rightarrow L\{t^1 f(t)\}$$

$$L\{t^1 f(t)\} = (-1)^1 \frac{d}{ds} F(s)$$

Let equation (1) is true for " n ", we prove it for " $n + 1$ ",

$$\begin{aligned}
 L\{t^n f(t)\} &= (-1)^n \frac{d^n}{ds^n} F(s) \Rightarrow \int_0^{\infty} e^{-st} \{t^n f(t)\} dt = (-1)^n \frac{d^n}{ds^n} F(s) \\
 \frac{d}{ds} \int_0^{\infty} e^{-st} \{t^n f(t)\} dt &= \frac{d}{ds} \left[(-1)^n \frac{d^n}{ds^n} F(s) \right] \\
 \int_0^{\infty} \frac{\partial}{\partial s} (e^{-st}) \{t^n f(t)\} dt &= (-1)^n \frac{d^{n+1}}{ds^{n+1}} [F(s)] \\
 \int_0^{\infty} (-t) e^{-st} t^n f(t) dt &= (-1)^n \frac{d^{n+1}}{ds^{n+1}} [F(s)] \\
 \int_0^{\infty} (-1)t e^{-st} t^n f(t) dt &= (-1)^n \frac{d^{n+1}}{ds^{n+1}} [F(s)] \\
 \int_0^{\infty} (-1)e^{-st} t^{n+1} f(t) dt &= (-1)^n \frac{d^{n+1}}{ds^{n+1}} [F(s)]
 \end{aligned}$$

So, finally

$$L\{t^n f(t)\} = \int_0^{\infty} e^{-st} t^{n+1} f(t) dt = (-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}} [F(s)]$$

This implies; truth of given proportion for " n " also implies its truth for " $n + 1$ ".

LECTURE NO. 112

Problem:

$$\int_0^{\infty} t e^{-3t} \sin t dt = ?$$

Solution: As given

$$\int_0^{\infty} t e^{-3t} \sin t dt = \int_0^{\infty} e^{-3t} (t \sin t) dt$$

Here $\{f(t) = \sin t \Rightarrow F(s) = \frac{1}{s^2+1}\}$. Using relation

$$\begin{aligned}
 \int_0^{\infty} e^{-st} \{t f(t)\} dt &= (-1) \frac{d}{ds} F(s) \\
 \int_0^{\infty} e^{-st} \{t \sin t\} dt &= (-1) \frac{d}{ds} \left(\frac{1}{1+s^2} \right) = (-1)(-1) \frac{2s}{(1+s^2)^2} = \frac{2s}{(1+s^2)^2}
 \end{aligned}$$

Put $s = 3$;

$$\int_0^{\infty} e^{-st} \{t \sin t\} dt = \frac{2(3)}{(1+3^2)^2} = \frac{3}{50}$$

This is our required result.

LECTURE NO. 113

Problem:

$$L^{-1} \left\{ \ln \left(\frac{s+a}{s+b} \right) \right\} ; \quad \text{whee } a, b > 0$$

Solution: Here

$$F(s) = \ln\left(\frac{s+a}{s+b}\right) \Rightarrow F(s) = \ln(s+a) - \ln(s+b)$$

$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

Applying Inverse Transform,

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right)$$

Using result

$$\begin{cases} L(e^{-at}) = \frac{1}{s+a} \Rightarrow L^{-1}\left(\frac{1}{s+a}\right) = e^{-at} \\ L\{tf(t)\} = -\frac{d}{ds}F(s) \Rightarrow L^{-1}\{F'(s)\} = (-1)t f(t) \end{cases}$$

Above relation becomes,

$$\begin{aligned} (-1)t f(t) &= e^{-at} - e^{-bt} \\ f(t) &= \frac{e^{-bt} - e^{-at}}{t} \end{aligned}$$

So,

$$L^{-1}(F(s)) = L^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\} = \frac{e^{-bt} - e^{-at}}{t}$$

This is our required result.

LECTURE NO. 114

Integration of Transforms:

Theorem: If $L\{f(t)\} = F(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)dx$, provided that $\lim_{t \rightarrow 0} \left\{\frac{f(t)}{t}\right\}$ exists.

Proof: Let

$$g(t) = \frac{f(t)}{t} \Rightarrow f(t) = t g(t)$$

Applying Laplace Transform,

$$L\{f(t)\} = L\{t g(t)\} = L\{t^1 g(t)\} = (-1) \frac{d}{ds} [G(s)]$$

$$F(s) = -\frac{d}{ds} [G(s)] = -\frac{d}{ds} L\{g(t)\}$$

$$\int_s^\infty F(s)ds = -[L\{g(t)\}]_s^\infty = -\lim_{s \rightarrow \infty} L\{g(t)\} + L\{g(t)\}$$

Since

$$\lim_{s \rightarrow \infty} L\{g(t)\} = \lim_{s \rightarrow \infty} \int_0^\infty e^{-st} g(t)dt = 0$$

So, above relation becomes

$$\begin{aligned} \int_s^\infty F(s)ds &= L\left\{\frac{f(t)}{t}\right\} = L\{g(t)\} = \int_s^\infty F(x)dx \\ L^{-1}\left\{\int_s^\infty F(x)dx\right\} &= \frac{f(t)}{t} \end{aligned}$$

This is our required result.

LECTURE NO. 115

Problem:

$$\text{Show that } \int_s^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

Solution: Since we know from previous lecture,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})d\tilde{s} \Rightarrow \int_s^\infty e^{-st} \frac{\sin t}{t} dt = \int_s^\infty \frac{1}{1+(\tilde{s})^2} d\tilde{s} = [\tan^{-1}]_s^\infty$$

$$\int_0^\infty \frac{1}{e^{st}} \frac{\sin t}{t} dt = \tan^{-1}(\infty) - \tan^{-1}(s)$$

Taking $\lim_{s \rightarrow \infty} \frac{1}{e^{st}} \Rightarrow \frac{1}{e^{st}} \rightarrow 1$ and $s \rightarrow 0 \Rightarrow \tan^{-1}(s) \rightarrow 0^-$

$$\int_0^\infty \frac{1}{e^{st}} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1}(0) \Rightarrow \int_0^\infty 1 \frac{\sin t}{t} dt = \frac{\pi}{2} - (0) = \frac{\pi}{2}$$

Hence prove.

LECTURE NO. 116

Theorem:

If $f(t)$ is a piecewise continuous function satisfying $\lim_{t \rightarrow 0} \frac{f(t)}{t} < \infty$, then show that

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tilde{s})d\tilde{s}$$

Proof: Let

$$g(t) = \int_0^t f(\tilde{s})d\tilde{s} \Rightarrow g(0) = 0 \quad \& \quad g'(t) = f(t)$$

$$L\{g(t)\} = L\{f(t)\}$$

$$sL\{g(t)\} - g(0) = F(s) \Rightarrow L\{g(t)\} = \frac{F(s)}{s}$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = g(t) = \int_0^t f(\tilde{s})d\tilde{s}$$

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^y f(\tilde{s})d\tilde{s} d\tilde{s}$$

Generalized form,

$$L^{-1}\left\{\frac{F(s)}{s^n}\right\} = \int_0^t \int_0^t \dots \int_0^t f(\tilde{s})d\tilde{s}^n$$

This is our required result.

LECTURE NO. 117

Problem:

$$L^{-1}\left\{\ln\left(\frac{s+b}{s+a}\right)^{-s^{-1}}\right\}; \quad a, b > 0$$

Proof:

$$\ln\left(\frac{s+b}{s+a}\right)^{-s^{-1}} = \ln\left(\frac{s+b}{s+a}\right)^{-\frac{1}{s}} = \ln\left(\frac{s+a}{s+b}\right)^{\frac{1}{s}} = \frac{1}{s} \ln\left(\frac{s+a}{s+b}\right) \dots (1)$$

This implies,

$$F(s) = \ln\left(\frac{s+a}{s+b}\right) = \ln(s+a) - \ln(s+b)$$

$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

Applying Inverse Transform,

$$\begin{aligned} L^{-1}(F'(s)) &= L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right) \\ (-1)^1 t f(t) &= e^{-at} - e^{-bt} \\ f(t) &= \frac{e^{-bt} - e^{-at}}{t} \end{aligned}$$

Hence,

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tilde{s}) d\tilde{s} = \int_0^t \frac{e^{-b\tilde{s}} - e^{-a\tilde{s}}}{\tilde{s}} d\tilde{s}$$

This is our required result.

LECTURE NO. 118

Heaviside's Expansion Theorem:

Let $F(s)$ and $G(s)$ be two polynomials on " s " where " $\text{Degree of } F(s) < \text{Degree of } G(s)$ ". If $G(s)$ has " n " distinct zeroes, " $\alpha_r; r = 1, 2, \dots, n$ " i.e. $G(s) = (s - \alpha_1)(s - \alpha_2) \dots (s - \alpha_n)$, then

$$L^{-1}\left[\frac{F(s)}{G(s)}\right] = \sum_{r=1}^n \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}$$

Proof: As given that, " $\text{Degree of } F(s) < \text{Degree of } G(s)$ " and $G(s)$ has " n " distinct zeroes, i.e. " $\alpha_r; r = 1, 2, \dots, n$ "

$$\begin{aligned} \frac{F(s)}{G(s)} &= \frac{F(s)}{(s - \alpha_1)(s - \alpha_2) \dots (s - \alpha_r) \dots (s - \alpha_n)} ; \quad 1 \leq r \leq n \in N \\ \frac{F(s)}{G(s)} &= \frac{A_1}{s - \alpha_1} + \frac{A_2}{s - \alpha_2} + \dots + \frac{A_r}{s - \alpha_r} + \dots + \frac{A_n}{s - \alpha_n} \end{aligned}$$

Multiplying with " $s - \alpha_r$ " on both sides,

$$\frac{F(s)}{G(s)}(s - \alpha_r) = \frac{A_1}{s - \alpha_1}(s - \alpha_r) + \frac{A_2}{s - \alpha_2}(s - \alpha_r) + \dots + \frac{A_r}{s - \alpha_r}(s - \alpha_r) + \dots + \frac{A_n}{s - \alpha_n}(s - \alpha_r)$$

Taking " $\lim_{s \rightarrow \alpha_r} (s - \alpha_r) = 0$ "

So, above relation becomes,

$$\begin{aligned} \lim_{s \rightarrow \alpha_r} \frac{F(s)}{G(s)}(s - \alpha_r) &= A_r \\ A_r &= F(\alpha_r) \lim_{s \rightarrow \alpha_r} \frac{(s - \alpha_r)}{G(s)} = F(\alpha_r) \lim_{s \rightarrow \alpha_r} \left[\frac{1}{G'(s)} \right] = \frac{F(\alpha_r)}{G'(\alpha_r)} \end{aligned}$$

Now;

$$\frac{F(s)}{G(s)} = \left(\frac{F(\alpha_1)}{G'(\alpha_1)} \right) \cdot \frac{1}{(s - \alpha_1)} + \left(\frac{F(\alpha_2)}{G'(\alpha_2)} \right) \cdot \frac{1}{(s - \alpha_2)} + \dots + \left(\frac{F(\alpha_r)}{G'(\alpha_r)} \right) \cdot \frac{1}{(s - \alpha_r)} + \dots + \left(\frac{F(\alpha_n)}{G'(\alpha_n)} \right) \cdot \frac{1}{(s - \alpha_n)}$$

Applying Inverse Transform and solving, we have

$$L^{-1}\left(\frac{F(s)}{G(s)}\right) = \frac{F(\alpha_1)}{G'(\alpha_1)} e^{\alpha_1 t} + \frac{F(\alpha_2)}{G'(\alpha_2)} e^{\alpha_2 t} + \dots + \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t} + \dots + \frac{F(\alpha_n)}{G'(\alpha_n)} e^{\alpha_n t}$$

We can write it as,

$$L^{-1}\left(\frac{F(s)}{G(s)}\right) = \sum_{r=1}^n \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}$$

This is our required result.

LECTURE NO. 119

Problem:

Use Heaviside's expansion theorem;

$$L^{-1} \left\{ \frac{2s^2 + 5s - 4}{s^3 + s^2 - 2s} \right\} = ?$$

Solution: Since we know from previous lecture

$$L^{-1} \left(\frac{F(s)}{G(s)} \right) = \sum_{r=1}^n \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t} \quad \dots (1)$$

Here

$$F(s) = 2s^2 + 5s - 4 \quad \& \quad G(s) = s^3 + s^2 - 2s = s(s-1)(s+2)$$

$$G'(s) = 3s^2 + 2s - 2,$$

And G(s) has distinct zeros i.e. $\alpha_1 = 0, 1, -2$

$$G'(0) = -2, \quad G'(1) = 3, \quad G'(-2) = 6$$

$$F(0) = -4, \quad F(1) = 3, \quad F(-2) = -6$$

Equation (1) becomes for given problem,

$$L^{-1} \left(\frac{F(s)}{G(s)} \right) = \sum_{r=1}^n \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t} = \frac{F(0)}{G'(0)} e^{0t} + \frac{F(1)}{G'(1)} e^{1t} + \frac{F(-2)}{G'(-2)} e^{-2t}$$

By putting values,

$$L^{-1} \left(\frac{F(s)}{G(s)} \right) = \frac{-4}{-2} (1) + \frac{3}{3} e^t + \frac{-6}{6} e^{-2t}$$

$$L^{-1} \left(\frac{F(s)}{G(s)} \right) = 2 + e^t - e^{-2t}$$

This is our required result.

LECTURE NO. 120

Laplace Transform of Rodriguez Formula of Laguerre's Polynomial:

Laguerre's Polynomial is given by;

$$l_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t})$$

Applying Laplace Transform,

$$L\{l_n(t)\} = L \left\{ \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) \right\} = ?$$

Calculation;

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{e^{-t} t^n\} = \frac{n!}{(s+1)^{n+1}}$$

$$L \left\{ \frac{d^n}{dt^n} (e^{-t} t^n) \right\} = s^n L\{e^{-t} t^n\} = \frac{s^n n!}{(s+1)^{n+1}}$$

$$\therefore F(s) \rightarrow F(s-1)$$

$$L \left\{ e^t \frac{d^n}{dt^n} (e^{-t} t^n) \right\} = n! \frac{(s-1)^n}{s^{n+1}}$$

$$\frac{1}{n!} L \left\{ e^t \frac{d^n}{dt^n} (e^{-t} t^n) \right\} = \frac{(s-1)^n}{s^{n+1}}$$

$$L \left\{ \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) \right\} = \frac{(s-1)^n}{s^{n+1}}$$

This is our required result.

LECTURE NO. 121

Solving ODE with Variable Coefficients (Laguerre's Equation)

$$ty'' + (1-t)y' + ny = 0; \quad n = 0, 1, 2, 3 \dots \quad \& \quad y(0) = y'(0) = 0$$

Solution: Taking Laplace Transform,

$$\begin{aligned} L\{ty''\} + L\{(1-t)y'\} + nL\{y\} &= 0 \\ L\{ty''\} + L(y') - L(ty') + nL\{y\} &= 0 \dots (1) \end{aligned}$$

Calculation;

$$\begin{aligned} L(y) &= Y \\ L(y') &= sL(y) - y(0) = sY = F'(s) \\ L(ty') &= -\frac{d}{ds}(F'(s)) = -\frac{d}{ds}[sY] = -Y - s\frac{dy}{ds} \\ L(ty'') &= -\frac{d}{ds}(F''(s)) = -\frac{d}{ds}[s^2Y] = -2sY - s^2\frac{dy}{ds} \end{aligned}$$

By putting values, equation (1) becomes,

$$\begin{aligned} \left(-2sY - s^2\frac{dy}{ds}\right) + sY - \left(-Y - s\frac{dy}{ds}\right) + nY &= 0 \\ (s - s^2)\frac{dy}{ds} + (n + 1 - s)Y &= 0 \\ s(1 - s)\frac{dy}{ds} &= (s - n - 1)Y \end{aligned}$$

Now separating the variables,

$$\begin{aligned} \frac{dy}{Y} &= \frac{(s - n - 1)}{s(1 - s)} ds = \left[\frac{1}{1 - s} - \frac{n + 1}{s(1 - s)}\right] ds \\ \frac{dy}{Y} &= \frac{1}{1 - s} ds - (n + 1) \left[\frac{1}{1 - s} + \frac{1}{s}\right] ds \\ \frac{dy}{Y} &= \left[\frac{1 - n - 1}{(1 - s)} - \frac{(n + 1)}{s}\right] ds = \left[\frac{n}{s - 1} - \frac{(n + 1)}{s}\right] ds \end{aligned}$$

Integrating;

$$\begin{aligned} \ln y &= n \ln(s - 1) - (n + 1) \ln s = \ln(s - 1)^n - \ln s^{n+1} \\ \ln y &= \ln \left[\frac{(s - 1)^n}{s^{n+1}} \right] \\ y &= \frac{(s - 1)^n}{s^{n+1}} = F(s) \end{aligned}$$

Applying Inverse Transform, this implies

$$y(t) = L^{-1}(y) = L^{-1}(F(s)) = L^{-1} \left[\frac{(s - 1)^n}{s^{n+1}} \right] = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t})$$

This is our required result.

LECTURE NO. 122

System of ODEs:

$$x' - y' - 2x - y = 6e^{3t} \quad ; \quad 2x' + y' - 3x - 3y = 6e^{3t} \quad ; \quad x(0) = 3, y(0) = 0$$

Applying Laplace Transform,

$$L(x') - L(y') - 2L(x) - L(y) = 6L(e^{3t}) \quad ; \quad 2L(x') + L(y') - 3L(x) - 3L(y) = 6L(e^{3t})$$

Let

$$L(x) = X, \quad L(y) = Y$$

Now, simplifying

$$\begin{aligned} sX - x(0) - sY + y(0) - 2X - Y &= 6\frac{1}{s - 3} \quad ; \quad 2sX - 2x(0) + sY - y(0) - 3X - 3Y = 6\frac{1}{s - 3} \\ (s - 2)X - (s + 1)Y &= 3 + \frac{6}{s - 3} \quad ; \quad (2s - 3)X + (s - 3)Y = 6 + \frac{6}{s - 3} \\ \frac{3s - 3}{s - 3} \quad ; \quad \frac{6s - 12}{s - 3} \end{aligned}$$

Solving Simultaneously,

$$\begin{cases} [(s-3)(s-2) + (s+1)(2s-3)]X = \frac{(3s-3)(s-3)}{(s-3)} + \frac{(s+1)(6s-12)}{(s-3)} \\ [(s-2)(s-3) - (s+3)(2s-3)]Y = \frac{(6s-12)(s-2)}{(s-3)} - \frac{(3s-3)(2s-3)}{(s-3)} \end{cases}$$

Simplify and Solve by Partial Fraction,

$$\begin{cases} X = \frac{3s^2 - 6s - 1}{(s-3)(s-1)^2} = \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{2}{(s-3)} \\ Y = -\frac{3s+5}{(s-3)(s-1)^2} = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-3)} \end{cases}$$

Applying Inverse Transform,

$$\begin{cases} x(t) = L^{-1}(X) = e^t + 2t e^t + 2e^{3t} \\ y(t) = L^{-1}(Y) = e^t - t e^t - e^{3t} \end{cases}$$

This is our required result.

LECTURE NO. 123

Laplace Transform of Some Partial Derivatives:

Theorem: If $u(x, t)$ is a function of “ x & t ”, then show that

$$\begin{cases} i): L\left(\frac{\partial u}{\partial t}\right) = s\tilde{u}(x, s) - u(x, 0) \\ ii): L\left(\frac{\partial^2 u}{\partial t^2}\right) = s^2\tilde{u}(x, s) - su(x, 0) - u_t(x, 0) \\ iii): L\left(\frac{\partial u}{\partial x}\right) = \frac{d}{dx}\tilde{u}(x, s) = \frac{d\tilde{u}}{dx} \\ iv): L\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{d^2\tilde{u}}{dx^2} \end{cases}$$

I):

$$\begin{aligned} L\left(\frac{\partial u}{\partial t}\right) &= \int_0^\infty e^{-st} \left(\frac{\partial u}{\partial t}\right) dt = e^{-st} [u(x, t)]_0^\infty - (-s) \int_0^\infty u(x, t) e^{-st} dt \\ L\left(\frac{\partial u}{\partial t}\right) &= 0 - 1 \cdot u(x, 0) + sL(u(x, t)) = s\tilde{u}(x, s) - u(x, 0) \\ L\left(\frac{\partial u}{\partial t}\right) &= s\tilde{u} - u(x, 0) \end{aligned}$$

II):

$$\begin{aligned} L\left(\frac{\partial^2 u}{\partial t^2}\right) &= L\left(\frac{\partial v}{\partial t}\right) \quad \text{Here let } \frac{\partial u}{\partial t} = v \\ L\left(\frac{\partial^2 u}{\partial t^2}\right) &= sL(v) - v(x, 0) = sL\left(\frac{\partial u}{\partial t}\right) - v(x, 0) \\ L\left(\frac{\partial^2 u}{\partial t^2}\right) &= s[sL(u) - u(x, 0)] - v(x, 0) = s^2L(u) - su(x, 0) - u_t(x, 0) \\ L\left(\frac{\partial^2 u}{\partial t^2}\right) &= s^2\tilde{u}(x, s) - su(x, 0) - u_t(x, 0) \end{aligned}$$

III):

$$\begin{aligned} L\left(\frac{\partial u}{\partial x}\right) &= \int_0^\infty e^{-st} \left(\frac{\partial u}{\partial x}\right) dt = \frac{d}{dx} \int_0^\infty (e^{-st} u) dt = \frac{d}{dx} L(u(x, t)) \\ L\left(\frac{\partial u}{\partial x}\right) &= \frac{d}{dx} \tilde{u}(x, s) = \frac{d\tilde{u}}{dx} \end{aligned}$$

IV):

$$L\left(\frac{\partial^2 u}{\partial x^2}\right) = L\left(\frac{\partial z}{\partial x}\right) \quad \text{Here let } \frac{\partial u}{\partial x} = z$$

$$L\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{d}{dx} [L(z)] = \frac{d}{dx} \left(\frac{d}{dx} \tilde{u}\right) = \frac{d^2}{dx^2} (\tilde{u})$$

$$L\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{d^2}{dx^2} \{\tilde{u}(x, s)\}$$

Prove completed.

LECTURE NO. 124

Problem:

$$\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u \quad ; \quad u(x, 0) = 6e^{3x} \quad \text{which is bounded for } x > 0, t > 0.$$

Solution: Applying Laplace Transform,

$$L\left(\frac{\partial u}{\partial x}\right) = 2L\left(\frac{\partial u}{\partial t}\right) + L(u)$$

$$\frac{\partial}{\partial x} \tilde{u} = 2[s\tilde{u}(x, s) - u(x, 0)] + \tilde{u}(x, s)$$

Putting given value and simplify,

$$\frac{\partial \tilde{u}}{\partial x} - (2s + 1)\tilde{u}(x, s) = -12e^{-3x} \quad \dots (1)$$

Now,

$$I.F = e^{-\int (2s+1)dx} = e^{-(2s+1)x}$$

So, solution of equation (1) is given by,

$$(I.F) \times \tilde{u} = c + \int (I.F) \times (R.H.S) dx$$

$$(e^{-(2s+1)x}) \times \tilde{u} = c + \int (e^{-(2s+1)x}) \times (-12e^{-3x}) dx$$

$$(e^{-(2s+1)x}) \times \tilde{u} = c - 12 \int e^{-(2s+1)x} e^{-3x} dx = c - 12 \int e^{-(2s+4)x} dx$$

$$(e^{-(2s+1)x}) \times \tilde{u} = c - \frac{12}{-(2s+4)} e^{-(2s+4)x}$$

$$\tilde{u} = c e^{(2s+1)x} + \frac{6}{(s+2)} e^{-3x} \quad \dots (2)$$

Given that $u(x, t)$ bounded as $x \rightarrow \infty, t \rightarrow \infty$.

Put $c = 0$, in equation (2)

$$L(u) = \tilde{u} = \frac{6}{(s+2)} e^{-3x}$$

$$u(x, t) = L^{-1}(u) = L^{-1}\left(\frac{6}{(s+2)} e^{-3x}\right)$$

$$u(x, t) = 6e^{-3x} L^{-1}\left(\frac{1}{s+2}\right) = 6e^{-3x} e^{-2t}$$

$$u(x, t) = 6e^{-3x-2t}$$

This is our required result.

{- THE END -}

"Dear, Good Luck For Exam."