

September 2021

STA642

Q No. 1

$$f(x, y) = \begin{cases} 2 & , 0 < x < y, 0 < y < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

(a) Find the marginal distributions:

(i) Marginal distribution of X :

$$f_x(x) = \int_x^1 2 dy = 2[y]_x^1 = 2(1-x),$$

(ii) Marginal distribution of Y :

$$f_y(y) = \int_0^y 2 dx = 2[x]_0^y = 2(y-0) = 2y$$

(b) Find the conditional distributions:

$$f(x|y) = \frac{f(x, y)}{f_y(y)} = \frac{2}{2y} = \frac{1}{y}$$

$$\text{and } f(y|x) = \frac{f(x, y)}{f_x(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}$$

Q No. 2 Define Incomplete Gamma Function

Ans: In mathematics, the upper and lower incomplete gamma functions are types of special functions.

The upper incomplete gamma function is defined as:

$$\Gamma(\alpha, y) = \int_y^{\infty} t^{\alpha-1} e^{-t} dt$$

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The lower incomplete gamma function is defined as:

$$\gamma(\alpha, y) = \int_0^y t^{\alpha-1} e^{-t} dt$$

It is interesting to note that

$$\gamma(\alpha, y) + \Gamma(\alpha, y) = \Gamma(\alpha)$$

Q No. 3: In which conditions the poisson distribution is applicable?

Ans: Conditions of poisson distribution:

- (i) An event can occur any number of times during a time period
- (ii) Events occur independently.

Q No. 4 Write down the formula of cumulative distribution function of normal distribution.

Ans:

$$F(x) = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{x - \mu}{\sigma \sqrt{2}} \right) \right] \text{ for } -\infty < x < \infty$$

erf is the error function.

Q No. 5: The mean and variance of poisson distribution is 4. Find ~~Covariance~~.

Given $\mu = \sigma^2 = 4$

$$COV = \frac{\sigma}{\mu} \times 100\% = \frac{\sqrt{4}}{4} \times 100\% = \frac{2}{4} \times 100\% = 50\%$$

Q No. 6: Derivation of the mean of exponential distribution:

Sol. The PDF of exponential distribution
 $f(x) = \lambda e^{-\lambda x} \quad x \geq 0, \lambda > 0$

$$\begin{aligned}\text{Mean} = E(X) &= \int_0^{\infty} x f(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx \\&= \left[-x e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \right]_0^{\infty} \\&= - \left[x e^{-\lambda x} \right]_0^{\infty} - \left[\frac{1}{\lambda} e^{-\lambda x} \right]_0^{\infty} \\&= - [0 - 0] - \left[0 - \frac{1}{\lambda} \right] \\&= 0 - \left[-\frac{1}{\lambda} \right] = \frac{1}{\lambda}\end{aligned}$$

$$\text{Mean} = E(X) = \mu = \frac{1}{\lambda}$$

Derivation of variance of exponential distribution

$$E(X^2) = \int_0^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \dots = \frac{2}{\lambda^2}$$

$$E(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\text{S.D.}(X) = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda}$$

Q No. 7 General Concept of Ellipse

Ans: We may assume that one of the many diameters of the circle is being "elongated"

As soon as we "elongate" one of the diameters, we get a major axis as well as a minor axis.

Q No. 8 The concept of Contours

Ans: By the term "contour", we mean the set of points on the X, Y floor corresponding to which the ordinates are of equal height.

Q No. 9: Drive the mean of normal distribution.

Sol:
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{let } z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + z\sigma \Rightarrow dx = \sigma dz$$

$$\Rightarrow E(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + z\sigma) e^{-\frac{z^2}{2}} (\sigma dz)$$

$$E(X) = \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz + \underbrace{\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz}_{\text{odd function} = 0}$$

$$E(X) = \mu \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz + 0$$

$$E(X) = \mu [1] + 0 = \mu$$

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$$

The area under a normal curve is unity (1).

μ is the mean of Normal Distribution.

Q NO. 10. What do you mean by statistically Independent variables?

Ans: Statistical models normally specify how one set of variables, called dependent variables, functionally depend on another set of variables, called independent variables.

OR Two random variables X and Y are independent, statistically independent if

$$f_{XY}(x, y) = f_X(x) f_Y(y).$$

Q No. 11: Write down PDF of exponential distribution with $\theta = 1.5$.

Ans. PDF of exponential distribution.

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta} \quad x \geq 0, \theta > 0$$

$$\theta = 1.5$$

$$f(x; 1.5) = \frac{1}{1.5} e^{-x/1.5} \quad 0 < x < \infty$$

Q No. 12: Derive the CV of exponential distribution.

Ans:

$$CV = \frac{\sigma}{\mu}$$

$$\sigma = \frac{1}{\lambda} \quad \text{and} \quad \mu = \frac{1}{\lambda}$$

$$CV = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}} = 1$$

$$CV = 1$$

Coefficient of variation of exponential distribution is always equal to 1.

Q No. 13: Write down MGF of Binomial distribution.

Ans: MGF of Binomial distribution:

$$M(t) = [(1-p) + pe^t]^n \quad \forall \text{ value of } t.$$

Q No 14: If $p = 0.2$, then find mean of the geometric distribution.

Sol $p = 0.2$

$$\text{Mean} = E(X) = \mu = \frac{1}{p} = \frac{1}{0.2} = 5$$

Q No 15: Covariance of Multinomial distribution.

Ans: The covariance of multinomial distribution is

$$\text{Cov}(X_i, X_j) = -np_i p_j \quad i \neq j$$

Q No 16: Theorem: If X and Y have a bivariate normal distribution with correlation coefficient ρ . Then X and Y are independent if and only if $\rho = 0$.

$$\text{i.e. } f_{xy}(x, y) = f_x(x) f_y(y)$$

proof:

$$f_{xy}(x, y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 - 2\rho \left(\frac{x-\mu_x}{\sigma_x} \right) \left(\frac{y-\mu_y}{\sigma_y} \right) \right]}$$

Substituting $\rho = 0$ in the pdf of the bivariate normal distribution, we have.

$$f_{xy}(x, y) = \frac{1}{2\pi \sigma_x \sigma_y} e^{-\frac{1}{2} \left[\left(\frac{x-\mu_x}{\sigma_x} \right)^2 + \left(\frac{y-\mu_y}{\sigma_y} \right)^2 \right]}$$

Further simplifying, we have:

$$f_{xy}(x, y) = \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\delta_x\delta_y} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\delta_x}\right)^2} \cdot e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\delta_y}\right)^2}$$

$$f_{xy}(x, y) = \frac{1}{\sqrt{2\pi}\delta_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\delta_x}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\delta_y} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\delta_y}\right)^2}$$

$$f_{xy}(x, y) = f_x(x) f_y(y)$$

(Hence proved)

Q No 17: PDF of bivariate distribution.

$$N(\mu_x, \mu_y, \delta_x^2, \delta_y^2, \rho)$$

Ans: The probability density function of bivariate distribution (x, y) is

$$f_{xx}(x, y) = \frac{1}{2\pi\delta_x\delta_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\delta_x}\right)^2 + \left(\frac{y-\mu_y}{\delta_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\delta_x}\right)\left(\frac{y-\mu_y}{\delta_y}\right)\right]}$$

where $-\infty < x < \infty$ and $-\infty < y < \infty$

and the parameters are such that

$$-\infty < \mu_x, \mu_y < \infty,$$

$$\delta_x, \delta_y > 0$$

$$-1 < \rho < 1$$

Solved By Mahar Afaq Safdar Muhammadi

Q No. 18: properties of Gamma Function.

Ans:

(i) If $\alpha = 1$, clearly $\Gamma(1) = \int_0^{\infty} e^{-y} dy = 1$

(ii) If $\alpha > 1$, an Integration by parts show that

$$\Gamma(\alpha) = (\alpha-1) \int_0^{\infty} y^{\alpha-2} e^{-y} dy$$

$$\Gamma(\alpha) = (\alpha-1) \Gamma(\alpha-1)$$

Accordingly,

if α is positive integer greater than 1.

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2) \cdots (3)(2)(1) \Gamma(1)$$

$$= (\alpha-1)! (1)$$

$$\Gamma(\alpha) = (\alpha-1)!$$

Since $\Gamma(1) = 1$, this suggests we take

$$\Gamma(1) = 0! = 1$$

It is important to note that the value of a gamma function is always positive.

Q No. 19 properties of Beta distribution

Ans:

$$(i) \text{ Mean } = \mu = E(X) = \frac{\alpha}{\alpha + \beta}$$

$$(ii) \text{ Variance } = \sigma^2 = E(X^2) - [E(X)]^2 \\ = \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}$$

$$(iii) \text{ Mode } = \frac{\alpha - 1}{\alpha + \beta - 2} \quad \alpha, \beta > 1$$

Q No. 20 Example: If it is known that, on the average, two minor accidents occur at a busy crossing per day, what is the probability that, at the most two minor accidents will occur during an entire week?

Sol:

This is a poisson process with Occurance Rate $= \lambda = 2$ per day

Also Time ~~one~~ week = 7 days $\Rightarrow t = 7$

Hence $\lambda t = 2 \times 7 = 14$

$$P(X = x) = \begin{cases} \frac{e^{-\lambda t} (\lambda t)^x}{x!} = \frac{e^{-14} (14)^x}{x!}, & x = 0, 1, 2, \dots \\ 0 & \text{elsewhere} \end{cases}$$

Solved By Mahar Afaq Safdar Muhammadi

Now we require $X \leq 2$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{e^{-14} (14)^0}{0!} + \frac{e^{-14} (14)^1}{1!} + \frac{e^{-14} (14)^2}{2!}$$

$$= e^{-14} + 14e^{-14} + 98e^{-14}$$

$$= 113e^{-14}$$

$$= 0.00009396$$

$$P(X \leq 2) = 9.396 \times 10^{-5}$$

We can say that the probability that at most 2 accidents will occur at a busy crossing in a week, this probability is almost 0, which is "Nent to impossible".

Q No. 21: Write down the mean of the multinomial distribution.

Ans:

The mean of multinomial distribution is

$$E(X_i) = np_i$$

The variance of multinomial distribution is

$$\text{Var}(X_i) = np_i(1-p_i)$$

$$= np_i q_i$$

Q No. 22: Find the first derivative of

$$K(t) = \mu t + \frac{\sigma^2 t^2}{2}$$

Sol: Differentiate with respect to "t"

$$K'(t) = \frac{d}{dt} \left(\mu t + \frac{\sigma^2 t^2}{2} \right)$$

$$= \mu \frac{d}{dt} (t) + \frac{\sigma^2}{2} \frac{d}{dt} (t^2)$$

$$= \mu (1) + \frac{\sigma^2}{2} (2t)$$

$$K'(t) = \mu + \sigma^2 t$$

Q No. 23: prove the theorem Bivariate Normal Distribution.

Statement: If X and Y have a bivariate normal distribution with correlation ρ . Then X and Y are independent if and only if $\rho = 0$

Proof: ~~If X and Y are independent, then $\rho = 0$~~ The statement iff means:

(a) If X and Y are independent, then $\rho = 0$

(b) If $\rho = 0$, then X and Y are independent

(a) In general, if X and Y are independent random variables, then their correlation coefficient is 0.

As such, there is no need to prove the first part of statement.

(b) We know that

$$f_{xy}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}$$

Substituting $\rho=0$ in the pdf of the B.N.D. we have

$$f_{xy}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$\Rightarrow f_{xy}(x, y) = \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \cdot e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$\Rightarrow f_{xy}(x, y) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$\Rightarrow f_{xy}(x, y) = f_x(x) f_y(y) \quad (\text{Hence proved})$$

X and Y are independent.

Q No. 24: Example:

If X is $b(n, 1/3)$, then $P(Y \geq 1) = 1 - (2/3)^n$

Find the smallest value of n that yields
 $P(Y \geq 1) > 0.80$

Sol: let us first see how

$$\begin{aligned} P(Y \geq 1) &= 1 - P(Y < 1) \\ &= 1 - P(Y = 0) \\ &= 1 - \left(\frac{2}{3}\right)^n \end{aligned}$$

We know that

$$P(Y=y) = \binom{n}{y} p^y (1-p)^{n-y}$$

$$P(Y=0) = \binom{n}{0} p^0 (1-p)^{n-0}$$

$$P(Y=0) = 1(1) (1-p)^n = (1-p)^n$$

Here: $p = 1/3$

$$P(Y=0) = \left(1 - \frac{1}{3}\right)^n = \left(\frac{2}{3}\right)^n$$

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Now we required to find the smallest value of n .

$$P(Y \geq 1) > 0.80$$

$$1 - \left(\frac{2}{3}\right)^n > 0.80$$

$$1 - 0.80 > \left(\frac{2}{3}\right)^n$$

$$0.20 > \left(\frac{2}{3}\right)^n$$

Taking "ln"

$$\ln(0.20) > \ln\left(\left(\frac{2}{3}\right)^n\right)$$

$$-1.6094 > n \ln\left(\frac{2}{3}\right)$$

$$\Rightarrow -1.60947 - 0.4055n$$

$$1.6094 < 0.4055n$$

$$\frac{1.6094}{0.4055} \leq n$$

$$n > 3.97$$

$$n \leq 4$$

Q No. 25: Write two properties of normal curve.

Ans: properties of normal curve are:

(i) The normal curve is asymptotic to the x-axis.

(ii) The total area of normal curve is unity.

(iii) The maximum ordinate of the Normal Distribution at $x = \mu$ is equal to $\frac{1}{\sigma\sqrt{2\pi}}$.

Q No. 26: Write the formula of variance of the Hypergeometric distribution.

Ans: The variance of Hypergeometric distribution is:

$$\text{Variance} = n \cdot \frac{K}{N} \cdot \frac{(N-K)}{N} \cdot \frac{N-n}{N-1}$$

or $\text{Variance} = npq \left(\frac{N-n}{N-1} \right), p = \frac{K}{N}, q = \frac{N-K}{N}$

Q No 27: Write PMF of Binomial distribution

Ans: PMF of the Binomial distribution is

$$P(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \dots, n \\ 0 & \text{elsewhere.} \end{cases}$$

Q No 28: Derive variance of Continuous Uniform distribution.

Ans: The PDF of C.U.D

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 \longrightarrow \textcircled{1}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_a^b x \cdot \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$E(X) = \frac{1}{2(b-a)} (b^2 - a^2) = \frac{a+b}{2}$$

$$E(X^2) = \int_{-a}^{\infty} x^2 f(x) dx$$

$$= \int_a^b x^2 \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b$$

$$= \frac{1}{3(b-a)} [b^3 - a^3]$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$$

From ①

$$\text{Var}(X) = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2} \right)^2$$

$$= \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12}$$

$$= \frac{a^2 - 2ab + b^2}{12}$$

$$\text{Var}(X) = \frac{(a-b)^2}{12}$$

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Q No. 29: What is another name of rectangle distribution?

Ans: Uniform distribution is another name of rectangle distribution.

Q No. 30: MGF of Multinomial distribution.

Ans: The moment generating function of the multinomial distribution is

$$M_{\underline{X}}(t) = M_{\underline{X}}(t_1, t_2, \dots, t_k) \\ = \left(\sum_{i=1}^K p_i e^{t_i} \right)^n$$

Q No. 31: MGF of Normal distribution.

Ans: The MGF of Normal distribution is

$$M(t) = e^{\mu t + \frac{1}{2} \sigma^2 t^2} \quad \text{for } -\infty < t < \infty$$

Q No. 32: Write CDF of continuous uniform distribution.

Ans: The CDF of the continuous uniform distribution is:

$$f(x) = \begin{cases} \frac{1}{b-a} & ; a \leq x \leq b \\ 0 & ; \text{elsewhere} \end{cases}$$

Q No. 33: Write the formula of variance in discrete uniform distribution.

Ans: The Variance of discrete uniform distribution is

$$\text{Var}(X) = \sigma^2 = \frac{n^2 - 1}{12}$$

Q No. 34: Write the formula of mean in discrete uniform distribution.

Ans: The mean of discrete uniform distribution is

$$\text{Mean} = E(X) = \mu = \sum_{x=1}^n x \cdot p(x)$$

$$\text{Mean} = \mu = \sum_{x=1}^n x \cdot \frac{1}{n} = \frac{1}{n} (1 + 2 + 3 + \dots + n)$$

$$\text{Mean} = \mu = \frac{1}{n} \times \frac{n(n+1)}{2}$$

$$\text{Mean} = \mu = \frac{n+1}{2}$$

Q No 35: Derivation of variance of the discrete uniform distribution:

Ans:

$$\text{Var}(X) = E(X^2) - [E(X)]^2 \rightarrow \text{①}$$

$$\begin{aligned} E(X^2) &= \sum x^2 p(x) = \sum x^2 \cdot \frac{1}{n} \\ &= \frac{1}{n} \sum x^2 \end{aligned}$$

$$E(X^2) = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$\begin{aligned} E(X) &= \sum x p(x) = \sum x \cdot \frac{1}{n} \\ &= \frac{1}{n} \sum x \\ &= \frac{1}{n} \cdot \frac{n(n+1)}{2} \end{aligned}$$

$$E(X) = \frac{n+1}{2}$$

From ①

$$\begin{aligned} \text{Var}(X) &= \frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2} \right)^2 \\ &= \frac{4n(n+1)(2n+1) - 3(n+1)^2}{24} \end{aligned}$$

$$\text{Var}(X) = \frac{4(2n^2 + n + 2n + 1) - 6(n^2 + 2n + 1)}{24}$$

$$= \frac{2[4n^2 + 2n + 4n + 2 - 3n^2 - 6n - 3]}{24}$$

$$= \frac{2[n^2 - 1]}{24}$$

$$\text{Var}(X) = \frac{n^2 - 1}{12}$$

Q No. 36: A oil company conducts a geological study that indicates that an exploratory oil well should have a 20% chance of striking oil. The company is waiting success twice (2) and is willing to try upto 5 times.

What is the probability that the second success comes on the fifth well drilled.

Sol:

We know that, the pmf of X is.

$$P(X=x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, r+2, \dots$$

$$p = 20\% = 0.20 \quad \text{and} \quad r = 2$$

$$P(X=x) = \binom{x-1}{1} (0.20)^2 (1-0.20)^{x-2}, \quad x = 2, 3, 4, \dots$$

$$P(X=x) = \binom{x-1}{1} (0.20)^2 (0.80)^{x-2}, \quad x = 2, 3, 4, \dots$$

We need to find $P(X=5)$

$$P(X=5) = \binom{5-1}{1} (0.2)^2 (0.8)^{5-2}$$

$$= \binom{4}{1} (0.2)^2 (0.8)^3$$

$$= 4 (0.04) (0.512)$$

$$P(X=5) = 0.08192$$

$$P(X=5) = 8.192 \%$$

Q No. 37: Find Coefficient of variation in Poisson distribution.

Ans: Mean and Variance of Poisson distribution is μ .

$$E(X) = \mu, \text{ Var}(X) = \mu$$

$$COV = \frac{S.D}{\text{Mean}} = \frac{\sqrt{\mu}}{\mu} = \frac{1}{\sqrt{\mu}}$$

Q No. 38: What is PDF of normal random variable X or standard normal distribution of X .

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

Q No. 39: The names of 5 men and 5 women are written on slips of paper and placed in a hat. Four names are drawn. What is probability that 2 are men and 2 are women?

Sol:

Let X denote the number of men.

Then, $N = 5 + 5 = 10$ names to be drawn from.

$K = 5$ and $n = 4$

and here the possible values of x are

0, 1, 2, 3, 4, i.e. n .

Hence the hypergeometric distribution is

$$h(x; N, n, K) = \frac{\binom{K}{x} \binom{N-K}{n-x}}{\binom{N}{n}}$$
$$h(x; 10, 4, 5) = \frac{\binom{5}{x} \binom{10-5}{4-x}}{\binom{10}{4}}$$

and the required probability i.e. $P(X=2)$ is

$$h(2; 10, 4, 5) = \frac{\binom{5}{2} \binom{5}{4-2}}{\binom{10}{4}} = \frac{\binom{5}{2} \binom{5}{2}}{\binom{10}{4}}$$

$$= \frac{(10)(10)}{210} = \frac{10}{21} \text{ Ans}$$

Q No. 40: The probability that a planted radish seed germinates is 0.80. A gardener plants nine seeds.

- (i) What is the average number of seeds.
- (ii) What is the standard deviation of X .
- (iii) What is the probability that, out of nine seeds, at least 8 will germinate?

Sol:

Let X denote the number of radish seeds that successfully germinate.

(i) We know that the mean of X is np

$$\text{Mean} = \mu = np$$

$$n = 9 \text{ and } p = 0.80$$

$$\text{Mean} = \mu = 9 \times 0.80 = 7.2 \text{ seeds}$$

(ii) We know that the variance of X is $np(1-p)$

$$\text{Variance}(X) = np(1-p)$$

$$= 9(0.80)(1-0.80)$$

$$= 9(0.80)(0.20)$$

$$\text{Var}(X) = 1.44$$

$$\text{S.D.}(X) = \sqrt{1.44} = 1.20$$

(iii) We know that

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$n=9, \quad p=0.80$$

$$P(X=x) = \binom{9}{x} (0.80)^x (0.20)^{9-x}$$

$$P(\text{at least } 8) = P(X \geq 8) = P(X=8) + P(X=9)$$

$$= \binom{9}{8} (0.80)^8 (0.20)^{9-8} + \binom{9}{9} (0.80)^9 (0.20)^{9-9}$$

$$= (9)(0.80)^8 (0.20) + (1)(0.80)^9 (1)$$

$$= 9 \times 0.1678 \times 0.20 + 0.1342$$

$$= 0.3020 + 0.1342$$

$$= 0.4362$$

$$= 43.62\%$$

Q No. 41: Let the joint pdf of X_1 and X_2 be

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & 0 < x_1, x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Show that X_1 and X_2 dependent.

Sol: Here the marginal probability density function are obtained as follows:

$$f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2$$

$$= \int_0^1 (x_1 + x_2) dx_2$$

$$= \left[x_1 x_2 + \frac{x_2^2}{2} \right]_0^1$$

$$= (x_1)(1) - (x_1)(0) + \frac{(1)^2}{2} - \frac{(0)^2}{2}$$

$$= x_1 - 0 + \frac{1}{2} - 0$$

So $f(x_1) = x_1 + \frac{1}{2}, \quad 0 < x_1 < 1$

Now $f(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_1$

$$= \int_0^1 (x_1 + x_2) dx_1$$

$$= \left[\frac{x_1^2}{2} + x_1 x_2 \right]_0^1$$

$$= \frac{(1)^2}{2} - \frac{(0)^2}{2} + (1)(x_2) - (0)(x_2)$$

$$f(x_2) = \frac{1}{2} + x_2, \quad 0 < x_2 < 1$$

Now $f(x_1) f(x_2) = \left(x_1 + \frac{1}{2}\right) \left(x_2 + \frac{1}{2}\right)$

$$f(x_1) f(x_2) = x_1 x_2 + \frac{1}{2} x_1 + \frac{1}{2} x_2 + \frac{1}{4}$$

$$f(x_1) f(x_2) \neq f(x_1, x_2)$$

So the random variables X_1 and X_2 are dependent (i.e. not independent).

Q No. 42: Write down the PDF of joint normal distribution of X_1 and X_2 .

Sol: The PDF of joint normal distribution of X_1 and X_2 is:

$$f(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1 - \mu_1}{\sigma_1} \right)^2 + \left(\frac{x_2 - \mu_2}{\sigma_2} \right)^2 - 2\rho \left(\frac{x_1 - \mu_1}{\sigma_1} \right) \left(\frac{x_2 - \mu_2}{\sigma_2} \right) \right]}$$

for $-\infty < x_1 < \infty$ and $-\infty < x_2 < \infty$

where $-\infty < \mu_1, \mu_2 < \infty$, $\sigma_1 > 0$, $\sigma_2 > 0$

and $-1 \leq \rho \leq 1$

Q No. 43: In Bivariate Distribution; What is meant by the " X_1, X_2 floor"?

Ans: Consider the three-dimensional space for example the room in which you are sitting right now.

★ Now focus on the floor of the room.

★ One "edge" of the floor will be called the X_1 axis.

★ The other "edge" of the floor will be called the X_2 axis.

And as such, the floor will be called the " X_1, X_2 floor" (actually it is the X_1, X_2 -plane)

Q No. 44: Drive MGF of the exponential distribution.

Sol:
$$M_0(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$
$$= \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty} = \lambda \left[\frac{-e^{-(\lambda-t)x}}{\lambda-t} \right]_0^{\infty}$$

Now if $t < \lambda$ then $\lambda - t > 0$, we have

$$M_0(t) = \lambda \left[\frac{-e^{-\infty}}{\lambda - t} - \frac{-e^0}{\lambda - t} \right] = \lambda \left[\frac{0}{\lambda - t} + \frac{1}{\lambda - t} \right]$$

$$M_0(t) = \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda$$

Q No. 45: Define poisson process

Ans: The poisson process is one of the most widely used counting processes.

Whenever a particular event of interest occurs randomly over a time scale, we say that we are dealing with a poisson process.

Q No. 46: Write down the PDF of normal distribution.

Ans. A random variable X is said to be a normal distribution if its PDF is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

Where μ is the mean and $\sigma > 0$ is the standard deviation of the distribution $\sim N(\mu, \sigma^2)$.

Q No. 47: A statistical class takes two exams Exam 1 (Mid term Exam) and Exam 2 (Final term Exam), follow a bivariate normal distribution with parameters:

$$\mu_x = 70, \mu_y = 60$$
$$\sigma_x = 10, \sigma_y = 15$$
$$\rho = 0.6$$

(1) What is the probability that the student scores over 75 on Exam 1 (Midterm Exam)?

Sol: mean = $\mu_x = 70$

Variance = $\sigma_x = 10$

i.e. $X \sim N(70, 10)$

Hence the probability that a randomly selected student scores over 75 in Exam 1 is given by

$$P(\text{over } 75) = P(X > 75) = P\left(Z > \frac{x - \mu_x}{\sigma_x}\right)$$

$$= P\left(Z > \frac{75 - 70}{10}\right) = P(Z > 0.5)$$

$$= 1 - P(Z < 0.5)$$

$$= 1 - 0.6915$$

$$= 0.3085$$

$$P(\text{over } 75) = 30.85\% \approx 31\%$$

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Q No. 48: The duration of long distance telephone calls is found to be exponentially distributed with a mean of 3 minutes.

What is the probability that a call will last

(i) more than 3 minutes.

(ii) more than 5 minutes.

Sol: Let X be the exponential r.v with parameter θ . Then $\text{mean} = \theta = 3$
The pdf can be written as

$$f(x) = \int_3^{\infty} \left(\frac{1}{\theta}\right) e^{-x/\theta} dx$$

$$\begin{aligned} \text{(i)} \quad P(\text{more than 3}) &= P(X > 3) = \int_3^{\infty} \left(\frac{1}{3}\right) e^{-x/3} dx \\ &= \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3} \right]_3^{\infty} = -e^{-\infty} + e^{-3/3} \end{aligned}$$

$$P(X > 3) = 0 + e^{-1} = e^{-1} = 0.3679$$

$$\text{(ii)} \quad P(\text{more than 5}) = P(X > 5) = \int_5^{\infty} \left(\frac{1}{3}\right) e^{-x/3} dx$$

$$P(X > 5) = \int_5^{\infty} \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3} \right]_5^{\infty} = -e^{-\infty} + e^{-5/3}$$

$$P(X > 5) = 0 + e^{-5/3} = e^{-1.7} = 0.1827$$

Solved By Mahar Afaq Safdar Muhammadi